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BRST Conformal Symmetry and Quantum Gravity

Ken-ji HAMADA
http://research.kek.jp/people/hamada/

References

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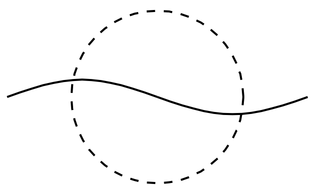
- K. Hamada, "*BRST Analysis of Physical Fields and States for 4D Quantum Gravity on R x S^3*", Rhys. Rev. D86 (2012) 124006.

Introduction

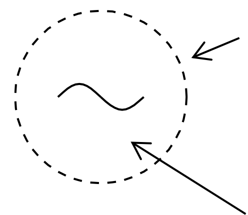
Why we need Background Free Nature

First of all, particle picture conflicts with Einstein gravity, because it is a black hole beyond Planck scale

Compton wave length
= typical size of particle
$\sim 1/m$



$m < m_{\rm pl}$



Horizon size $\sim m/m_{\rm pl}^2$

Particle information is lost

$m > m_{\rm pl}$

In order to solve this information loss problem,
I consider fully fluctuated spacetime such that
the concept of distance itself is lost beyond Planck scale
➔ Background-metric independence

Such a spacetime has no scale and no singularity

How to realize it

If conformal invariance is a “gauge symmetry”, it implies that all spacetime connected by conformal transformations are gauge-equivalent !!

$$ds^2 \cong \Omega^2 ds^2$$

BRST Conformal Symmetry

- This is an algebraic representation of the background-metric independence
- I will show that such a symmetry is included in diffeomorphism invariance

Diffeomorphism Inv. in D Dimensions

$$\delta_\xi g_{\mu\nu} = g_{\mu\lambda}\nabla_\nu \xi^\lambda + g_{\nu\lambda}\nabla_\mu \xi^\lambda$$

ξ^μ : gauge parameter

Metric field is now expanded as

$$g_{\mu\nu} = \underline{\underline{e^{2\phi}}}\,\bar{g}_{\mu\nu} \qquad \bar{g}_{\mu\nu} = (\hat{g}\,e^{th})_{\mu\nu} = \hat{g}_{\mu\lambda}\left(\delta^\lambda{}_\nu + t h^\lambda{}_\nu + \frac{t^2}{2}(h^2)^\lambda{}_\nu + \cdots\right)$$

Conformal factor
Exactly

Traceless tensor field
Perturbatively

gauge-fixed later

Diffeomorphism is then decomposed as

$$\begin{aligned}
\delta_\xi \phi &= \xi^\lambda \partial_\lambda \phi + \frac{1}{D}\hat{\nabla}_\lambda \xi^\lambda, \\
\delta_\xi h_{\mu\nu} &= \frac{1}{t}\left(\hat{\nabla}_\mu \xi_\nu + \hat{\nabla}_\nu \xi_\mu - \frac{2}{D}\hat{g}_{\mu\nu}\hat{\nabla}_\lambda \xi^\lambda\right) + \xi^\lambda \hat{\nabla}_\lambda h_{\mu\nu} + \frac{1}{2}h_{\mu\lambda}\left(\hat{\nabla}_\nu \xi^\lambda - \hat{\nabla}^\lambda \xi_\nu\right) \\
&\quad + \frac{1}{2}h_{\nu\lambda}\left(\hat{\nabla}_\mu \xi^\lambda - \hat{\nabla}^\lambda \xi_\mu\right) + o(t)
\end{aligned}$$

two modes completely decoupled!

BRST Conf. Inv. Arise As A Part of Diff. Inv.

Consider gauge parameter satisfying conformal Killing vectors

$$\hat{\nabla}_\mu \zeta_\nu + \hat{\nabla}_\nu \zeta_\mu - \frac{2}{D} \hat{g}_{\mu\nu} \hat{\nabla}_\lambda \zeta^\lambda = 0$$

Gauge transformations with ζ^μ at $t = 0$ (UV limit) become

characteristic of diff.

$$\delta_\zeta \phi = \zeta^\lambda \partial_\lambda \phi + \underline{\frac{1}{D} \hat{\nabla}_\lambda \zeta^\lambda},$$

dimensionless scalar with shift term

$$\delta_\zeta h_{\mu\nu} = \zeta^\lambda \hat{\nabla}_\lambda h_{\mu\nu} + \frac{1}{2} h_{\mu\lambda} \left(\hat{\nabla}_\nu \zeta^\lambda - \hat{\nabla}^\lambda \zeta_\nu \right) + \frac{1}{2} h_{\nu\lambda} \left(\hat{\nabla}_\mu \zeta^\lambda - \hat{\nabla}^\lambda \zeta_\mu \right)$$

dimensionless tensor

In the following

Traceless tensor fields are gauge-fixed properly such that gauge d.o.f. reduce to conformal Killing vectors

Changing ζ^μ with ghost c^μ, we obtain BRST conformal symmetry

Plan of This Talk

I will present two models with BRST conformal symmetry, and only these two are known at present

1. Brief Summary of 2D Quantum Gravity on RxS^1

2. 4D Quantum Gravity on R x S^3

3. Conclusion

and also, at last, discuss how this conformal symmetry is breaking by the coupling *t* at low energies

Note:

- Due to BRST conformal invariance, we can choose any background as far as it is conformally flat
- Here, the cylindrical background $R\times S^{D-1}$ is used because we can define primary states in Lorentzian CFT on it, unlike on M^4

Of course, we can also choose Minkowski background M^4

K.H., Phys. Rev. D85 (2012)024028

Brief Summary of 2D Quantum Gravity on RxS^1

To remember Kato-Ogawa-type BRST symmetry

This is the simplest example of BRST conforml symmetry, which is known as the reparametrization inv. in world-sheet theory of string

The Action of 2D Quantum Gravity

Incorporating a contribution from the path integral measure, quantum gravity can be described as

Jacobian to ensure diff. inv.
=Wess-Zumino action

$$Z = \int [dgdf]_{g} e^{iI(f,g)} = \int [d\phi dh df]_{\hat{g}} e^{iS(\phi,\hat{g}) + iI(f,g)}$$

