

Dimensional reduction of Einstein's equation

Shun'ya Mizoguchi

2002.3.12.

1 The Einstein-Hilbert action

$$\begin{aligned} c_{A,BC} &= E_B^M E_C^N c_{A,MN}, \quad c_{A,MN} = 2\partial_{[M} E_{N]A}, \\ \omega_{M,AB} &= -\frac{1}{2}(c_{M,AB} - c_{A,BM} - c_{B,MA}), \end{aligned} \quad (1)$$

in particular

$$\omega_{,AB}^B = -c_{,AB}^B = -E^{-1}\partial_M(EE^M_A), \quad (2)$$

where $E = \det E_M^A$. The curvatures are

$$\begin{aligned} R_{MNAB} &= 2(\partial_{[M}\omega_{N],AB} + \omega_{[M,AC}\omega_{N]}^C), \\ R &= E^{MA}E^{NB}R_{MNAB}. \end{aligned} \quad (3)$$

Then

Proposition 1.

$$\begin{aligned} ER &= -\frac{1}{4}E(c_{A,BCC}c^{A,BC} - 2c_{A,BC}c^{B,CA} - 4c_{,AC}^A c_{,B}^B) \\ &\quad + 2\partial_M(EE^{MA}\omega_{,AB}^B). \end{aligned} \quad (4)$$

Proof.

$$\begin{aligned} R &= E^{MA}E^{NB}R_{MNAB} \\ &= E^{MA}E^{NB}(2\partial_{[M}\omega_{N]AB} + \omega_{M,AC}\omega_{N,B}^C - \omega_{N,AC}\omega_{M,B}^C) \\ &= 2E^{MA}E^{NB}\partial_{[M}\omega_{N]AB} + \omega_{,AC}^A \omega_{,B}^B - \omega_{,AC}^B \omega_{,B}^A. \end{aligned} \quad (5)$$

$$ER = 2EE^{MA}E^{NB}\partial_{[M}\omega_{N],AB} + E(\omega_{,AC}^A \omega_{,B}^B - \omega_{,AC}^B \omega_{,B}^A).$$

Since $[MN]$ can be replaced with $[AB]$, this anti-symmetrization is already incorporated in the indices of ω . Thus

$$= 2\partial_M(EE^{MA}E^{NB}\omega_{N,AB}) - 2\partial_M(EE^{MA}E^{NB})\omega_{N,AB}$$

$$\begin{aligned}
& + E(\omega_{,AC}^A \omega_{,B}^{B,C} - \omega_{,AC}^B \omega_{,B}^{A,C}) \\
= & 2\partial_M(EE^{MA}E^{NB}\omega_{N,AB}) \\
& - 2\partial_M(EE^{MA})\omega_{,AB}^B - 2EE^{MA}\partial_M E^{NB} \cdot \omega_{N,AB} \\
& + E(\omega_{,AC}^A \omega_{,B}^{B,C} - \omega_{,AC}^B \omega_{,B}^{A,C}). \tag{6}
\end{aligned}$$

$$\text{The second term (6)} = -2Ec^C_{,C} \omega_{,AB}^B, \tag{7}$$

where we have used

$$\partial_M(EE^{MA}) = Ec^B_{,B}. \tag{8}$$

The latter is because

$$\begin{aligned}
c^B_{,B} &= E^{AM}E^{BN}c_{B,MN} \\
&= E^{AM}E^{BN}(\partial_M E_{NB} - \partial_N E_{MB}) \\
&= E^{-1}\partial^A E + E^{AM}E_{MB}\partial_N E^{BN} \\
&= E^{-1}\partial^A E + \partial_N E^{AN}. \tag{9}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\text{The third term (6)} &= +2EE^{MA}(E_C^N \partial_M E_P^C \cdot E_P^B) \omega_{N,AB} \\
&= 2EE^{MA}E^{PB}\omega_{N,AB}\partial_M E_P^C \cdot E_C^N \\
&\stackrel{[MP]}{\rightarrow} = E\omega_{C,AB}c^C_{,AB}. \tag{10}
\end{aligned}$$

Using (7)(10), we find

$$\begin{aligned}
ER &= 2\partial_M(EE^{MA}\omega_{,AB}^B) - 2Ec^C_{,C}\omega_{,AB}^B + Ec^C_{,B}\omega_{,C,AB} \\
&\quad + E(\omega_{,AC}^A \omega_{,B}^{B,C} - \omega_{,AC}^B \omega_{,B}^{A,C}) \\
&= 2\partial_M(EE^{MA}\omega_{,AB}^B) + E[\omega_{,AB}^B(-2c^C_{,C} + \omega_{,C}^C) \\
&\quad + \omega_{C,AB}(c^C_{,AB} - \omega_{,C}^B)] \\
&= 2\partial_M(EE^{MA}\omega_{,AB}^B) + E[\omega_{,AC}^C(-2c^B_{,B} + \frac{1}{2}(c_B^{AB} - c_B^{BA})) \\
&\quad + \omega_{C,AB}(c^C_{,AB} + \underbrace{\frac{1}{2}(c^A_{,B} - c^B_{,C})}_{AB \text{ symmetric} \Rightarrow 0} - c_C^{AB})] \\
&= 2\partial_M(EE^{MA}\omega_{,AB}^B) + E[-\omega_{,AC}^C c^B_{,B} + \frac{1}{2}\omega_{C,AB}c^C_{,AB}] \\
&= 2\partial_M(EE^{MA}\omega_{,AB}^B) + E\left[\frac{1}{2}(c_{,AC}^C - c_{,CA}^C)c^B_{,B} - \frac{1}{4}(c_{C,AB} - \underbrace{c_{A,BC} - c_{B,CA}}_{\nwarrow = \nearrow})c^{C,AB}\right] \\
&= 2\partial_M(EE^{MA}\omega_{,AB}^B) \\
&+ E\left[-\frac{1}{4}c_{C,ABC}c^{C,AB} + \frac{1}{2}c_{A,BCC}c^{C,AB} + c_{,AC}^C c^B_{,B}\right]. \tag{11} \quad \square
\end{aligned}$$

