

D=3、4 共形場理論の最近の発展 について

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参考文献

- *Bounding Scalar Operator Dimensions in 4D CFT*,
R. Rattazzi, V. Rychkov, E. Tonni and A. Vichi, *JHEP* 0812 (2008) 031 [arXiv:0807.0004].
- *Solving the 3D Ising Model with Conformal Bootstrap*,
S. El-Showk, M. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi,
Phys. Rev. D 86 (2012) 025022 [arXiv:1203.6064].

第一章 はじめに

Conformal Field Theory (CFT)

1. Critical theory of statistical model

For $T \neq T_c$ $\langle O(x)O(0) \rangle \sim e^{-|x|/\xi}$ (exponential damping)

At $T = T_c$ $\xi \rightarrow \infty$ $\langle O(x)O(0) \rangle = \frac{1}{|x|^{2\Delta}}$ (power-law damping)

Critical pointでconformal inv.が現れる

Critical theory = CFT S_{CFT} [In general, action is unknown]

Critical phenomena (=small perturbation from critical point)

$$S_{\text{CFT}} \rightarrow S_{\text{CFT}} + t a^{\Delta-D} \int d^D x O(x) \quad t = (T - T_c)/T_c (\ll 1)$$

Critical exponent (a is UV cutoff)

臨界現象の分類 = conformal field O の分類

(Ex. for $O = \varepsilon$, $t a^{\Delta_\varepsilon - D} = \xi^{\Delta_\varepsilon - D}$ ($\xi = a t^{-\nu}$) $\rightarrow \Delta_\varepsilon = D - 1/\nu$)

2. Background Free Quantum Gravity

Conformal invariance = gauge invariance (diff. inv.)

(cf. in usual CFT, vacuum is inv, but fields are variant)

- ◆ 2D Quantum gravity (Liouville gravity)
Polyakov, KPZ, DK, David
- ◆ 4D Quantum gravity
Riegert, Antoniadis-Mazur-Mottola, Hamada

BRST invariance → physical state = “primary (composite) scalar”
[tensor state is forbidden]



スケール不変なスカラーゆらぎしかない！

Primordial spectrum of the Universe

Hamada-Horata-Yukawa, PRD81(2010)083533

第二章

CFTの基礎

CFT (on Euclidean Space)

Generators: $P_\mu, M_{\mu\nu}, D, K_\mu$

Translation, Lorentz transf., Dilatation, Special conformal transf.

spacetime dim.

Conformal Algebra $SO(D+1, 1)$

$$\begin{aligned} [P_\mu, P_\nu] &= 0, & [M_{\mu\nu}, P_\lambda] &= -i(\delta_{\mu\lambda}P_\nu - \delta_{\nu\lambda}P_\mu), \\ [M_{\mu\nu}, M_{\lambda\sigma}] &= -i(\delta_{\mu\lambda}M_{\nu\sigma} + \delta_{\nu\sigma}M_{\mu\lambda} - \delta_{\mu\sigma}M_{\nu\lambda} - \delta_{\nu\lambda}M_{\mu\sigma}), \\ [D, P_\mu] &= -iP_\mu, & [D, M_{\mu\nu}] &= 0, & [D, K_\mu] &= iK_\mu, \\ [M_{\mu\nu}, K_\lambda] &= -i(\delta_{\mu\lambda}K_\nu - \delta_{\nu\lambda}K_\mu), & [K_\mu, K_\nu] &= 0, \\ [K_\mu, P_\nu] &= 2i(\delta_{\mu\nu}D + M_{\mu\nu}) \end{aligned}$$

Hermiticity

$$P_\mu^\dagger = K_\mu, \quad D^\dagger = -D \quad (\text{注 Minkowskiとは異なる})$$

Primary fields

Spin l fields = symmetric traceless tensors $O_{\mu_1 \dots \mu_l}$

Primary field conditions:

$$\begin{aligned} i [P_\mu, O_{\lambda_1 \dots \lambda_l}(x)] &= \partial_\mu O_{\lambda_1 \dots \lambda_l}(x), \\ i [M_{\mu\nu}, O_{\lambda_1 \dots \lambda_l}(x)] &= (x_\mu \partial_\nu - x_\nu \partial_\mu - i \Sigma_{\mu\nu}) O_{\lambda_1 \dots \lambda_l}(x), \\ i [D, O_{\lambda_1 \dots \lambda_l}(x)] &= (x_\mu \partial_\mu + \Delta) O_{\lambda_1 \dots \lambda_l}(x), \\ i [K_\mu, O_{\lambda_1 \dots \lambda_l}(x)] &= (x^2 \partial_\mu - 2x_\mu x_\nu \partial_\nu - 2\Delta x_\mu + 2ix_\nu \Sigma_{\mu\nu}) O_{\lambda_1 \dots \lambda_l}(x) \end{aligned}$$

$$\Sigma_{\mu\nu} O_{\lambda_1 \dots \lambda_l} = i \sum_{j=1}^l (\delta_{\mu\lambda_j} \delta_{\nu\sigma} - \delta_{\nu\lambda_j} \delta_{\mu\sigma}) O_{\lambda_1 \dots \lambda_{j-1} \sigma \lambda_{j+1} \dots \lambda_l}$$