In 2 dimensions, we can take conformal gauge

$$h_{\mu\nu} = 0$$

➔ residual gauge symmetry = BRST conformal symmetry

2D quantum gravity action is $S_{\mathrm{2DQG}} = S_{\mathrm{L}} + I_{\mathrm{M}} + I_{\mathrm{gh}}$

$$S_{\mathrm{L}} = -\frac{b_{\mathrm{L}}}{4\pi} \int d^2x \sqrt{-\hat{g}} \left(\hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \underline{\hat{R}\phi} \right) \qquad b_{\mathrm{L}} = \frac{25 - c_{\mathrm{M}}}{6}$$

This WZ action is called Liouville action

generate shift term in diffeomorphism $\delta_\zeta \phi$

central charge of matter field

Virasoro Algebra and BRST Operator

Virasoro generator (= generator of conformal transformation)

$$L_n^{\pm} = L_n^{\mathrm{L}\pm} + L_n^{\mathrm{M}\pm} + L_n^{\mathrm{gh}\pm}$$

Left and right movers on S^1

satisfy Virasoro algebra

$$\left[L_n^{\pm}, L_m^{\pm}\right] = (n-m)L_{n+m}^{\pm} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \qquad [L_n^{+}, L_m^{-}] = 0$$

with vanishing central charge such as

$$c = 1 + 6b_{\mathrm{L}} + c_{\mathrm{M}} - 26 = 0 \longleftarrow$$

conformally invariant
at the quantum level

Nilpotent BRST operator is constructed as

bc ghost modes

$$Q_{\mathrm{BRST}} = Q^{+} + Q^{-}$$

$$Q^{\pm} = \sum_{n\in\mathbf{Z}} c_{-n}^{\pm}\left(L_n^{\mathrm{L}\pm} + L_n^{\mathrm{M}\pm}\right) - \frac{1}{2}\sum_{n,m\in\mathbf{Z}}(n-m) : c_{-n}^{\pm}c_{-m}^{\pm}b_{n+m}^{\pm} :$$

BRST Algebra and Physical Fields

BRST conformal transformations (= diff. in 2D) are

$$i[Q_{\mathrm{BRST}}, \phi] = c^\mu \partial_\mu \phi + \frac{1}{2}\partial_\mu c^\mu \qquad i\{Q_{\mathrm{BRST}}, c^\mu\} = c^\nu \partial_\nu c^\mu$$

The simplest gravitationally-dressed operator corresponding to $\sqrt{-g}\Phi_\Delta$ is

matter field

$$V_\Delta = :\omega e^{\gamma\phi}\Phi_\Delta: \qquad \omega = c^+ c^- (= \epsilon_{\mu\nu} c^\mu c^\nu / 2)$$

BRST invariance determines the (Liouville) charge γ as

$$i[Q_{\mathrm{BRST}}, V_\Delta] = \frac{1}{2}\left(\gamma - \frac{\gamma^2}{2b_{\mathrm{L}}} + 2\Delta - 2\right) V_\Delta = 0$$

quantum corrections

$$\gamma = 2b_{\mathrm{L}}\left(1 - \sqrt{1 - \frac{4-4\Delta}{b_{\mathrm{L}}}}\right) = \underline{2 - 2\Delta} + \frac{2(\Delta-1)^2}{b_{\mathrm{L}}} + \cdots$$

classical value of $\sqrt{-g}\Phi_\Delta$

Physical States

※ There are many physical states with derivatives called "discreet states"
BMP, Comm. Math. Phys. 145 (1992)541

BRST invariance condition is

$$Q_{\mathrm{BRST}}|\gamma\rangle = 0$$

Gravitationally dressed state is given by

conformally inv. vacuum

$$|\gamma\rangle = e^{\gamma\phi_0}|\Omega\rangle_{\mathrm{L}} \otimes |\Delta\rangle_{\mathrm{M}} \otimes c_1^+ c_1^- |0\rangle_{\mathrm{gh}} \qquad L_n^{\mathrm{L}\pm}|\Omega\rangle_{\mathrm{L}} = 0 \ (n \geq -1)$$

State-operator correspondence are given by

$$|\gamma\rangle = \lim_{\eta \to i\infty} V_{\Delta}(\eta,\sigma)|\Omega\rangle_{\mathrm{L}} \otimes |0\rangle_{\mathrm{M}} \otimes |0\rangle_{\mathrm{gh}}$$

Physical states are given by "primary real scalars" in terms of CFT

Since Virasoro generators are BRST trivial like $\{Q_{\mathrm{BRST}}, b_n^{\pm}\} = L_n^{\pm}$
decendant states generated by $L_{-n}^{\pm}$ become BRST trivial

General Comments on Theoretical Structure of BRST Conformal Symmetry

First, the kinetic (UV) terms of both matter and gravitational fields must have "classical conformal invariance".

feedback

When only matter fields are quantized (= curved space theory), conformal invariance is always violated through Wess-Zumino actions associated with conformal anomalies.

However

When gravity is quantized further incorporating Wess-Zumino action properly, conformal invariance recovers exactly at the quantum level.

Thus, "conformal anomalies" are now necessary elements to preserve exact conformal invariance, namely diffeomorphism inv.

In order to construct the BRST operator at the quantum level, classical conformal invariance of the kinetic terms are necessary.

➔ This symmetry exists only in even dimensions, but not in odd

4D Quantum Gravity on RxS^3

I will show that the similar structure mentioned before appears in this case

The Action of 4D Quantum Gravity

The path integral measure can be written as

Wess-Zumino actions

$$Z = \int [dgdf]_{\underline{g}} e^{iI(f,g)} = \int [d\phi dh df]_{\underline{\hat{g}}} e^{iS(\phi,\hat{g})+iI(f,g)}$$

denotes later

The traceless tensor fields are gauge-fixed in radiation+ gauge

➔ residual gauge sym. reduces to BRST conformal sym.