2 Reduction to $D(\geq 3)$ dimensions

Parameterize

$$E_M^A = \begin{bmatrix} e^{-\frac{1}{D-2}} E^{(D)\alpha}_\mu & B_\mu^m e_m^a \\ 0 & e_m^a \end{bmatrix}, \quad E_A^M = \begin{bmatrix} e^{\frac{1}{D-2}} E^{(D)\mu}_\alpha & -e^{\frac{1}{D-2}} E^{(D)\mu}_\alpha B_\mu^m \\ 0 & e_a^m \end{bmatrix}, \quad (12)$$

where $E^{(D)} = \det E^{(D)\alpha}_\mu$, $e = \det e_m^a$. Assume that the fields depend only on x^μ . $c_{A,BC}$ are explicitly written as follows:

$$\begin{aligned} c_{\alpha,\beta\gamma} &= E_\beta^M E_\gamma^N (\partial_M E_{N\alpha} - \partial_N E_{M\alpha}) \\ &= (E_\beta^M E_\gamma^N - E_\beta^N E_\gamma^M) \partial_M E_{N\alpha} \\ &= (E_\beta^\mu E_\gamma^\nu - E_\beta^\nu E_\gamma^\mu) \partial_\mu E_{\nu\alpha}, \\ c_{\alpha,\beta c} &= 0, \\ c_{\alpha,bc} &= 0, \\ c_{a,\beta\gamma} &= (E_\beta^M E_\gamma^N - E_\beta^N E_\gamma^M) \partial_M E_{Na} \\ &= (E_\beta^\mu E_\gamma^\nu - E_\beta^\nu E_\gamma^\mu) \partial_\mu E_{Na} \\ &= 2(E_{[\beta}^\mu E_{\gamma]}^\nu \partial_\mu E_{\nu a} + E_{[\beta}^\mu E_{\gamma]}^n \partial_\mu E_{na}) \\ &= 2(E_{[\beta}^\mu E_{\gamma]}^\nu \partial_\mu (B_\nu^n e_{na}) - E_{[\beta}^\mu E_{\gamma]}^n B_\nu^\nu \partial_\mu e_{na}) \\ &= 2E_{[\beta}^\mu E_{\gamma]}^\nu e_{na} \partial_\mu B_\nu^n \\ &= e_{na} E_\beta^\mu E_\gamma^\nu F_{\mu\nu}^n, \\ c_{a,b\gamma} &= (E_b^M E_\gamma^N - E_b^N E_\gamma^M) \partial_M E_{Na} \\ &= -E_b^m E_\gamma^\mu \partial_\mu E_{ma} \\ &= -E_\gamma^\mu e_b^m \partial_\mu e_{ma}, \\ c_{a,bc} &= 0. \end{aligned} \quad (13)$$

$$\begin{aligned} c_{A,BC} c^{A BC} &= c_{\alpha,\beta\gamma} c^{\alpha \beta\gamma} + c_{a,\beta\gamma} c^a \beta\gamma + 2 \underbrace{c_{\alpha,b\gamma} c^{\alpha b\gamma}}_{=0} \\ &\quad + 2c_{a,b\gamma} c^a b\gamma + \underbrace{c_{a,b\gamma} c^a b\gamma}_{=0} + \underbrace{c_{a,bc} c^a bc}_{=0} \\ &= c_{\alpha,\beta\gamma} c^{\alpha \beta\gamma} + g_{mn} e^{\frac{4}{D-2}} G^{(D)\mu\rho} G^{(D)\nu\sigma} F_{\mu\nu}^m F_{\rho\sigma}^n \\ &\quad + 2g^{mn} e^{\frac{2}{D-2}} G^{(D)\mu\nu} \partial_\mu e_{ma} \partial_\nu e_n^a, \\ c_{A,BC} c^{B CA} &= c_{\alpha,\beta\gamma} c^{\beta \gamma\alpha} + \underbrace{c_{a,\beta\gamma} c^{\beta \gamma a}}_{=0} + \underbrace{c_{\alpha,b\gamma} c^b \gamma\alpha}_{=0} + c_{a,b\gamma} c^b \gamma a \\ &\quad + \underbrace{c_{\alpha,\beta c} c^{\beta c\alpha}}_{=0} + \underbrace{c_{a,\beta c} c^{\beta c\alpha}}_{=0} + \underbrace{c_{\alpha,bc} c^b c\alpha}_{=0} + \underbrace{c_{a,\beta c} c^b c\alpha}_{=0} \\ &= c_{\alpha,\beta\gamma} c^{\beta \gamma\alpha} - E_\gamma^\mu e_b^m \partial_\mu e_{ma} \cdot E^{\gamma\nu} e^{an} \partial_\mu e_n^b \\ &= c_{\alpha,\beta\gamma} c^{\beta \gamma\alpha} - e^{\frac{2}{D-2}} G^{(D)\mu\nu} e_b^m e^{an} \partial_\mu e_{ma} \partial_\nu e_n^b, \end{aligned}$$

