

CMB Power Spectrum Formula in the Background-Field Method

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Sec 1. Introduction (1/2) History

Cosmic Microwave Background Radiation Observation Data is accumulating

▶ WMAP-5year

- Dark Matter, Dark Energy (\sim Cosmological Term)
- 'Micro' Theory of Gravity : **Divergence** Problem (Infra-red, Ultra-violet)
- Quantum Field Theory on dS_4 is not defined
 - '01 E. Witten, inf-dim Hilbert space
 - '03 J. Maldacena, Non-Gaussian ...
 - '06 S. Weinberg, in-in formalism
 - Schwinger-Keldysh formalism in '07 A.M. Polyakov
 - '09- T. Tanaka & Y. Urakawa
 - '11- H. Kitamoto & Y. Kitazawa

Sec 1. Introduction (2/2) Recent Words and References

- A.M. Polyakov, '09
Dark energy, like the black body radiation 150 years ago, hides secrets of fundamental physics
- E. Verlinde, '10
Emergent Gravity
- A. Strominger et al, '11
From Navier-Stokes to Einstein, arXiv:1101.2451
From Petrov-Einstein to Navier-Stokes, arXiv:1104.5502

Sec 2. Background Field Formalism (1/2)

B.S. DeWitt, 1967; G. 'tHooft, 1973; I.Y. Aref'eva, A.A. Slavnov & L.D. Faddeev, 1974

$\Phi(x)$: Scalar Field, $g_{\mu\nu}(x)$: Gravitational Field, $V(\Phi) = \frac{\sigma}{4!} \Phi^4$, $\sigma > 0$

$$S[\Phi; g_{\mu\nu}] = \int d^4x \sqrt{g} \left(\frac{-(R - 2\lambda)}{16\pi G_N} - \frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi - \frac{m^2}{2} \Phi^2 - V(\Phi) \right) \quad (1)$$

Background Expansion: $\Phi = \Phi_{cl} + \varphi$, NOT expand $g_{\mu\nu}$ (2)

Sec.2 Background Field Formalism (2/2)

$$e^{i\Gamma[\Phi_{cl}; g_{\mu\nu}]} = \int \mathcal{D}\varphi \exp i \left\{ S[\Phi_{cl} + \varphi; g_{\mu\nu}] - \frac{\delta S[\Phi_{cl}; g_{\mu\nu}]}{\delta \Phi_{cl}} \varphi \right\} \Gamma[\Phi_{cl}; g_{\mu\nu}] ;$$

Φ_{cl} is perturbatively solved, at the tree level, as

$$\Phi_{cl}(x) = \Phi_0(x) + \int d^4x' D(x-x') \sqrt{g} \frac{\delta V(\Phi_{cl})}{\delta \Phi_{cl}} \Big|_{x'} ,$$

$$\sqrt{g}(\nabla^2 - m^2)\Phi_0 = 0 \quad , \quad \sqrt{g}(\nabla^2 - m^2)D(x-x') = \delta^4(x-x') \quad . \quad (4)$$

$\Phi_0(x)$: [asymptotic fields](#) for n-point function (see later part)

Sec.2 Background Field Formalism (2'/2)

Aref'eva, Slavnov & Faddeev 1974

Harmonic Oscillator (Feynman's text '72)

Density Matrix

$$\rho(x_2, x_1; \beta) = \int \mathcal{D}x(\tau) \exp \left[-\frac{1}{\hbar} \int_0^\beta \left(\frac{\dot{x}^2}{2} + \frac{\omega^2}{2} x^2 \right) d\tau \right]_{x(0)=x_1, x(\beta)=x_2}$$

Background Field Expansion: $x(\tau) = x_{cl}(\tau) + y(\tau)$

$$\rho(x_2, x_1; \beta) = \sqrt{\frac{1}{2\pi\hbar\beta}} \exp \left[-\frac{1}{\hbar} \int_0^\beta \left(\frac{\dot{x}_{cl}^2}{2} + \frac{\omega^2}{2} x_{cl}^2 \right) d\tau \right] \quad . \quad (6)$$

Transition probability is given by

$$\frac{\delta}{\delta x_{cl}(0)} \frac{\delta}{\delta x_{cl}(\beta)} \rho(x_2, x_1; \beta) \quad . \quad (7)$$

Sec 3. dS₄ Geometry (1/3)