Quantum gravity system in the UV limit (t=0) is then described as

$$S_{\rm 4DQG} = S_{\rm RWZ} + I_{\rm W}\big|_{t\to 0}$$ + “finite” gauge ghosts (+ matters)

$$I_{\rm W} = -\frac{1}{t^2}\int d^4x\sqrt{-g}C^2_{\mu\nu\lambda\sigma}$$

Weyl action (kinetic term only at t=0)

$$S_{\rm RWZ} = -\frac{b_1}{(4\pi)^2}\int d^4x\sqrt{-\hat{g}}\left[2\phi\hat{\Delta}_4\phi + \left(\hat{G}_4 - \frac{2}{3}\hat{\nabla}^2\hat{R}\right)\underline{\phi}\right]$$

Riegert-Wess-Zumino action

$$g_{\mu\nu} = e^{2\phi}(\hat{g}_{\mu\nu} + th_{\mu\nu} + \cdots)$$

Gauge-Fixed Action of 4DQG

Let us take radiation gauge $h_{00} = \hat{\nabla}^i h_{0i} = \nabla^i h^{\rm tr}_{ij} = 0$

$\Box_3 = \hat{\nabla}^i \hat{\nabla}_i$

Then, the action on R x S^3 (radius=1) is given by

$$
\begin{aligned}
S_{\rm 4DQG} \quad = \quad & \int d\eta \int_{S^3} d\Omega_3 \Big\{ -\frac{2b_1}{(4\pi)^2} \phi \left(\partial_\eta^4 - 2\Box_3 \partial_\eta^2 + \Box_3^2 + 4\partial_\eta^2 \right) \phi \\
& -\frac{1}{2} h^{\rm TT}_{ij} \left(\partial_\eta^4 - 2\Box_3 \partial_\eta^2 + \Box_3^2 + 8\partial_\eta^2 - 4\Box_3 + 4 \right) h^{ij}_{\rm TT} \\
& + h^{\rm T}_i \underline{\left(\Box_3 + 2 \right)} \left(-\partial_\eta^2 + \Box_3 - 2 \right) h^i_{\rm T} \Big\}
\end{aligned}
$$

Furthermore, we remove the mode satisfying $(\Box_3 + 2)\, h^i_{\rm T} = 0 \quad \Leftrightarrow \quad h^{\rm T}_i \big|_{J=\frac{1}{2}} = 0$

= radiation+ gauge ➔ residual gauge d.o.f. reduces conformal Killing vectors

The coeff. of RWZ action is positive and more than 4:

$-7/90 + 87/20$

(Riegert + Weyl)

$$
b_1 = \frac{1}{360}\left(N_S + 11 N_F + 62 N_A \right) + \underline{\frac{769}{180}} \; (> 4)
$$

Mode Expansion and Quantization

Scalar harmonics on *S^3* 🡐 *(J,J)* rep. of *SO(4)=SU(2)xSU(2)* isometry

$$\Box_3 Y_{JM} = -2J(2J+2)Y_{JM}, \qquad Y_{JM} = \sqrt{\frac{2J+1}{V_3}} D^J_{mm'} \qquad M=(m,m')$$

: multiplicity index

Wigner D-function

Mode expansion

$$\begin{aligned}\phi \;=\; & \frac{\pi}{2\sqrt{b_1}}\Bigg\{ 2(\hat{q}+\hat{p}\eta)Y_{00} \\ & + \sum_{J\geq\frac{1}{2}}\sum_{M} \frac{1}{\sqrt{J(2J+1)}}\left(a_{JM} e^{-i2J\eta} Y_{JM} + a^\dagger_{JM} e^{i2J\eta} Y^*_{JM} \right) \\ & + \sum_{J\geq 0}\sum_{M} \frac{1}{\sqrt{(J+1)(2J+1)}}\left(b_{JM} e^{-i(2J+2)\eta} Y_{JM} + b^\dagger_{JM} e^{i(2J+2)\eta} Y^*_{JM} \right)\Bigg\}\end{aligned}$$

Commutation relations

negative-metric

$$[\hat{q},\hat{p}] = i, \quad [a_{J_1M_1}, a^\dagger_{J_2M_2}] = \delta_{J_1J_2}\delta_{M_1M_2}, \quad [b_{J_1M_1}, b^\dagger_{J_2M_2}] = -\delta_{J_1J_2}\delta_{M_1M_2}$$

Later, I will show that all these modes are not gauge-inv. alone

STT spherical tensor harmonics ← $(J+\varepsilon_n, J-\varepsilon_n)$ rep. of *SU(2)xSU(2)*

$$\Box_3 Y^{i_1\cdots i_n}_{J(M\varepsilon_n)} = [-2J(2J+2)+n] Y^{i_1\cdots i_n}_{J(M\varepsilon_n)} \qquad J \geq n/2 \qquad M=(m,m')$$

Mode expansions

$\varepsilon_n = \pm n/2$: polarization index

$$\begin{aligned} h^{ij}_{\mathrm{TT}} &= \frac{1}{4}\sum_{J\geq 1}\sum_{M,x}\frac{1}{\sqrt{J(2J+1)}}\left\{c_{J(Mx)}e^{-i2J\eta}Y^{ij}_{J(Mx)} + c^\dagger_{J(Mx)}e^{i2J\eta}Y^{ij*}_{J(Mx)}\right\} \\ &+\frac{1}{4}\sum_{J\geq 1}\sum_{M,x}\frac{1}{\sqrt{(J+1)(2J+1)}}\left\{d_{J(Mx)}e^{-i(2J+2)\eta}Y^{ij}_{J(Mx)}\right. \\ &\qquad \left.+d^\dagger_{J(Mx)}e^{i(2J+2)\eta}Y^{ij*}_{J(Mx)}\right\}, \end{aligned}$$

$x = \varepsilon_2 = \pm 1$

$$\begin{aligned} h^i_{\mathrm{T}} &= i\frac{1}{2}\sum_{J\geq 1}\sum_{M,y}\frac{1}{\sqrt{(2J-1)(2J+1)(2J+3)}} \\ &\quad \times\left\{e_{J(My)}e^{-i(2J+1)\eta}Y^i_{J(My)} - e^\dagger_{J(My)}e^{i(2J+1)\eta}Y^{i*}_{J(My)}\right\} \end{aligned}$$