$$\begin{aligned}
c^A_{,AC} c^B_{,B}^C &= (c^{\alpha}_{,\alpha\gamma} + c^a_{,a\gamma})(c^{\beta}_{,\beta}{}^\gamma + c^b_{,b}{}^\gamma) \\
&\quad + (\underbrace{c^{\alpha}_{,\alpha c}}_{=0} + \underbrace{c^a_{,ac}}_{=0})(\underbrace{c^{\beta}_{,\beta}{}^c}_{=0} + \underbrace{c^b_{,b}{}^c}_{=0}) \\
&= c^{\alpha}_{,\alpha\gamma} c^{\beta}_{,\beta}{}^\gamma + 2c^{\alpha}_{,\alpha\gamma} (-E^{\gamma\mu} e^{-1} \partial_\mu e) \\
&\quad + e^{\frac{2}{D-2}} G^{(D)\mu\nu} e^{-2} \partial_\mu e \partial_\nu e.
\end{aligned} \tag{14}$$

Therefore, up to a total derivative (denoted by $\overset{\nabla}{=}$) we obtain

$$\begin{aligned}
ER &\overset{\nabla}{=} -\frac{1}{4} E(c_{\alpha,\beta\gamma} c^{\alpha}{}^{\beta\gamma} - 2c_{\alpha,\beta\gamma} c^{\beta}{}^{\gamma\alpha} - 4c^{\alpha}_{,\alpha\gamma} c^{\beta}_{,\beta}{}^\gamma) \\
&\quad - \frac{1}{4} E(g_{mn} e^{\frac{4}{D-2}} G^{(D)\mu\rho} G^{(D)\nu\sigma} F_{\mu\nu}^m F_{\rho\sigma}^n \\
&\quad \quad + 2g^{mn} e^{\frac{2}{D-2}} G^{(D)\mu\nu} \partial_\mu e_m \partial_\mu e_n^a) \\
&\quad + \frac{1}{2} E(-e^{\frac{2}{D-2}} G^{(D)\mu\nu} e^{bm} e^{an} \partial_\mu e_m \partial_\nu e_n) \\
&\quad + E(-2e^{\frac{1}{D-2}} E^{(D)\mu\gamma} c^{\alpha}_{,\alpha\gamma} e^{-1} \partial_\mu e + e^{\frac{2}{D-2}} G^{(D)\mu\nu} e^{-2} \partial_\mu e \partial_\nu e) \\
&= -\frac{1}{4} E(c_{\alpha,\beta\gamma} c^{\alpha}{}^{\beta\gamma} - 2c_{\alpha,\beta\gamma} c^{\beta}{}^{\gamma\alpha} - 4c^{\alpha}_{,\alpha\gamma} c^{\beta}_{,\beta}{}^\gamma) \\
&\quad - \frac{1}{4} E g_{mn} e^{\frac{4}{D-2}} G^{(D)\mu\rho} G^{(D)\nu\sigma} F_{\mu\nu}^m F_{\rho\sigma}^n \\
&\quad + \frac{1}{2} E e^{\frac{2}{D-2}} G^{(D)\mu\nu} \underbrace{(-g^{mn} \partial_\mu e_m \partial_\nu e_n^a + \partial_\mu e_m \partial_\nu e_n^a)}_{!} \\
&\quad + E \left(-2e^{\frac{1}{D-2}} E^{(D)\mu\gamma} \cdot e^{\frac{1}{D-2}} (c^{(D)\alpha}_{,\alpha\gamma} + \frac{D-1}{D-2} E^{(D)\mu}_{\gamma} e^{-1} \partial_\mu e) e^{-1} \partial_\mu e \right. \\
&\quad \quad \left. + e^{\frac{2}{D-2}} G^{(D)\mu\nu} e^{-2} \partial_\mu e \partial_\nu e \right) \\
&= -\frac{1}{4} E(c_{\alpha,\beta\gamma} c^{\alpha}{}^{\beta\gamma} - 2c_{\alpha,\beta\gamma} c^{\beta}{}^{\gamma\alpha} - 4c^{\alpha}_{,\alpha\gamma} c^{\beta}_{,\beta}{}^\gamma) \\
&\quad - \frac{1}{4} E g_{mn} e^{\frac{4}{D-2}} G^{(D)\mu\rho} G^{(D)\nu\sigma} F_{\mu\nu}^m F_{\rho\sigma}^n \\
&\quad + \frac{1}{4} E e^{\frac{2}{D-2}} G^{(D)\mu\nu} \partial_\mu g^{mn} \partial_\nu g_{mn} \\
&\quad - 2\underbrace{E e^{\frac{2}{D-2}} c^{(D)\alpha}_{,\alpha\gamma} E^{(D)\mu\gamma} e^{-1} \partial_\mu e}_{=E^{(D)}} \\
&\quad - \frac{D}{D-2} \underbrace{E e^{\frac{2}{D-2}} G^{(D)\mu\nu} e^{-2} \partial_\mu e \partial_\nu e}_{=E^{(D)}}, \tag{15}
\end{aligned}$$