- background field $g_{\mu\nu} : \text{dS}_4$
 $ds^2 = -dt^2 + e^{2H_0 t}(dx^2 + dy^2 + dz^2) \equiv g_{\mu\nu}^{inf} dx^\mu dx^\nu$
- time variable: $t \rightarrow \eta$ (conformal time)
- $ds^2 = \frac{1}{(H_0\eta)^2}(-d\eta^2 + dx^2 + dy^2 + dz^2)$
 $= \tilde{g}_{\mu\nu}(\chi) d\chi^\mu d\chi^\nu, (\chi^0, \chi^1, \chi^2, \chi^3) = (\eta, x, y, z)$

To regularize IR behavior, we introduce

$$Z_2 \text{ Symmetry : } t \leftrightarrow -t, \quad \text{Periodicity : } t \rightarrow t + 2\pi / H_0, \quad (8)$$

Sec 3. dS₄ Geometry (2/3)

The perturbative solution Φ_{cl} , (4), is given by

$$\begin{aligned} \Phi_{cl}(\chi) = \Phi_0(\chi) + \int \tilde{D}(\chi, \chi') \frac{1}{(H_0\eta')^4} \frac{\delta V(\Phi_{cl})}{\delta \Phi_{cl}} \Big|_{\chi'} d^4\chi' , \\ \sqrt{-\tilde{g}}(\tilde{\nabla}^2 - m^2)\Phi_0 = \\ - \left\{ \partial_\eta \frac{1}{(H_0\eta)^2} \partial_\eta + \frac{m^2}{(H_0\eta)^4} - \frac{1}{(H_0\eta)^2} \tilde{\nabla}^2 \right\} \Phi_0 = 0. \end{aligned} \quad (9)$$

Switch to the **spacially-Fourier-transformed** expression:

$$\Phi_0(\eta, \vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \phi_{\vec{p}}(\eta), \quad \tilde{D}(\chi, \chi') = \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} \tilde{D}_{\vec{p}}(\eta, \eta'), (1)$$

$\tilde{D}_{\vec{p}}(\eta, \eta')$: 'Momentum/Position propagator'

Sec 3. dS₄ Geometry (3/3)

$\phi_{\vec{p}}(\eta)$ satisfies the following **Bessel** eigenvalue equation.

$$\left\{ \partial_\eta^2 - \frac{2}{\eta} \partial_\eta + \frac{m^2}{(H_0 \eta)^2} + M^2 \right\} \phi_M(\eta) =$$

$$\{s(\eta)^{-1} \hat{L}_\eta + M^2\} \phi_M(\eta) = 0,$$

$$M^2 \equiv \vec{p}^2, \quad s(\eta) \equiv \frac{1}{(H_0 \eta)^2}, \quad \hat{L}_\eta \equiv \partial_\eta s(\eta) \partial_\eta + \frac{m^2}{(H_0 \eta)^4} \quad . \quad (11)$$

Sec 4. Bunch-Davies Vacuum (1/2)

Boundary Condition for Free Wave Function

$$\begin{aligned} \Phi_0 &= 0 && \text{Dirichlet for } P = - \\ \partial_\eta \Phi_0 &= 0 && \text{Neumann for } P = + \end{aligned} \quad (12)$$

Bunch-Davies Vacuum: the complete and orthonormal eigenfunctions $\phi_n(\eta)$ of the operator $s^{-1}\hat{L}_\eta$.

Sec 4. Bunch-Davies Vacuum (2/2)

$$\begin{aligned}
 \phi_n(\eta) &\equiv (n|\eta) = (\eta|n) \quad , \quad \{s(\eta)^{-1}\hat{L}_\eta + M_n^2\}\phi_n(\eta) = 0 \quad , \\
 \left(\int_{-1/H_0}^{-1/\omega} + \int_{1/\omega}^{1/H_0} \right) \frac{d\eta}{(H_0\eta)^2} (n|\eta)(\eta|k) &= 2 \int_{-1/H_0}^{-1/\omega} \frac{d\eta}{(H_0\eta)^2} (n|\eta)(\eta|k) \\
 &= (n|k) = \delta_{n,k} \quad , \\
 (\eta|\eta') &= \begin{cases} (H_0\eta)^2 \epsilon(\eta) \epsilon(\eta') \hat{\delta}(|\eta| - |\eta'|) & \text{for } P = - \\ (H_0\eta)^2 \delta(|\eta| - |\eta'|) & \text{for } P = + \end{cases} \\
 \left(\int_{-1/H_0}^{-1/\omega} + \int_{1/\omega}^{1/H_0} \right) \frac{d\eta}{(H_0\eta)^2} |\eta)(\eta| &= 2 \int_{-1/H_0}^{-1/\omega} \frac{d\eta}{(H_0\eta)^2} |\eta)(\eta| = 1 \quad , \\
 \sum |n)(n| &= 1 \quad , (13)
 \end{aligned}$$