2-point function

$$\langle O_{\mu_1 \dots \mu_l}(x) O_{\nu_1 \dots \nu_l}(0) \rangle = \frac{1}{|x|^{2\Delta}} \left[\frac{1}{l!} (I_{\mu_1 \nu_1} \dots I_{\mu_l \nu_l} + \text{perms}) - \text{traces} \right]$$

Positive sign by unitarity

$$I_{\mu\nu} = \delta_{\mu\nu} - 2 \frac{x_\mu x_\nu}{x^2}$$

Spin l primary states:

$$\begin{aligned} M_{\mu\nu}|\Delta, l\rangle &= \Sigma_{\mu\nu}|\Delta, l\rangle, \\ iD|\Delta, l\rangle &= \Delta|\Delta, l\rangle, \\ K_{\mu}|\Delta, l\rangle &= 0 \end{aligned} \quad \Rightarrow \quad |\Delta, l\rangle = O_{\mu_1 \dots \mu_l}(0)|0\rangle$$

Descendant states:

$$P_{\nu_1} \cdots P_{\nu_n} |\Delta, l\rangle \quad (\Leftrightarrow \partial_{\nu_1} \cdots \partial_{\nu_n} O_{\mu_1 \dots \mu_l}(x))$$

Unitarity bounds:

$$\|P_{\nu_1} \cdots P_{\nu_n} |\Delta, l\rangle\|^2 > 0 \quad \Rightarrow \quad \begin{cases} \Delta \geq \frac{D}{2} - 1 & \text{for } l = 0, \\ \Delta \geq D - 2 + l & \text{for } l \geq 1 \end{cases}$$

↑
spacetime dim.

Ferrara-Gatto-Grillo, PRD9(1974)3564,
G. Mack, CMP55(1977)1

2 & 3-Point Functions

2点、3点相関関数の形はconformal symmetryより以下のように決まる

2-point function (normalization)

$$\langle \phi(x)\phi(0) \rangle = \frac{1}{|x|^{2d}}$$

$$I_{\mu\nu} = \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}$$

$$\langle O_{\mu_1 \dots \mu_l}(x) O_{\nu_1 \dots \nu_l}(0) \rangle = \frac{1}{|x|^{2\Delta}} \left[\frac{1}{l!} (I_{\mu_1 \nu_1} \dots I_{\mu_l \nu_l} + \text{perms}) - \text{traces} \right]$$

Unitarity \rightarrow positive overall sign & unitarity bounds

3-point function

$$\langle \phi(x_1)\phi(x_2)O_{\mu_1 \dots \mu_l}(0) \rangle = \frac{f_{\Delta,l}}{|x_1 - x_2|^{2d-\Delta+l}|x_1|^{\Delta-l}|x_2|^{\Delta-l}} (Z_{\mu_1} \dots Z_{\mu_l} - \text{traces})$$

$$f_{\Delta,l} : \text{structure const.} \quad Z_\mu = \frac{(x_1)_\mu}{x_1^2} - \frac{(x_2)_\mu}{x_2^2}$$

Unitarity $\rightarrow f_{\Delta,l}$ is real !

Operator Product Expansion

ここでは次の形のOPEを考える

$$\begin{aligned} \phi(x)\phi(y) &= \frac{1}{|x-y|^{2d}} + \sum_{l=2n} f_{\Delta,l} \left[\frac{(x-y)_{\mu_1} \cdots (x-y)_{\mu_l}}{|x-y|^{2d-\Delta+l}} O_{\mu_1 \cdots \mu_l}(y) + \text{descendants} \right] \\ &= \frac{1}{|x-y|^{2d}} + \sum_{l=2n} \frac{f_{\Delta,l}}{|x-y|^{2d-\Delta}} C_{\Delta,l}(x-y, \partial_y) O_{\Delta,l}(y) \\ &\quad \parallel \\ &\quad O_{\mu_1 \cdots \mu_l} \end{aligned}$$

簡単のため $l=0$ の場合のみを与える

$$\langle \phi(x)\phi(y)O(z) \rangle = \frac{f_{\Delta,0}}{|x-y|^{2d-\Delta}} C_{\Delta,0}(x-y, \partial_y) \langle O(y)O(z) \rangle$$

$$\rightarrow C_{\Delta,0}(x-y, \partial_y) \frac{1}{|y-z|^{2\Delta}} = \frac{1}{|x-z|^\Delta |y-z|^\Delta}$$

Feynmanパラメータ
積分で書き換える

$$\begin{aligned} C_{\Delta,0}(x-y, \partial_y) &= \frac{1}{B(\frac{\Delta}{2}, \frac{\Delta}{2})} \int_0^1 dt [t(1-t)]^{\frac{\Delta}{2}-1} \\ &\quad \times \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n n!} \frac{[t(1-t)a^2]^n}{(\Delta+1-D/2)_n} (\partial_y^2)^n e^{ta \cdot \partial_y} \Big|_{a=x-y} \end{aligned}$$