where in the second expression we have used

$$c^{\alpha}_{,\alpha\gamma} = e^{\frac{1}{D-2}} (c^{(D)\alpha}_{,\alpha\gamma} + \frac{D-1}{D-2} E^{(D)\mu}_{\gamma} e^{-1} \partial_\mu e), \tag{16}$$

which is derived from

$$\begin{aligned}
c^{\alpha,\beta\gamma} &= (E_{\beta}^{\mu} E_{\gamma}^{\nu} - E_{\beta}^{\nu} E_{\gamma}^{\mu}) \partial_{\mu} E_{\nu\alpha} \\
&= e^{\frac{2}{D-2}} (E_{\beta}^{(D)\mu} E_{\gamma}^{(D)\nu} - E_{\beta}^{(D)\nu} E_{\gamma}^{(D)\mu}) \partial_{\mu} (e^{-\frac{1}{D-2}} E_{\nu\alpha}^{(D)}) \\
&= e^{\frac{1}{D-2}} (c_{\alpha,\beta\gamma}^{(D)} - \frac{1}{D-2} (E_{\beta}^{(D)\mu} \eta_{\gamma\alpha} - E_{\gamma}^{(D)\mu} \eta_{\beta\alpha}) e^{-1} \partial_{\nu} e), \\
&= e^{\frac{1}{D-2}} (c_{\alpha,\beta\gamma}^{(D)} + \frac{2}{D-2} \eta_{\alpha[\beta} E_{\gamma]}^{(D)\mu} e^{-1} \partial_{\mu} e).
\end{aligned} \tag{17}$$

Therefore,

$$\begin{aligned}
c_{\alpha,\beta\gamma} c^{\alpha,\beta\gamma} &= e^{\frac{2}{D-2}} \left(c_{\alpha,\beta\gamma}^{(D)} + \frac{2}{D-2} \eta_{\alpha[\beta} E_{\gamma]}^{(D)\mu} e^{-1} \partial_{\mu} e \right) \\
&\quad \cdot \left(c^{(D)\alpha,\beta\gamma} + \frac{2}{D-2} \eta^{\alpha[\beta} E^{(D)\gamma]\nu} e^{-1} \partial_{\nu} e \right) \\
&= e^{\frac{2}{D-2}} \left(c_{\alpha,\beta\gamma}^{(D)} c^{(D)\alpha,\beta\gamma} + \frac{4}{D-2} c_{\alpha,\beta\gamma}^{(D)} \eta^{\alpha[\beta} E^{(D)\gamma]\nu} e^{-1} \partial_{\nu} e \right. \\
&\quad \left. + \frac{4}{(D-2)^2} \eta_{\alpha[\beta} E_{\gamma]}^{(D)\mu} \eta^{\alpha[\beta} E^{(D)\gamma]\nu} e^{-2} \partial_{\mu} e \partial_{\nu} e \right) \\
&= e^{\frac{2}{D-2}} \left(c_{\alpha,\beta\gamma}^{(D)} c^{(D)\alpha,\beta\gamma} + \frac{4}{D-2} c_{\alpha,\gamma}^{(D)\alpha} E^{(D)\gamma\nu} e^{-1} \partial_{\nu} e \right. \\
&\quad \left. + \frac{2}{(D-2)^2} \eta_{\alpha\beta} E_{\gamma}^{(D)\mu} (\eta^{\alpha\beta} E^{(D)\gamma\nu} - \eta^{\alpha\gamma} E^{(D)\beta\nu}) e^{-2} \partial_{\mu} e \partial_{\nu} e \right) \\
&= e^{\frac{2}{D-2}} \left(c_{\alpha,\beta\gamma}^{(D)} c^{(D)\alpha,\beta\gamma} + \frac{4}{D-2} c_{\alpha,\gamma}^{(D)\alpha} E^{(D)\mu\gamma} e^{-1} \partial_{\mu} e \right. \\
&\quad \left. + \frac{2(D-1)}{(D-2)^2} G^{(D)\mu\nu} e^{-2} \partial_{\mu} e \partial_{\nu} e \right), \tag{18}
\end{aligned}$$