Sec 5. Casimir Energy (1/2)

Casimir energy: free part of the effective action in (3)

$$\begin{aligned} \exp\{-H_0^{-3}E_{Cas}^{dS4}\} &= \int \mathcal{D}\varphi \exp i \int d^4x \sqrt{g} \left(-\frac{1}{2} \nabla_\mu \varphi \nabla^\mu \varphi - \frac{m^2}{2} \varphi^2 \right) \\ &= \exp \left[\int \frac{d^3\vec{p}}{(2\pi)^3} 2 \int_{-1/H_0}^{-1/\omega} d\eta \left\{ -\frac{1}{2} \ln(-s(\eta)^{-1} \hat{L}_\eta - \vec{p}^2) \right\} \right] \quad (14) \end{aligned}$$

From the formula: $\int_0^\infty (e^{-t} - e^{-tM})/t \, dt = \ln M$, $\det M > 0$,

$$-H_0^{-3}E_{Cas}^{dS4} = \int_0^\infty \frac{d\tau}{\tau} \frac{1}{2} \text{Tr} H_{\vec{p}}(\eta, \eta'; \tau) \quad , \quad (15)$$

Sec 5. Casimir Energy (2/2)

where $H_{\vec{p}}(\eta, \eta'; \tau)$ is the **Heat-Kernel**:

$$\begin{aligned}
 \left\{ \frac{\partial}{\partial \tau} - (s^{-1} \hat{L}_\eta + \vec{p}^2) \right\} H_{\vec{p}}(\eta, \eta'; \tau) &= 0 \quad , \\
 H_{\vec{p}}(\eta, \eta'; \tau) &= (\eta | e^{(s^{-1} \hat{L}_\eta + \vec{p}^2) \tau} | \eta') \\
 = e^{\vec{p}^2 \tau} \sum_n e^{-M_n^2 \tau} \phi_n(\eta) \phi_n(\eta') &\rightarrow (\eta | \eta') \text{ as } \tau \rightarrow +0 \quad . \quad (16)
 \end{aligned}$$

Sec 6. Spatial Wick Rotation

$$-H_0^{-3} E_{Cas}^{dS4} = \int \frac{d^3 \vec{p}}{(2\pi)^3} 2 \int_{-1/H_0}^{-1/\omega} d\eta \left\{ \frac{1}{2} \int_0^\infty \frac{d\tau}{\tau} (\eta | e^{\tau(s(\eta)^{-1} \hat{L}_\eta + \vec{p}^2)} | \eta) \right\} \quad (17)$$

Diverges very badly ! To regularize it, we do

Wick rotation for **space-components** of momentum

$$p_x, p_y, p_z \longrightarrow ip_x, ip_y, ip_z \quad (18)$$

The regularized expression is Casimir energy for **AdS₄**. The **finiteness** (both for IR and for UV) is shown. S.I. arXiv:0812.1263, 0801.3064

Sec 7. Metric Fluctuation (1/3)

Metric field $g_{\mu\nu}(x)$: the background one ($dS_4, g_{\mu\nu}^{inf}$). This is regarded as the variational solution of the effective action $\Gamma[\Phi_{cl}; g_{\mu\nu}]$ (3).

$\Phi_{cl}(x), g_{\mu\nu}^{inf}(x)$: fixed function of x^μ

$$\Gamma[\Phi_{cl}(x); g_{\mu\nu}^{inf}(x)] \equiv \int d^4x \mathcal{L}^{eff}[x^\mu] = \int dt d^3\vec{x} \mathcal{L}^{eff}[t, \vec{x}] \quad . \quad (19)$$

The action for a quantum mechanical system: dynamical variables x^i ($i = 1, 2, 3$) and time $x^0 = t$. The small fluctuation of x^i , keeping $x^0 = t$ fixed, in the dS_4 geometry $g_{\mu\nu}^{inf}(x)$.

$$x^i \rightarrow x^i + \sqrt{\epsilon} f^i(\vec{x}, t) = x^{i'} \quad , \quad t = t' \quad (x^0 = x^{0'}) \quad , \quad (20)$$

where $\vec{x} = (x^i)$. ϵ : a small parameter.