$y = \varepsilon_1 = \pm 1/2$

radiation+ $e_{\frac{1}{2}(My)} = 0$

Commutation relations

$$[c_{J_1(M_1x_1)}, c^\dagger_{J_2(M_2x_2)}] = -[d_{J_1(M_1x_1)}, d^\dagger_{J_2(M_2x_2)}] = \delta_{J_1J_2}\delta_{M_1M_2}\delta_{x_1x_2}$$

$$[e_{J_1(M_1y_1)}, e^\dagger_{J_2(M_2y_2)}] = -\delta_{J_1J_2}\delta_{M_1M_2}\delta_{y_1y_2}$$

I will show that all these modes are also not gauge-inv. alone

15 Conformal Killing Vectors on RxS^3

1 Time translation: $\zeta_H^\mu = (1,0,0,0)$ (= dilatation on *RxS^3*)

6 Rotations on S^3: $\zeta_R^\mu = (0, \zeta_{MN}^\mu)$

$$\zeta_{MN}^j = i\frac{\mathrm{V}_3}{4}\left\{Y^*_{\frac{1}{2}M}\hat{\nabla}^j Y_{\frac{1}{2}N} - Y_{\frac{1}{2}N}\hat{\nabla}^j Y^*_{\frac{1}{2}M}\right\}$$

4 Special conformal: $\zeta_S^\mu = (\zeta_M^0, \zeta_M^i)$ with

$$\zeta_M^0 = \frac{\sqrt{\mathrm{V}_3}}{2}e^{i\eta}Y^*_{\frac{1}{2}M}, \quad \zeta_M^j = -i\frac{\sqrt{\mathrm{V}_3}}{2}e^{i\eta}\hat{\nabla}^j Y^*_{\frac{1}{2}M}$$

4 Translations: $\zeta_T^\mu = \zeta_S^{\mu\dagger}$ (= hermite conjugate of special conf.)

characteristic features of conf. algebra on RxS^3
(not on M^4)

Radius of *S^3* =1, the volume is $V_3 = 2\pi^2$

Indices *M, N* without *J* denote 4-vectors of *SO(4)* with *J=1/2*

Conformal Algebra on RxS^3

Generators of conformal transformations

Riegert + Weyl

$$Q_\zeta = \int_{S^3} d\Omega_3 \zeta^\mu : \hat{T}_{\mu 0} : \qquad \delta_\zeta F = i[Q_\zeta, F]$$

conformal Killing vectors

$$\hat{T}_{\mu\nu} = \frac{2}{\sqrt{-\hat{g}}} \frac{\delta S_{4DQG}}{\delta \hat{g}_{\mu\nu}}$$

S(4,2) conformal algebra

15 generators

$$\begin{aligned}
\left[Q_M, Q_N^\dagger\right] &= 2\delta_{MN} H + 2R_{MN}, \\
[H, Q_M] &= -Q_M, \quad [H, R_{MN}] = 0, \\
[Q_M, Q_N] &= 0, \quad [Q_M, R_{NL}] = \delta_{ML} Q_N - \epsilon_N \epsilon_L \delta_{M-N} Q_{-L}, \\
[R_{MN}, R_{LK}] &= \delta_{MK} R_{LN} - \epsilon_M \epsilon_N \delta_{-NK} R_{L-M} \\
&\quad -\delta_{NL} R_{MK} + \epsilon_M \epsilon_N \delta_{-ML} R_{-NK}.
\end{aligned}$$

← feature of RxS^3

- H : Hamiltonian
- R_{MN} : 6 S^3 rotation
- Q_M : 4 special conf.
- $Q_M^\dagger$: 4 translation

In CFT,

H R_{MN} Q_M define primary states, while $Q_M^\dagger$ generates descendant states

$$R_{MN} = -\epsilon_M \epsilon_N R_{-N-M}, \qquad R_{MN}^\dagger = R_{NM}$$

Generators for Riegert sector

diagonal form

$$H = \frac{1}{2}\hat{p}^2 + b_1 + \sum_{J\geq 0}\sum_{M}\{2J a^{\dagger}_{JM} a_{JM} - (2J+2) b^{\dagger}_{JM} b_{JM}\}$$

Casimir effect on R x S^3

off-diagonal form

$$Q_M = \left(\sqrt{2b_1} - i\hat{p}\right) a_{\frac{1}{2}M} + \sum_{J\geq 0}\sum_{M_1,M_2} \mathbf{C}^{\frac{1}{2}M}_{JM_1,J+\frac{1}{2}M_2}\Big\{\sqrt{2J(2J+2)}\epsilon_{M_1} a^{\dagger}_{J-M_1} a_{J+\frac{1}{2}M_2}$$
$$-\sqrt{(2J+1)(2J+3)}\epsilon_{M_1} b^{\dagger}_{J-M_1} b_{J+\frac{1}{2}M_2} + \underline{\epsilon_{M_2} a^{\dagger}_{J+\frac{1}{2}-M_2} b_{JM_1}}\Big\}$$

The Q_M gauge transformation mixes positive- and negative-metric modes

➔ Each mode is not gauge invariant !

$$\left(\left[Q_M, b^{\dagger}_{JN_1}\right] = \sqrt{2J(2J+2)}\sum_{N_2}\epsilon_{N_2}\mathbf{C}^{\frac{1}{2}M}_{JN_1,J-\frac{1}{2}-N_2} b^{\dagger}_{J-\frac{1}{2}N_2} - \sum_{N_2}\epsilon_{N_2}\mathbf{C}^{\frac{1}{2}M}_{JN_1,J+\frac{1}{2}-N_2}\underline{a^{\dagger}_{J+\frac{1}{2}N_2}}\right)$$

where *SU(2)xSU(2)* Clebsch-Gordan coeff. Is defined by

$$\mathbf{C}^{JM}_{J_1M_1,J_2M_2} = \sqrt{2}\pi\int_{S^3} d\Omega_3 Y^{*}_{JM} Y_{J_1M_1} Y_{J_2M_2} = \sqrt{\frac{(2J_1+1)(2J_2+1)}{2J+1}} C^{Jm}_{J_1m_1,J_2m_2} C^{Jm'}_{J_1m'_1,J_2m'_2}$$

Generators for Weyl sector ($i[Q_\zeta, h_{\mu\nu}] = \delta_\zeta h_{\mu\nu}$)

$$
\begin{aligned}
H \;=\;& \sum_{J(\geq 1)} \sum_{M,x} \left\{ 2J c^\dagger_{J(Mx)} c_{J(Mx)} - (2J+2) d^\dagger_{J(Mx)} d_{J(Mx)} \right\} \\
& - \sum_{J(\geq 1)} \sum_{M,y} (2J+1) e^\dagger_{J(My)} e_{J(My)}
\end{aligned}
$$

SU(2)^2 CG coeff.