$$\begin{aligned}
c_{\alpha,\beta\gamma} c^{\beta,\gamma\alpha} &= e^{\frac{2}{D-2}} \left(c_{\alpha,\beta\gamma}^{(D)} + \frac{2}{D-2} \eta_{\alpha[\beta} E_{\gamma]}^{(D)\mu} e^{-1} \partial_{\mu} e \right) \\
&\quad \cdot \left(c^{(D)\beta,\gamma\alpha} + \frac{2}{D-2} \eta^{\beta[\gamma} E^{(D)\alpha]\nu} e^{-1} \partial_{\nu} e \right) \\
&= e^{\frac{2}{D-2}} \left(c_{\alpha,\beta\gamma}^{(D)} c^{(D)\beta,\gamma\alpha} \right. \\
&\quad \left. + \frac{1}{D-2} c_{\alpha,\beta\gamma}^{(D)} (\eta^{\beta\gamma} E^{(D)\alpha\nu} - \eta^{\beta\alpha} E^{(D)\gamma\nu}) e^{-1} \partial_{\nu} e \right. \\
&\quad \left. + \frac{1}{D-2} c_{\alpha,\beta\gamma}^{(D)} (\eta_{\alpha\beta} E_{\gamma}^{(D)\mu} - \eta_{\alpha\gamma} E_{\beta}^{(D)\mu}) e^{-1} \partial_{\mu} e \right. \\
&\quad \left. + \frac{1}{(D-2)^2} (\eta_{\alpha\beta} E_{\gamma}^{(D)\mu} - \underbrace{\eta_{\alpha\gamma} E_{\beta}^{(D)\mu}}_{=0}) (\eta^{\beta\gamma} E^{(D)\alpha\nu} - \eta^{\beta\alpha} E^{(D)\gamma\nu}) \right. \\
&\quad \left. \cdot e^{-2} \partial_{\mu} e \partial_{\nu} e \right)
\end{aligned}$$

$$\begin{aligned}
&= e^{\frac{2}{D-2}} \left(c_{\alpha,\beta\gamma}^{(D)} c^{(D)\beta\gamma\alpha} \right. \\
&\quad + \underbrace{\frac{1}{D-2} c_{\alpha,\gamma}^{(D)\gamma\alpha} E_{\gamma}^{(D)\mu} e^{-1} \partial_{\mu} e}_{||} \\
&\quad - \underbrace{\frac{1}{D-2} c_{\alpha,\gamma}^{(D)\alpha} E^{(D)\gamma\nu} e^{-1} \partial_{\nu} e}_{||} \\
&\quad \left. + \frac{1-D}{(D-2)^2} G^{(D)\mu\nu} e^{-2} \partial_{\mu} e \partial_{\nu} e \right), \tag{19}
\end{aligned}$$

$$\begin{aligned}
c^{\alpha}_{,\alpha\gamma} c^{\beta\gamma}_{,\beta} &= e^{\frac{2}{D-2}} \left(c^{(D)\alpha}_{,\alpha\gamma} + \frac{2}{D-2} \eta^{\alpha}_{[\alpha} E^{(D)\mu}_{\gamma]} e^{-1} \partial_{\mu} e \right) \\
&\quad \cdot \left(c^{(D)\beta}_{,\beta} + \frac{2}{D-2} \eta^{\beta}_{\beta} E^{(D)\gamma\nu} e^{-1} \partial_{\nu} e \right) \\
&= e^{\frac{2}{D-2}} \left(c^{(D)\alpha}_{,\alpha\gamma} + \frac{D-1}{D-2} E^{(D)\mu}_{\gamma} e^{-1} \partial_{\mu} e \right) \\
&\quad \cdot \left(c^{(D)\beta}_{,\beta} + \frac{D-1}{D-2} E^{(D)\gamma\nu} e^{-1} \partial_{\nu} e \right) \\
&= e^{\frac{2}{D-2}} \left[c^{(D)\alpha}_{,\alpha\gamma} c^{(D)\beta}_{,\beta} + \frac{2(D-1)}{D-2} c^{(D)\alpha}_{,\alpha\gamma} E^{(D)\gamma\nu} e^{-1} \partial_{\nu} e \right. \\
&\quad \left. + \frac{(D-1)^2}{(D-2)^2} G^{(D)\mu\nu} e^{-2} \partial_{\mu} e \partial_{\nu} e \right]. \tag{20}
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow -\frac{1}{4} E(c_{\alpha,\beta\gamma} c^{\alpha\beta\gamma} - 2c_{\alpha,\beta\gamma} c^{\beta\gamma\alpha} - 4c^{\alpha}_{,\alpha\gamma} c^{\beta\gamma}_{,\beta}) \\
&= -\frac{1}{4} E^{(D)} \left[c_{\alpha,\beta\gamma}^{(D)} c^{(D)\alpha\beta\gamma} + \frac{4}{D-2} c^{(D)\alpha}_{,\alpha\gamma} E^{(D)\mu\gamma} e^{-1} \partial_{\mu} e \right. \\
&\quad + \frac{2(D-1)}{(D-2)^2} G^{(D)\mu\nu} e^{-2} \partial_{\mu} e \partial_{\nu} e \\
&\quad - 2c_{\alpha,\beta\gamma}^{(D)} c^{(D)\beta\gamma\alpha} + \frac{4}{D-2} c^{(D)\alpha}_{,\alpha\gamma} E^{(D)\mu\gamma} e^{-1} \partial_{\mu} e \\
&\quad + \frac{2(D-1)}{(D-2)^2} G^{(D)\mu\nu} e^{-2} \partial_{\mu} e \partial_{\nu} e \\
&\quad - 4c^{(D)\alpha}_{,\alpha\gamma} c^{(D)\beta\gamma}_{,\beta} - \frac{8(D-1)}{D-2} c^{(D)\alpha}_{,\alpha\gamma} E^{(D)\gamma\nu} e^{-1} \partial_{\nu} e \\
&\quad \left. - 4\frac{(D-1)^2}{(D-2)^2} G^{(D)\mu\nu} e^{-2} \partial_{\mu} e \partial_{\nu} e \right] \\
&= -\frac{1}{4} E^{(D)} \left[c_{\alpha,\beta\gamma}^{(D)} c^{(D)\alpha\beta\gamma} - 2c_{\alpha,\beta\gamma}^{(D)} c^{(D)\beta\gamma\alpha} - 4c^{(D)\alpha}_{,\alpha\gamma} c^{(D)\beta\gamma}_{,\beta} \right. \\
&\quad \left. - 8c^{(D)\alpha}_{,\alpha\gamma} E^{(D)\mu\gamma} e^{-1} \partial_{\mu} e - \frac{4(D-1)}{D-2} G^{(D)\mu\nu} e^{-2} \partial_{\mu} e \partial_{\nu} e \right], \tag{21}
\end{aligned}$$

