

# Computation of conformal anomalies: Loop calculations

A free massless scalar field in two dimensions:

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{ab} \partial_a \phi \partial_b \phi. \quad (1)$$

We define  $h^{ab}$  by

$$g^{ab} = \delta^{ab} + h^{ab}. \quad (2)$$

We also set a rule that the indices of  $h^{ab}$  are lowered by  $\delta_{ab}$ .

$$\implies g_{ab} = \delta_{ab} - h_{ab} + h_a^c h_{cb} - \dots, \quad (3)$$

$$\sqrt{g} = 1 - \frac{1}{2} h_a^a + \dots. \quad (4)$$

$$\begin{aligned} \implies T^{ab} &= -\frac{1}{\alpha'} \left( \partial^a \phi \partial^b \phi - \frac{1}{2} g^{ab} g^{cd} \partial_c \phi \partial_d \phi \right) \\ &= -\frac{1}{\alpha'} \left( \partial^a \phi \partial^b \phi - \frac{1}{2} \delta^{ab} \delta^{cd} \partial_c \phi \partial_d \phi - \frac{1}{2} (\delta^{ab} h^{cd} + \delta^{cd} h^{ab}) \partial_c \phi \partial_d \phi + \dots \right). \end{aligned} \quad (5)$$

$\alpha'$  can be absorbed by the rescaling

$$\frac{1}{\sqrt{2\pi\alpha'}} \phi \longrightarrow \phi \quad (6)$$

so that

$$\begin{aligned} S &= \int d^2\sigma \frac{1}{2} \sqrt{g} g^{ab} \partial_a \phi \partial_b \phi \\ &= \int d^2\sigma \left[ \frac{1}{2} \delta^{ab} \partial_a \phi \partial_b \phi + \frac{1}{2} (-h^{ab} + \frac{1}{2} h \delta^{ab}) \partial_a \phi \partial_b \phi \right. \\ &\quad \left. + \underbrace{\frac{1}{2} \left( h_a^c h^{cb} - \frac{1}{2} h h^{ab} + \left( -\frac{1}{4} h_c^d h_d^c + \frac{1}{8} h^2 \right) \delta^{ab} \right)}_{\equiv H_2^{ab}} \partial_a \phi \partial_b \phi + \dots \right], \end{aligned} \quad (7)$$

where  $h \equiv h_a^a$ . The one-loop effective action of  $\phi$  is

$$\begin{aligned} \Gamma &\equiv -\log \int [d\phi] e^{-S} \\ &= -\log \int [d\phi] e^{-\int d^2\sigma \frac{1}{2} \phi (-\square) \phi} \exp \left[ -\int d^2\sigma \left( \underbrace{\frac{1}{2} (-h^{ab} + \frac{1}{2} h \delta^{ab})}_{\equiv -\tilde{h}^{ab}} \partial_a \phi \partial_b \phi \right. \right. \\ &\quad \left. \left. + \frac{1}{2} H_2^{ab} \partial_a \phi \partial_b \phi + \dots \right) \right] \\ &= -\log \int [d\phi] e^{-\int d^2\sigma \frac{1}{2} \phi (-\square) \phi} \left[ 1 + \int d^2\sigma \frac{1}{2} \tilde{h}^{ab} \partial_a \phi \partial_b \phi - \int d^2\sigma \frac{1}{2} H_2^{ab} \partial_a \phi \partial_b \phi \right. \\ &\quad \left. + \frac{1}{2!} \frac{1}{4} \int d^2\sigma \int d^2\sigma' (-\tilde{h}^{ab}(\sigma) \partial_a \phi(\sigma) \partial_b \phi(\sigma)) (-\tilde{h}^{cd}(\sigma') \partial_c \phi(\sigma') \partial_d \phi(\sigma')) + \dots \right]. \end{aligned} \quad (8)$$

The propagator :

$$\begin{aligned}\langle\phi(\sigma)\phi(\sigma')\rangle &= -\square^{-1}(\sigma-\sigma') \\ &= \int \frac{d^2k}{(2\pi)^2} \frac{e^{ik_a(\sigma^a-\sigma'^a)}}{k^2}.\end{aligned}\quad (9)$$

Note that the tadpole

$$\langle\partial_a\phi(\sigma)\partial_b\phi(\sigma)\rangle = \int \frac{d^2k}{(2\pi)^2} \frac{-k_ak_b}{k^2} \quad (10)$$

does not contain any dimensionful parameter so is treated as zero in the dimensional regularization. Therefore