$\mathbf{E}$: STT type

$\mathbf{H}$: STV type

$\mathbf{D}$: SVV type

$$
\begin{aligned}
Q_M \;=\;& \sum_{J\geq 1} \sum_{M_1,x_1} \sum_{M_2,x_2} \mathrm{E}^{\frac{1}{2}M}_{J(M_1x_1),J+\frac{1}{2}(M_2x_2)} \Big\{ \sqrt{2J(2J+2)}\,\epsilon_{M_1} c^\dagger_{J(-M_1x_1)} c_{J+\frac{1}{2}(M_2x_2)} \\
& - \sqrt{(2J+1)(2J+3)}\,\epsilon_{M_1} d^\dagger_{J(-M_1x_1)} d_{J+\frac{1}{2}(M_2x_2)} + \underline{\epsilon_{M_2} c^\dagger_{J+\frac{1}{2}(-M_2x_2)} d_{J(M_1x_1)}} \Big\} \\
& + \sum_{J\geq 1} \sum_{M_1,x_1} \sum_{M_2,y_2} \mathrm{H}^{\frac{1}{2}M}_{J(M_1x_1);J(M_2y_2)} \Big\{ A(J)\underline{\epsilon_{M_1} c^\dagger_{J(-M_1x_1)} e_{J(M_2y_2)}} \\
& \qquad + B(J)\underline{\epsilon_{M_2} e^\dagger_{J(-M_2y_2)} d_{J(M_1x_1)}} \Big\} \\
& + \sum_{J\geq 1} \sum_{M_1,y_1} \sum_{M_2,y_2} \mathrm{D}^{\frac{1}{2}M}_{J(M_1y_1),J+\frac{1}{2}(M_2y_2)} C(J)\,\epsilon_{M_1} e^\dagger_{J(-M_1y_1)} e_{J+\frac{1}{2}(M_2y_2)}
\end{aligned}
$$

cross terms

The Q_M gauge transformation mixes all modes, except the lowest of positive-metric $c^\dagger_{\underline{1}(Mx)}$

$$
A(J) = \sqrt{\frac{4J}{(2J-1)(2J+3)}}, \qquad B(J) = \sqrt{\frac{2(2J+2)}{(2J-1)(2J+3)}},
$$

$$
C(J) = \sqrt{\frac{(2J-1)(2J+1)(2J+2)(2J+4)}{2J(2J+3)}}
$$

BRST Operator

BRST transf. : $\zeta^\lambda \to c^\lambda \Longrightarrow \begin{cases} i[Q_{\mathrm{BRST}}, \phi] = c^\mu \hat{\nabla}_\mu \phi + \frac{1}{4} \hat{\nabla}_\mu c^\mu = \delta_B \phi \\ i[Q_{\mathrm{BRST}}, h_{\mu\nu}] = \delta_B h_{\mu\nu} \\ i\{Q_{\mathrm{BRST}}, c^\mu\} = c^\nu \hat{\nabla}_\nu c^\mu \end{cases}$

Gauge ghost fields (15 Grassmannian modes)

$$c^\mu = \mathrm{c}\eta^\mu + \sum_M \left(\mathrm{c}^\dagger_M \zeta^\mu_M + \mathrm{c}_M \zeta^{\mu *}_M \right) + \sum_{M,N} \mathrm{c}_{MN} \zeta^\mu_{MN}$$

$$\mathrm{c}^\dagger_{MN} = \mathrm{c}_{NM} \qquad \mathrm{c}_{MN} = -\epsilon_M \epsilon_N \mathrm{c}_{-N-M}$$

We set commutation relations as

$$\{\mathrm{b}, \mathrm{c}\} = 1, \quad \{\mathrm{b}^\dagger_M, \mathrm{c}_N\} = \{\mathrm{b}_M, \mathrm{c}^\dagger_N\} = \delta_{MN},$$

$$\{\mathrm{b}_{MN}, \mathrm{c}_{LK}\} = \delta_{ML}\delta_{NK} - \epsilon_M \epsilon_N \delta_{-MK}\delta_{-NL}$$

antighosts: $\mathrm{b},\ \mathrm{b}_{MN},\ \mathrm{b}_M,\ \mathrm{b}^\dagger_M$

satisfy the similar relations to c_{MN}

Nilpotent BRST operator

$$
\begin{aligned}
Q_{\mathrm{BRST}} &= \mathrm{c}\mathcal{H} + \sum_{M,N} \mathrm{c}_{MN}\mathcal{R}_{MN} - \mathrm{b}\mathcal{M} - \sum_{M,N} \mathrm{b}_{MN} Y_{MN} + \hat{Q} \\
\mathcal{M} &= 2\sum_M \mathrm{c}^\dagger_M \mathrm{c}_M, \qquad Y_{MN} = \mathrm{c}^\dagger_M \mathrm{c}_N + \sum_L \mathrm{c}_{ML}\mathrm{c}_{LN}, \\
\hat{Q} &= \sum_M \left(\mathrm{c}^\dagger_M \mathcal{Q}_M + \mathrm{c}_M \mathcal{Q}^\dagger_M \right)
\end{aligned}
$$