where we have used

$$E = e^{-\frac{2}{D-2}} E^{(D)} \quad (22)$$

in the second expression. Plugging (21) into (15), we obtain

Proposition 2.

$$\begin{aligned} ER &\stackrel{\Sigma}{=} -\frac{1}{4} E^{(D)} \left(c_{\alpha,\beta\gamma}^{(D)} c^{(D)\alpha,\beta\gamma} - 2c_{\alpha,\beta\gamma}^{(D)} c^{(D)\beta,\gamma\alpha} - 4c_{,\alpha\gamma}^{(D)\alpha} c^{(D)\beta,\gamma} \right) \\ &\quad - \frac{1}{4} E^{(D)} e^{\frac{2}{D-2}} g_{mn} G^{(D)\mu\rho} G^{(D)\nu\sigma} F_{\mu\nu}^m F_{\rho\sigma}^n \\ &\quad + \frac{1}{4} E^{(D)} G^{(D)\mu\nu} \partial_\mu g^{mn} \partial_\nu g_{mn} \\ &\quad - \frac{1}{D-2} E^{(D)} G^{(D)\mu\nu} e^{-2} \partial_\mu e \partial_\nu e. \end{aligned} \quad (23)$$

Remark 1. The first line is equal to $E^{(D)} R^{(D)}$ up to a(nother) total derivative (Proposition 1).

Remark 2. (23) does not depend on the signature of the flat local Lorenz metric η_{AB} . However, g_{MN} 's expression in terms of E_M^A does. For example, in the reduction from $D+1$ to D dimensions, $g_{DD} = \pm e^2$ depending on the signature.

3 Reduction to two dimensions

In two dimensions the ‘dilaton’ factor cannot be removed by a rescaling of the metric. Thus anticipating the application to the $D=2$ case, we parameterize the vielbein as

$$E_M^A = \begin{bmatrix} E_\mu^\alpha & B_\mu^m e_m^a \\ 0 & e_m^a \end{bmatrix}, \quad E_A^M = \begin{bmatrix} E_\alpha^\mu & -E_\alpha^\mu B_\mu^m \\ 0 & e_a^m \end{bmatrix}, \quad (24)$$

$$\eta_{AB} = \begin{bmatrix} \eta_{\alpha\beta} & 0 \\ 0 & \eta_{ab} \end{bmatrix}. \quad (25)$$

We can use (13) without modifications. Then similar calculations show that

$$\begin{aligned} c_{A,BCC}^{A\ BC} &= c_{\alpha,\beta\gamma} c^{\alpha\beta\gamma} + g_{mn} G^{\mu\rho} G^{\nu\sigma} F_{\mu\nu}^m F_{\rho\sigma}^n \\ &\quad + 2g^{mn} G^{\mu\nu} \partial_\mu e_m \partial_\nu e_n^a, \\ c_{A,BCC}^{B\ CA} &= c_{\alpha,\beta\gamma} c^{\beta\gamma\alpha} - G^{\mu\nu} e_b^m e^{an} \partial_\mu e_m \partial_\nu e_n^b, \\ c_{,AC}^A c_{,B}^B c_{,C}^C &= c_{,\alpha\gamma}^\alpha c_{,\beta}^\beta c_{,\gamma}^\gamma + 2c_{,\alpha\gamma}^\alpha (-E^{\gamma\mu} e^{-1} \partial_\mu e) \\ &\quad + G^{\mu\nu} e^{-2} \partial_\mu e \partial_\nu e. \end{aligned} \quad (26)$$

Plugging them into Proposition 1, we have

$$\begin{aligned}
ER = & -\frac{1}{4}E \left(c_{\alpha,\beta\gamma}c^{\alpha\beta\gamma} + g_{mn}G^{\mu\rho}G^{\nu\sigma}F_{\mu\nu}^m F_{\rho\sigma}^n + 2g^{mn}G^{\mu\nu}\partial_\mu e_{ma}\partial_\nu e_n^a \right. \\
& - 2(c_{\alpha,\beta\gamma}c^{\beta\gamma\alpha} - G^{\mu\nu}e_b^m e^{an}\partial_\mu e_{ma}\partial_\nu e_n^b) \\
& \left. - 4(c_{\alpha\gamma}^\alpha c_{,\beta}^\beta c^\gamma + 2c_{,\alpha\gamma}^\alpha (-E^{\gamma\mu}e^{-1}\partial_\mu e) + G^{\mu\nu}e^{-2}\partial_\mu e\partial_\nu e) \right) \\
& + 2\partial_\mu(EE^{\mu A}\omega^B_{,AB}). \tag{27}
\end{aligned}$$