Sec 7. Metric Fluctuation (2/3)

This fluctuation can be **translated** into the metric fluctuation (and the scalar-field fluctuation) as the requirement of the invariance of the line element (**general coordinate invariance**).

$$g_{\mu\nu}^{inf}(x)dx^{\mu'}dx^{\nu'} = g_{\mu\nu}'(x)dx^{\mu}dx^{\nu}, \quad g_{\mu\nu}'(x) = g_{\mu\nu}^{inf}(x) + \epsilon h_{\mu\nu}(x),$$

$$\Phi_{cl}(x') = \Phi_{cl}(x) + \delta\Phi_{cl}(x) \quad , \quad \delta\Phi_{cl} = \partial_i\Phi_{cl}\sqrt{\epsilon}f^i \quad , (21)$$

$$h_{00} = e^{2H_0t}\partial_0f^i \cdot \partial_0f^i \quad , \quad h_{0i} = h_{i0} = e^{2H_0t}\partial_if^j \cdot \partial_0f^j \quad ,$$

$$h_{ij} = e^{2H_0t}\partial_if^k \cdot \partial_jf^k \quad ,$$

$$\text{constraint :} \quad \left\{ \frac{1}{2}(\partial_if^j + \partial_jf^i)dx^j + 2\partial_0f^i dt \right\} dx^i = 0 \quad , \quad (22)$$

Sec 7. Metric Fluctuation (3/3)

We see the coordinates fluctuation produces the metric one (around the homogeneous and isotropic (dS_4) metric) and the scalar-field fluctuation , as far as the above constraint is preserved. The constraint comes from the difference in the perturbation order between the metric and coordinate fluctuations.

Cause of the fluctuation: the underlying unknown 'micro' dynamics (just like Brownian motion of nano-particles in liquid and gas in the days of the classical mechanics). We treat it as the statistical phenomena. The coordinates are fluctuating in a statistical ensemble. By choosing the statistical distribution in the geometric principle, we can compute the statistical average. (NOTE: not the quantum effect but the statistical one.)

Sec 8. Statistical Ensemble by Geometry (1/4)

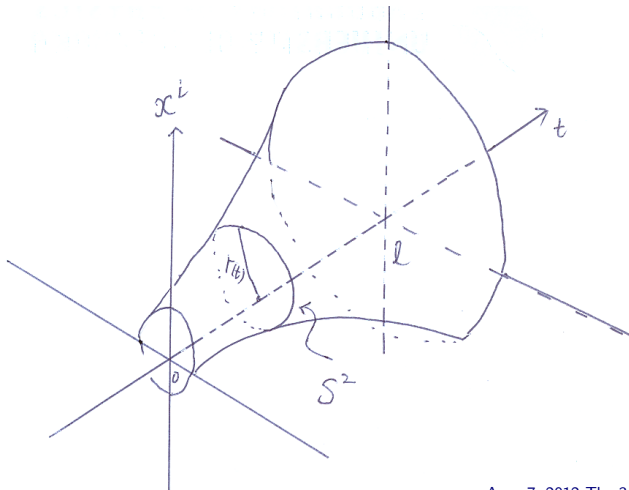
In order to specify the statistical ensemble in the **geometrical** way, we prepare the following **3 dimensional hypersurface** in dS_4 space-time based on the **isotropy** requirement for space (x, y, z) .

$$x^2 + y^2 + z^2 = r(t)^2 \quad , \quad (23)$$

$r(t)$: the radius of S^2 surrounding the 3D ball at a fixed t . $r(t)$ specify the 3D hypersurface. See Fig.1.

Sec 8. Statistical Ensemble by Geometry (2/4)

Figure: Hyper-surface in dS_4 space-time. Eq.(23).



Sec 8. Statistical Ensemble by Geometry (3/4)

Take the hyper-surface $\{r(t) : 0 \leq t \leq l\}$ as a (generalized) path.
On the path (23), the **induced metric** g_{ij} is given by

$$\begin{aligned} ds^2 &= g_{\mu\nu}^{infl} dx^\mu dx^\nu = -dt^2 + e^{2H_0 t} dx^i dx^i \\ &= \left(-\frac{1}{r^2 \dot{r}^2} x^i x^j + \delta^{ij} e^{2H_0 t}\right) dx^i dx^j \equiv g_{ij} dx^i dx^j \quad , \end{aligned} \quad (24)$$