Full generators of SO(4,2) including gauge ghost sector

$$
\begin{aligned}
\mathcal{H} &= H + H^{\mathrm{gh}}, \qquad \mathcal{R}_{MN} = R_{MN} + R^{\mathrm{gh}}_{MN}, \\
\mathcal{Q}_M &= Q_M + Q^{\mathrm{gh}}_M, \qquad \mathcal{Q}^\dagger_M = Q^\dagger_M + Q^{\mathrm{gh}\dagger}_M \\
H^{\mathrm{gh}} &= \sum_M \left(\mathrm{c}^\dagger_M \mathrm{b}_M - \mathrm{c}_M \mathrm{b}^\dagger_M \right)
\end{aligned}
$$

(see Ref. for R^{gh}_{MN} Q^{gh}_M)

Full generators become BRST trivial:

$$
\begin{aligned}
\{Q_{\mathrm{BRST}}, \mathrm{b}\} &= \mathcal{H}, \qquad \{Q_{\mathrm{BRST}}, \mathrm{b}_{MN}\} = 2\mathcal{R}_{MN}, \\
\{Q_{\mathrm{BRST}}, \mathrm{b}_M\} &= \mathcal{Q}_M, \qquad \left\{Q_{\mathrm{BRST}}, \mathrm{b}^\dagger_M\right\} = \mathcal{Q}^\dagger_M \longleftarrow
\end{aligned}
$$

Descendants are BRST trivial

First three define primary states in CFT

Physical State Conditions

Fock vacuum with Riegert charge γ

background charge

$$|\Omega\rangle = e^{-2b_1\phi_0(0)}|0\rangle$$

Riegert vacuum annihilated by all generators

$$|\gamma\rangle = e^{\gamma\phi_0(0)}|\Omega\rangle \otimes \underbrace{\prod_M \mathrm{c}_M |0\rangle_{\mathrm{gh}}}$$

ghost vacuum annihilated by c_M , b_M

Consider following states:

$$|\Psi\rangle = \mathcal{A}(\hat{p}, a^\dagger_{JM}, b^\dagger_{JM}, \cdots)|\gamma\rangle \qquad [H, \mathcal{A}] = l\mathcal{A}$$

level of state (even integer)

BRST invariance condition

$$Q_{\mathrm{BRST}}|\Psi\rangle = 0 \Rightarrow \begin{cases} [Q_M, \mathcal{A}] = [R_{MN}, \mathcal{A}] = 0 \\ \gamma - \dfrac{\gamma^2}{4b_1} + l - 4 = 0 \end{cases}$$

(no $Q^\dagger_M$ condition)

classical value

$$\rightarrow \gamma = \gamma_l \equiv 2b_1\left(1 - \sqrt{1 - \frac{4-l}{b_1}}\right) = \underline{4 - l} + o(1/b_1)$$

real due to $b_1 > 4$

Solve the Condition $[Q_M, \mathcal{A}] = 0$

All creation modes $a^\dagger_{JM}$ $b^\dagger_{JM}$ do not commute with Q_M

⇨ $\mathcal{A}$ is given by bilinear forms,

which gives building blocks of physical states

For $L \geq 1$ and integers

$$
\begin{aligned}
S^\dagger_{LN} &= \chi(\hat{p}, L) a^\dagger_{LN} + \sum_{K=\frac{1}{2}}^{L-\frac{1}{2}} \sum_{M_1} \sum_{M_2} x(L,K) \mathrm{C}^{LN}_{L-KM_1, KM_2} a^\dagger_{L-KM_1} a^\dagger_{KM_2} \\
\mathcal{S}^\dagger_{L-1N} &= \psi(\hat{p}) b^\dagger_{L-1N} + \sum_{K=\frac{1}{2}}^{L-\frac{1}{2}} \sum_{M_1} \sum_{M_2} x(L,K) \mathrm{C}^{L-1N}_{L-KM_1, KM_2} a^\dagger_{L-KM_1} a^\dagger_{KM_2} \\
&\quad + \sum_{K=\frac{1}{2}}^{L-1} \sum_{M_1, M_2} y(L,K) \mathrm{C}^{L-1N}_{L-K-1M_1, KM_2} b^\dagger_{L-K-1M_1} a^\dagger_{KM_2}
\end{aligned}
$$

where

$$
x(L,K) = \frac{(-1)^{2K}}{\sqrt{(2L-2K+1)(2K+1)}} \sqrt{\begin{pmatrix} 2L \\ 2K \end{pmatrix} \begin{pmatrix} 2L-2 \\ 2K-1 \end{pmatrix}} \qquad y(L,K) = -2\sqrt{(2L-2K-1)(2L-2K+1)}\, x(L,K)
$$

$$
\chi(\hat{p}, L) = \sqrt{2}(\sqrt{2b_1} - i\hat{p}) / \sqrt{(2L-1)(2L+1)} \qquad \psi(\hat{p}) = -\sqrt{2}(\sqrt{2b_1} - i\hat{p})
$$

Classification of Physical States

Table of building blocks for gravitational sector

traceless tensor field sector

rank of tensor	0	0	1	2	3	4
operators	$S^\dagger_{LN}$ $\mathcal{S}^\dagger_{L-1N}$	$A^\dagger_{LN}$ $\mathcal{A}^\dagger_{L-1N}$	$B^\dagger_{L-\frac{1}{2}(Ny)}$	$c^\dagger_{1(Nx)}$	$D^\dagger_{L-\frac{1}{2}(Nz)}$	$E^\dagger_{L(Nw)}$ $\mathcal{E}^\dagger_{L-1(Nw)}$
weights	$2L_{(\geq 2)}$	$2L_{(\geq 6)}$	$2L_{(\geq 6)}$	2	$2L_{(\geq 6)}$	$2L_{(\geq 6)}$