Here $\omega_{M,AB}$ in the last term is the connection made out of E_M^A . Since $E^{\mu A}$ is nonzero only if $A = \alpha$ and $\omega^B_{,\alpha B} = -c^B_{,\alpha B}$, this total derivative term can be written as

$$\begin{aligned}
2\partial_\mu(EE^{\mu A}\omega^B_{,AB}) = & 2\partial_\mu(EE^{\mu\alpha}\omega^B_{,\alpha B}) \\
= & -2\partial_\mu(EE^{\mu\alpha}c^B_{,\alpha B}) \\
= & -2\partial_\mu(EE^{\mu\alpha}c^{\beta}_{,\alpha\beta}) - 2\partial_\mu(EE^{\mu\alpha}c^b_{,\alpha b}) \\
= & 2\partial_\mu(EE^{\mu\alpha}\omega(E_\mu^\alpha)^\beta_{,\alpha\beta}) - 2\partial_\mu(EE^{\mu\alpha}c^b_{,\alpha b}) \\
= & 2\partial_\mu(EE^{\mu\alpha}\omega(E_\mu^\alpha)^\beta_{,\alpha\beta}) - 2\partial_\mu(EG^{\mu\nu}e^{-1}\partial_\nu e) \\
= & 2\partial_\mu(e(\det E_\mu^\alpha)E^{\mu\alpha}\omega(E_\mu^\alpha)^\beta_{,\alpha\beta}) - 2\partial_\mu(EG^{\mu\nu}e^{-1}\partial_\nu e) \\
= & 2e\partial_\mu((\det E_\mu^\alpha)E^{\mu\alpha}\omega(E_\mu^\alpha)^\beta_{,\alpha\beta}) + 2\partial_\mu e(\det E_\mu^\alpha)E^{\mu\alpha}\omega(E_\mu^\alpha)^\beta_{,\alpha\beta} \\
& - 2\partial_\mu(EG^{\mu\nu}e^{-1}\partial_\nu e) \\
= & 2e\partial_\mu((\det E_\mu^\alpha)E^{\mu\alpha}\omega(E_\mu^\alpha)^\beta_{,\alpha\beta}) - 2\partial_\mu e(\det E_\mu^\alpha)E^{\mu\alpha}c^{\beta}_{,\alpha\beta} \\
& - 2\partial_\mu(EG^{\mu\nu}e^{-1}\partial_\nu e) \\
= & 2e\partial_\mu((\det E_\mu^\alpha)E^{\mu\alpha}\omega(E_\mu^\alpha)^\beta_{,\alpha\beta}) + 2E c_{,\alpha\gamma}^\alpha E^{\mu\gamma}e^{-1}\partial_\mu e \\
& - 2\partial_\mu(EG^{\mu\nu}e^{-1}\partial_\nu e). \tag{28}
\end{aligned}$$

We have used

$$E = e \det E_\mu^\alpha. \tag{29}$$

$\omega(E_\mu^\alpha)_{\mu,\alpha\beta}$ is the connection made from E_μ^α . The second term cancels the second term of the third line in (27), while the first term becomes a part of $\det E_\mu^\alpha R(E_\mu^\alpha)$ together with the three cc terms in (27). In all, (27) is simplified to

Proposition 3. For $D \geq 2$ dimensions,

$$\begin{aligned}
ER = & e \det E_\mu^\alpha \left(R(E_\mu^\alpha) - \frac{1}{4}g_{mn}G^{\mu\rho}G^{\nu\sigma}F_{\mu\nu}^m F_{\rho\sigma}^n \right. \\
& \left. + \frac{1}{4}G^{\mu\nu}\partial_\mu g^{mn}\partial_\nu g_{mn} + G^{\mu\nu}e^{-2}\partial_\mu e\partial_\nu e \right) \\
& - 2\partial_\mu(\det E_\mu^\alpha \cdot G^{\mu\nu}\partial_\nu e). \tag{30}
\end{aligned}$$

Remark 3. Applying the Weyl rescaling formula to $\det E_\mu^\alpha \cdot R(E_\mu^\alpha)$, (30) reduces to Proposition 2.

We will now consider the $D = 2$ case.