The constraint in (22) reduces to

$$\begin{aligned} \left\{ \frac{1}{2} (\partial_i f^j + \partial_j f^i) v^j + 2 \partial_0 f^i \right\} v^i &= 0 \quad , \quad v^i \equiv \frac{dx^i}{dt} \quad , \\ \text{namely} \quad \vec{v} \cdot D_t \vec{f} &= 0 \quad , \quad D_t = \vec{v} \cdot \vec{\nabla} + \partial_0 \quad . \end{aligned} \quad (25)$$

cf. **fluid dynamics eq.**

Sec 8. Statistical Ensemble by Geometry (4/4)

As the geometrical quantity, we can take the **area** A of the hypersurface.

$$A[x^i, \dot{x}^i] = \int \sqrt{\det g_{ij}} d^3 \vec{x} = \frac{2\sqrt{2}}{3} \int_0^l e^{-3H_0 t} \sqrt{\dot{r}^2 - e^{-2H_0 t}} dt \quad .(26)$$

The **statistically averaged** action $\Gamma^{avg}[\Phi_{cl}; g_{\mu\nu}]$ is defined by the generalized path integral:

$$\Gamma^{avg}[\Phi_{cl}; g_{\mu\nu}] = \int_{1/\Lambda}^{1/\mu} d\rho \int_{r(0)=\rho, r(l)=\rho} \mathcal{D}x^i(t) \times \\ \Gamma[\Phi_{cl}(\vec{x}(\tilde{t}), \tilde{t}); g_{\mu\nu}(\vec{x}(\tilde{t}), \tilde{t})] \exp\left(-\frac{1}{2\alpha'} A[x^i, \dot{x}^i]\right) \quad . \quad (27)$$

μ, Λ : IR and UV cutoffs. $\frac{1}{\alpha'}$: surface tension parameter.

Sec 9. n-Point Function (1/2)

'CMB spectrum':

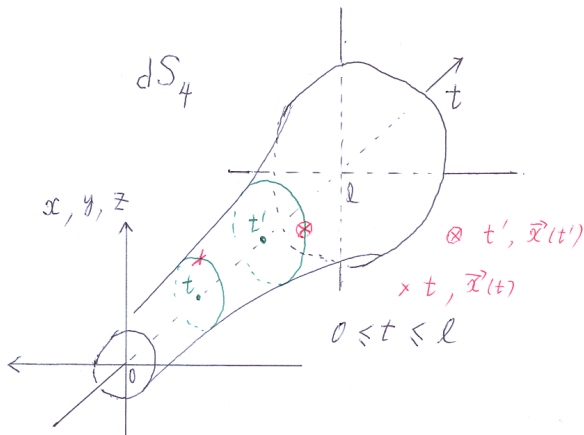
2-Point Function

$$\frac{\delta^2 \Gamma^{avg}}{\delta \tilde{\Phi}_0(t_1) \delta \tilde{\Phi}_0(t_2)} \quad , \quad \tilde{\Phi}_0(t_1) \equiv \Phi_0(\vec{x}(t_1), t_1) \quad (28)$$

$\vec{x}(t)^2 + t^2 = r(t)^2$, $\vec{x}(t')^2 + t'^2 = r(t')^2$. *Note* : $\vec{x}(t)^2 \neq \vec{x}(t')^2$ takes place only when $t \neq t'$, but $\vec{x}(t) \neq \vec{x}(t')$ can take place even when $t = t'$. See Fig.2.

Sec 9. n-Point Function (2/2)

Figure: Two points $\vec{x}(t), \vec{x}(t')$ in (28).



Sec 10. Extra Dimension Model (1/2)

1+4 Dim AdS_5 extra-dimension model:

$$\begin{aligned}
 & \langle \Phi_{cl}(x^\mu(w), w) \Phi_{cl}(x^\mu(w'), w') \rangle = \\
 & \quad \langle \Phi_{cl}(t(w), \vec{x}(w), w) \Phi_{cl}(t(w'), \vec{x}(w'), w') \rangle, \\
 & \quad \mu = 0, 1, 2, 3 \qquad t(w) = t(w') \qquad (29)
 \end{aligned}$$

See Fig.3 This is 2-point function for two **spacially-different** points at an **equal** time,

Sec 10. Extra Dimension Model (2/2)

Figure: Two points $(t(w), \vec{x}(w), w), (t(w'), \vec{x}(w'), w')$ in (29).
 $t(w) = t(w') \equiv i\tau$, $\vec{x}(w) \cdot \vec{x}(w) + \tau^2 = r^2(w)$, $\vec{x}(w') \cdot \vec{x}(w') + \tau^2 = r^2(w')$