The physical state is constructed by imposing $[R_{MN}, \mathcal{A}] = 0$ further

$$l = 0 \qquad |\gamma_0\rangle \quad \Longleftrightarrow \quad \sqrt{-g}$$

$$l = 2 \qquad \mathcal{S}^\dagger_{00}|\gamma_2\rangle \quad \Longleftrightarrow \quad \sqrt{-g}R$$

$$\gamma_l = 2b_1\left(1 - \sqrt{1 - \frac{4-l}{b_1}}\right) = 4 - l + o(1/b_1)$$

$$l = 4 \qquad \mathcal{S}^\dagger_{00}\mathcal{S}^\dagger_{00}|\gamma_4\rangle, \qquad \sum_N \epsilon_N S^\dagger_{1-N} S^\dagger_{1N}|\gamma_4\rangle, \qquad \sum_{N,x} \epsilon_N c^\dagger_{1(-Nx)} c^\dagger_{1(Nx)}|\gamma_4\rangle$$

$$\Updownarrow \qquad\qquad \Updownarrow \qquad\qquad \Updownarrow$$

$$\sqrt{-g}R^2 \qquad \sqrt{-g}(R_{\mu\nu} - g_{\mu\nu}R/4)^2 \qquad \sqrt{-g}C^2_{\mu\nu\lambda\sigma}$$

Physical Field Operators

Physical fields $[Q_{\mathrm{BRST}}, \omega O_\gamma] = 0$ where $\omega = \frac{1}{4!}\epsilon_{\mu\nu\lambda\sigma}c^\mu c^\nu c^\lambda c^\sigma$

➔ $\left[Q_{\mathrm{BRST}}, \int d\Omega_4 O_\gamma\right] = 0$ and $\lim_{\eta\to i\infty} e^{-4i\eta} O_\gamma|\Omega\rangle = |O_\gamma\rangle$

Ex. Quantum cosmological constant term($\alpha = \gamma_0$)

$$\boxed{\lim_{b_1\to\infty} V_\alpha = \sqrt{-g}}$$

$$V_\alpha =: e^{\alpha\phi}: \quad \text{with} \quad h_\alpha = 4$$

Quantum Ricci scalar curvature($\beta = \gamma_2$)

$$\boxed{\lim_{b_1\to\infty} W_\beta = -\sqrt{-g}R/6}$$

$$W_\beta \;=\; :e^{\beta\phi}\left(\hat{\nabla}^2\phi + \frac{\beta}{h_\beta}\hat{\nabla}_\mu\phi\hat{\nabla}^\mu\phi - \frac{h_\beta}{\beta}\right): \quad \text{with} \quad h_\beta = 2$$

$$|W_\beta\rangle = \lim_{\eta\to i\infty} e^{-4i\eta} W_\beta|\Omega\rangle = -\frac{\beta}{2\sqrt{2}b_1}\mathcal{S}_{00}^\dagger e^{\beta\phi_0(0)}|\Omega\rangle$$

$$h_\gamma = \gamma - \frac{\gamma^2}{4b_1}$$

Real Property of Fields and Positivity

Unitarity in CFT = Real property of fields

should be real

- positivity of 2-point function
- positivity of squared OPE coefficients

➔ used in recent conformal bootstrap arguments

In QG case

BRST conformal invariance makes all physical fields real
as well as all negative-metric modes unphysical

Since both Riegert and Weyl actions written in original gravitational fields are positive-definite, the path integral is well-defined such that real property of these fields are preserved

※ If the action unbounded below, the path integral diverges so that the real property of fields is sacrificed to regularize the divergence

Conclusion

We find that

- Quantum gravity is formulated as a usual quantum field theory on a background, and so we can quantize it easily
- BRST conformal symmetry arises as a part of diffeomorphism invariance when conformal-factor field is treated exactly
- This symmetry is nothing but an algebraic realization of background-metric independence, and so can use any background
- BRST operators of this symmetry can be constructed in 2 and 4 dimensions, in which the Wess-Zumino actions of conformal anomalies called Liouville and Riegert actions play a crucial role
- All negative-metric modes in 4D gravity become unphysical due to the presence of BRST conformal symmetry
- Physical states are given by real primary scalar states, and also their decendant states become BRST trivial

Renormalizable 4D quantum gravity can be formulated as a perturbation from such BRST conformal field theory:

K. H. and M. Matsuda, arXiv:1511.09161

BRST CFT + perturbation by the single coupling *t*

$$g_{\mu\nu} = \underline{e^{2\phi}}(\hat{g}_{\mu\nu} + \underline{t}h_{\mu\nu} + \cdots)$$

Beta function is negative

non-perturbative

- BRST conformal symmetry is realized at the UV limit of t=0.
- There exists an IR dynamical scale Λ_{QG}

➔ We propose that at the scale $10^{17}\mathrm{GeV}$, spacetime transits from background-free quantum gravity phase to ordinary our universe where gravitons and elementary particles are propagating

K. H., S. Horata and T. Yukawa, PRD81(2010)083533

Appendix

Wess-Zumino Condition and Background-metric Independence

Integral representation of Riegert-Wess-Zumino action

$$S_{\rm RWZ}(\phi,\hat{g}) = -\frac{b_1}{(4\pi)^2}\int d^4x \int_0^\phi d\phi \sqrt{-g} E_4 \qquad E_4 = G_4 - 2\nabla^2 R/3$$

Wess-Zumino consistency condition

$$S_{\rm RWZ}(\phi,\hat{g}) = S_{\rm RWZ}(\omega,\hat{g}) + S_{\rm RWZ}(\phi-\omega, e^{2\omega}\hat{g})$$

Proof of background-metric independence

$$\begin{aligned}
Z|_{e^{2\omega}\hat{g}} &= \int [d\phi dh]_{e^{2\omega}\hat{g}} \exp\left\{ iS_{\rm RWZ}(\phi, e^{2\omega}\hat{g}) + iI(e^{2\omega}g)\right\} \\
&= \int [d\phi dh]_{\hat{g}} \exp\left\{ iS_{\rm RWZ}(\omega,\hat{g}) + iS_{\rm RWZ}(\phi, e^{2\omega}\hat{g}) + iI(e^{2\omega}g)\right\} \\
&= \int [d\phi dh]_{\hat{g}} \exp\left\{ \underbrace{iS_{\rm RWZ}(\omega,\hat{g}) + iS_{\rm RWZ}(\phi-\omega, e^{2\omega}\hat{g})} + iI(g)\right\} \\
&= Z|_{\hat{g}}
\end{aligned}$$

use Wess-Zumino consistency condition

Adjoint of Physical State

cf. string theory
$\gamma = ip$ w/o BG charge
➔ $\langle O_{-ip} | O_{ip} \rangle = 1$

Remark: $\langle O_\gamma | O_\gamma \rangle$ is unnormalizable!