Lemma 1. In $D = 2$ the Einstein-Hilbert action is a total derivative and is given by

$$\det E_\mu^\alpha \cdot R(E_\mu^\alpha) = -2\partial_\mu(\det E_\mu^\alpha \cdot G^{\mu\nu} c_{,\nu\beta}^\beta). \quad (31)$$

Proof. The right hand side is the total derivative term of Proposition 1. Therefore, we have only to check that the cc terms vanish. This can be easily shown as

$$\begin{aligned} & c_{\alpha,\beta\gamma} c_{,\beta\gamma}^\alpha - 2c_{\alpha,\beta\gamma} c_{,\gamma\alpha}^\beta - 4c_{,\alpha\gamma}^\alpha c_{,\beta}^\beta \\ &= 2c_{\alpha,12} c_{,\alpha}^{12} - 2(c_{1,12} c_{,\alpha}^{12} + c_{2,21} c_{,\alpha}^{21}) - 4(c_{2,21} c_{,\alpha}^{21} + c_{1,12} c_{,\alpha}^{12}) \\ &= 0. \end{aligned}$$

□

Using Lemma 1 in Proposition 3 and integrating by parts, we obtain

Proposition 4. For $D = 2$,

$$\begin{aligned} ER &\stackrel{\nabla}{=} e \det E_\mu^\alpha \left(2G^{\mu\nu} e^{-1} \partial_\mu e \cdot c_{,\nu\beta}^\beta - \frac{1}{4} g_{mn} G^{\mu\rho} G^{\nu\sigma} F_{\mu\nu}^m F_{\rho\sigma}^n \right. \\ &\quad \left. + \frac{1}{4} G^{\mu\nu} \partial_\mu g^{mn} \partial_\nu g_{mn} + G^{\mu\nu} e^{-2} \partial_\mu e \partial_\nu e \right). \end{aligned} \quad (32)$$

4 The Kramer-Neugebauer involution

In two dimensions $\det E_\mu^\alpha \cdot G^{\mu\nu}$ is invariant under the scale transformation

$$E_\mu^\alpha \rightarrow \lambda E_\mu^\alpha. \quad (33)$$

On the other hand, $c_{,\nu\beta}^\beta = E_\beta^\mu (\partial_\nu E_\mu^\beta - \partial_\mu E_\nu^\beta)$ transforms as

$$\begin{aligned} c'_{,\nu\beta}^\beta &\equiv c(E'_\mu)_{,\nu\beta}^\beta \\ &= \lambda^{-1} E_\beta^\mu (\partial_\nu (\lambda E_\mu^\beta) - \partial_\mu (\lambda E_\nu^\beta)) \\ &= E_\beta^\mu (\partial_\nu E_\mu^\beta - \partial_\mu E_\nu^\beta) + (2-1)\lambda^{-1} \partial_\nu \lambda \\ &= c_{,\nu\beta}^\beta + \lambda^{-1} \partial_\nu \lambda. \end{aligned} \quad (34)$$

Therefore, the last term of Proposition 4 may be absorbed in the first term by setting $\lambda = e^{\frac{1}{2}}$. Thus we obtain

Proposition 5. For $D = 2$,

$$\begin{aligned} ER &\stackrel{\nabla}{=} eE' \left(2G'^{\mu\nu}e^{-1}\partial_\mu e \cdot c'^\beta_{,\nu\beta} - \frac{1}{4}eg_{mn}G'^{\mu\rho}G'^{\nu\sigma}F_{\mu\nu}^m F_{\rho\sigma}^n \right. \\ &\quad \left. + \frac{1}{4}G'^{\mu\nu}\partial_\mu g^{mn}\partial_\nu g_{mn} \right), \end{aligned} \quad (35)$$

where E_M^A is parameterized as (24) with $E'^\alpha_\mu = e^{\frac{1}{2}}E_\mu^\alpha$, and $G'^{\mu\nu} = E'_\alpha^\mu E'^{\alpha\nu}$, $E' = \det E'^\alpha_\mu$ and $c'^\beta_{,\nu\beta} = c(E'^\alpha_\mu)^\beta_{,\nu\beta}$.

Corollary 1. For $D = 2$,

$$ER \stackrel{\nabla}{=} e \det E_\mu^\alpha \left(2G^{\mu\nu}e^{-1}\partial_\mu e \cdot c^\beta_{,\nu\beta} + \frac{1}{4}G^{\mu\nu}\partial_\mu g^{mn}\partial_\nu g_{mn} \right), \quad (36)$$

where

$$E_M^A = \begin{bmatrix} e^{-\frac{1}{2}}E_\mu^\alpha & 0 \\ 0 & e_m^a \end{bmatrix}. \quad (37)$$

Corollary 2. For $D = 2$,

$$ER \stackrel{\nabla}{=} e \det E_\mu^\alpha \left(2G^{\mu\nu}e^{-1}\partial_\mu e \cdot c^\beta_{,\nu\beta} + \frac{1}{4}G^{\mu\nu}\partial_\mu \hat{g}^{mn}\partial_\nu \hat{g}_{mn} \right), \quad (38)$$

where

$$E_M^A = \begin{bmatrix} e^{-\frac{1}{4}}E_\mu^\alpha & 0 \\ 0 & e_m^a \end{bmatrix}, \quad (39)$$

and

$$\hat{g}_{mn} = e^{-1}g_{mn}. \quad (40)$$

Remark 4. Since $g_{mn} = e_m^a \eta_{ab} e_n^b$ and $\det e_m^a = e$, a 2×2 matrix with components $\{\eta^{mk}g_{kn}\}$ belongs to $SL(2, \mathbf{R})$. If $\eta_{ab} = \text{diag}[\pm 1, \pm 1]$, (38) is an $SL(2, \mathbf{R})/SO(2)$ non-linear sigma model (positive definite), while if $\eta_{ab} = \text{diag}[\pm 1, \mp 1]$, an $SL(2, \mathbf{R})/SO(1, 1)$ sigma model (indefinite).