$$\begin{aligned}\Gamma &= -\frac{1}{2!4} \int d^2\sigma \int d^2\sigma' \tilde{h}^{ab}(\sigma) \tilde{h}^{cd}(\sigma') \langle\partial_a\phi(\sigma)\partial_b\phi(\sigma)\partial_c\phi(\sigma')\partial_d\phi(\sigma')\rangle_{conn} + \dots \\ &= -\frac{1}{2!4} \int d^2\sigma \int d^2\sigma' \tilde{h}^{ab}(\sigma) \tilde{h}^{cd}(\sigma') \left( \frac{\partial}{\partial\sigma^a} \frac{\partial}{\partial\sigma'^c} \langle\phi(\sigma)\phi(\sigma)\rangle \frac{\partial}{\partial\sigma^b} \frac{\partial}{\partial\sigma'^d} \langle\phi(\sigma)\phi(\sigma)\rangle + (c \leftrightarrow d) \right) \\ &\quad + \dots \\ &= -\frac{1}{2!4} \int d^2\sigma \int d^2\sigma' \underbrace{\int \frac{d^2p}{(2\pi)^2} e^{ip\sigma} \tilde{h}^{ab}(p)}_{\tilde{h}^{ab}(\sigma)} \underbrace{\int \frac{d^2q}{(2\pi)^2} e^{iq\sigma'} \tilde{h}^{cd}(q)}_{\tilde{h}^{cd}(\sigma')} \\ &\quad \times \left( \int \frac{d^2k}{(2\pi)^2} \frac{k_ak_c e^{ik(\sigma-\sigma')}}{k^2} \cdot \int \frac{d^2k'}{(2\pi)^2} \frac{k'_bk'_d e^{ik'(\sigma-\sigma')}}{k'^2} + (c \leftrightarrow d) \right) + \dots \\ &= -\frac{1}{2!4} \int \frac{d^2p}{(2\pi)^2} \tilde{h}^{ab}(p) \tilde{h}^{cd}(-p) \left( \int \frac{d^2k}{(2\pi)^2} \frac{k_ak_c(p+k)_b(p+k)_d}{k^2(p+k)^2} + (c \leftrightarrow d) \right) + \dots,\end{aligned}\quad (11)$$

where we have kept terms up to quadratic in  $\tilde{h}$ .

### Some integration formulas

$$\frac{1}{k^2(p+k)^2} = \int_0^1 dx \frac{1}{((1-x)p^2 + 2kp(1-x) + k^2)^2}. \quad (12)$$

Therefore, to obtain

$$\int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2(p+k)^2}, \quad \int \frac{d^2k}{(2\pi)^2} \frac{k_a}{k^2(p+k)^2}, \dots \quad (13)$$

we first compute

$$\begin{aligned}&\int \frac{d^2k}{(2\pi)^2} \frac{1}{(m^2 + 2kp' + k^2)^\alpha}, \\ &\int \frac{d^2k}{(2\pi)^2} \frac{k_a}{(m^2 + 2kp' + k^2)^\alpha} = -\frac{1}{2(\alpha-1)} \frac{\partial}{\partial p'^2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{(m^2 + 2kp' + k^2)^{\alpha-1}}, \dots,\end{aligned}\quad (14)$$

set

$$m^2 = (1-x)p^2, \quad p' = (1-x)p \quad (15)$$

and then perform  $x$  integration  $\int_0^1 dx$ . In this way we find

$$\int \frac{d^n k}{(2\pi)^n} \frac{k^a k^b k^c k^d}{k^2(p+k)^2} = \frac{p^2}{4\pi\epsilon} \left( 2 \frac{p^a p^b p^c p^d}{p^4} - 2 \frac{\delta^{(ab} p^c p^{cd)}}{p^2} + \frac{1}{4} \delta^{(ab} \delta^{cd)} \right) + \dots, \quad (16)$$

$$\int \frac{d^n k}{(2\pi)^n} \frac{k^a k^b k^c}{k^2(p+k)^2} = \frac{p^2}{4\pi\epsilon} \left( -2 \frac{p^a p^b p^c}{p^4} + \frac{3}{2} \frac{\delta^{(ab} p^{c)}}{p^2} \right) + \dots, \quad (17)$$

$$\int \frac{d^n k}{(2\pi)^n} \frac{k^a k^b}{k^2(p+k)^2} = \frac{p^2}{4\pi\epsilon} \left( +2 \frac{p^a p^b}{p^4} - \frac{\delta^{ab}}{p^2} \right) + \dots, \quad (18)$$

$$(19)$$

where  $n \equiv 2 + \epsilon$ .

Using these formulas, we find

$$\begin{aligned} & \int \frac{d^n k}{(2\pi)^2} \frac{k^a k^c (p+k)^b (p+k)^d}{k^2(p+k)^2} \\ &= \frac{p^2}{4\pi\epsilon} \left( \frac{1}{4} \delta^{(ab} \delta^{cd)} + \frac{1}{6} \frac{\delta^{ab} p^c p^d - 2\delta^{ac} p^b p^d + \delta^{ad} p^b p^c + \delta^{cd} p^a p^b - 2\delta^{bd} p^a p^c + \delta^{bc} p^a p^d}{p^2} \right) \\ &+ \dots. \end{aligned} \quad (20)$$

Combining the ( $c \leftrightarrow d$ ) contribution together, we obtain

$$\begin{aligned} \Gamma \text{ quadratic in } h^2 &= -\frac{1}{2!4} \int \frac{d^2 p}{(2\pi)^2} \tilde{h}^{ab}(p) \tilde{h}^{cd}(-p) \frac{p^2}{4\pi\epsilon} \left( \frac{1}{6} (\delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc}) \right. \\ &\quad \left. - \frac{1}{6} \frac{\delta^{ac} p^b p^d + \delta^{ad} p^b p^c + \delta^{bd} p^a p^c + \delta^{bc} p^a p^d}{p^2} \right) + \dots. \end{aligned} \quad (21)$$

Comparing this with the expansion of  $R$ , we find

$$\Gamma = \frac{1}{24\pi\epsilon} \int d^2 \sigma \sqrt{g} R. \quad (22)$$

$$\implies \langle T_a^a \rangle = -\frac{1}{12} R. \quad (23)$$

The trace anomaly