Out-state $\langle \tilde{O}_\gamma |$ satisfying $\langle \tilde{O}_\gamma | O_\gamma \rangle = 1$ is defined by

due to duality $h_\gamma = h_{4b_1 - \gamma}$

Ex.
$$\langle \tilde{V}_\alpha | = \lim_{\eta \to -i\infty} e^{4i\eta} \langle \Omega | \tilde{V}_\alpha = \langle \Omega | e^{(4b_1 - \alpha)\phi_0(0)}$$
$$\langle \tilde{W}_\beta | = \lim_{\eta \to -i\infty} e^{4i\eta} \langle \Omega | \tilde{W}_\beta = \frac{4b_1 - \beta}{8\sqrt{2}} \langle \Omega | e^{(4b_1 - \beta)\phi_0(0)} \mathcal{S}_{00}$$
$$\tilde{V}_\alpha =: e^{(4b_1 - \alpha)\phi}: \qquad \tilde{W}_\beta = -\frac{b_1}{4} :e^{(4b_1 - \beta)\phi} \left(\hat{\nabla}^2 \phi + \frac{4b_1 - \beta}{h_\beta} \hat{\nabla}_\mu \phi \hat{\nabla}^\mu \phi - \frac{h_\beta}{4b_1 - \beta} \right):$$

No classical limit

Normalization

$\langle \Omega | e^{4b_1 \phi_0(0)} | \Omega \rangle = 1$ (🡐 Riegert charge conservation)

$${}_{\mathrm{gh}}\langle 0 | \prod \mathrm{c}_M^\dagger \vartheta \prod \mathrm{c}_M | 0 \rangle_{\mathrm{gh}} = 1 \qquad \vartheta = ic \prod \mathrm{c}_{MN}$$

Adjoint of physical state is given by $\langle \tilde{O}_\gamma | \otimes {}_{\mathrm{gh}}\langle 0 | \prod \mathrm{c}_M^\dagger \vartheta$

WZ action and Euler density

2D quantum gravity		4D quantum gravity
R	Euler density	$E_4 = G_4 - \frac{2}{3}\nabla^2 R$ (modified)
$\sqrt{-g}R = \sqrt{-\bar{g}}\left(2\bar{\Delta}_2\phi + \bar{R}\right)$	relation	$\sqrt{-g}E_4 = \sqrt{-\bar{g}}\left(4\bar{\Delta}_4\phi + \bar{E}_4\right)$
$\Delta_2 = -\nabla^2$	Conformlly inv. operator	$\Delta_4 = \nabla^4 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\nabla^2 + \frac{1}{3}\nabla^\mu R\nabla_\mu$
$-\frac{b_L}{4\pi}\int d^2x \int_0^\phi d\phi \sqrt{-g}R = -\frac{b_L}{4\pi}\int d^2x\sqrt{-\bar{g}}\left(\phi\bar{\Delta}_2\phi + \bar{R}\phi\right)$	WZ action	$-\frac{b_1}{(4\pi)^2}\int d^4x \int_0^\phi d\phi \sqrt{-g}E_4 = -\frac{b_1}{(4\pi)^2}\int d^4x\sqrt{-\bar{g}}\left(2\phi\bar{\Delta}_4\phi + \bar{E}_4\phi\right)$
Liouville action		Riegert-Wess-Zumino action

Physical Fields

Physical fields satisfy $\left[Q_{\mathrm{BRST}}, \int d\Omega_4 O\right] = 0 \Longleftrightarrow$ diff. inv.

The simplest one: $V_\alpha =: e^{\alpha\phi} := \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} : \phi^n := e^{\alpha\phi_>} e^{\alpha\phi_0} e^{\alpha\phi_<}$

This transforms as $i\left[Q_{\mathrm{BRST}}, V_\alpha\right] = c^\mu \hat{\nabla}_\mu V_\alpha + \frac{h_\alpha}{4} \hat{\nabla}_\mu c^\mu V_\alpha$

conf. weight $h_\alpha = \alpha - \underline{\underline{\frac{\alpha^2}{4b_1}}}$

spacetime dim.

For $\underline{\underline{h_\alpha = 4}}$, $\left[Q_{\mathrm{BRST}}, \int d\Omega_4 V_\alpha\right] = \int d\Omega_4 \hat{\nabla}_\mu \left(c^\mu V_\alpha\right) = 0$

quantum corrections

$$\alpha = 2b_1 \left(1 - \sqrt{1 - \frac{4}{b_1}}\right) = 4 + \frac{4}{b_1} + \cdots$$

The solution approaching canonical value 4 at large b_1 is selected

V_α with this Riegert charge ➔ cosmological constant term

Building Blocks of Physical States

rank of tensor index	0
creation op.	$\Phi^\dagger_{LN}$
level $(L \in \mathbf{Z}_{\geq 0})$	$2L+2$

rank of tensor index	0	1	2
creation op.	$\Psi^\dagger_{LN}$	$q^\dagger_{\frac{1}{2}(Ny)}$	$\Upsilon^\dagger_{L(Nx)}$
level $(L \in \mathbf{Z}_{\geq 2})$	$2L+2$	2	$2L+2$

rank of tensor	0	0	1	2	3	4
operators	$S^\dagger_{LN}$ $\mathcal{S}^\dagger_{L-1N}$	$A^\dagger_{LN}$ $\mathcal{A}^\dagger_{L-1N}$	$B^\dagger_{L-\frac{1}{2}(Ny)}$	$c^\dagger_{1(Nx)}$	$D^\dagger_{L-\frac{1}{2}(Nz)}$	$E^\dagger_{L(Nw)}$ $\mathcal{E}^\dagger_{L-1(Nw)}$
weights	$2L_{(\geq 2)}$	$2L_{(\geq 6)}$	$2L_{(\geq 6)}$	2	$2L_{(\geq 6)}$	$2L_{(\geq 6)}$