4-Point Functions (Conformal Blocks)

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{1}{|x_{12}|^{2d}|x_{34}|^{2d}} \left[1 + \sum_{\Delta, l} p_{\Delta, l} g_{\Delta, l}(u, v) \right]$$

$$p_{\Delta, l} = f_{\Delta, l}^2 \quad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

Conformal block for $l = 0$

$$g_{\Delta, 0}(u, v) = |x_{12}|^{\Delta} |x_{34}|^{\Delta} C_{\Delta, 0}(x_{12}, \partial_2) C_{\Delta, 0}(x_{34}, \partial_4) \frac{1}{|x_{24}|^{2\Delta}}$$
$$= u^{\frac{\Delta}{2}} \sum_{n, m=0}^{\infty} \frac{\left(\frac{\Delta}{2}\right)_n \left(\frac{\Delta}{2}\right)_n}{n! \left(\Delta + 1 - \frac{D}{2}\right)_n} \frac{\left(\frac{\Delta}{2}\right)_{n+m} \left(\frac{\Delta}{2}\right)_{n+m}}{m! (\Delta)_{2n+m}} u^n (1-v)^m$$

double series (Appell function)

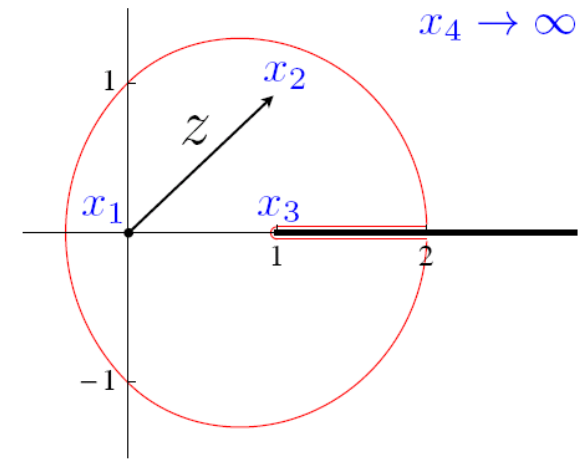
$l \neq 0$ の場合も同様に計算できる(漸化式を立てて計算 Dolan-Osborn 2001)

New variables

$$u = z\bar{z}, \quad v = (1 - z)(1 - \bar{z})$$

Gaussの超幾何級数

$$k_\beta(x) \equiv x^{\beta/2} \underline{\underline{{}_2F_1}}(\beta/2, \beta/2, \beta; x)$$



$D=\text{even} \rightarrow$ conformal blockは超幾何級数の`積`で書ける!

$$D=4 \quad g_{\Delta,l}(u, v)|_{D=4} = \frac{(-1)^l}{2^l} \frac{z\bar{z}}{z - \bar{z}} [k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - (z \leftrightarrow \bar{z})]$$

$$D=2 \quad g_{\Delta,l}(u, v)|_{D=2} = \frac{(-1)^l}{2^l} [k_{\Delta+l}(z)k_{\Delta-l}(\bar{z}) + (z \leftrightarrow \bar{z})]$$

$D=\text{odd} \rightarrow$ 一般式はまだ知られていない ($0 < z, \bar{z} < 1$)

$D=3 \quad z = \bar{z}$ の時は超幾何級数を用いて書くことが出来る

第三章

Positivityの条件

Conformal blocksが求まったからといって、物理が理解できたわけではない。臨界現象、すなわち共形場、を分類するには structure constantの情報が必要である。

Positivity (=unitarity) Conditions

2-point function → conformal dim. of primary field $O_{\Delta,l}$ ^{spin}

$$\text{Unitarity bounds} = \begin{cases} \Delta \geq \frac{D}{2} - 1 & \text{for } l = 0, \\ \Delta \geq D - 2 + l & \text{for } l \geq 1 \end{cases} \quad (\text{等式はfree field の場合})$$

3-point function → structure constant

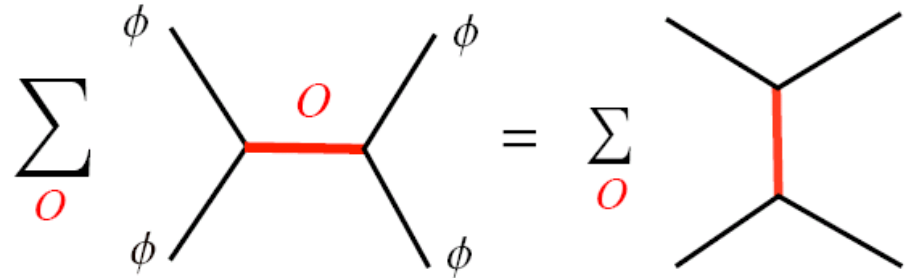
$f_{\Delta,l}$ is real

4-point function

$$p_{\Delta,l} = f_{\Delta,l}^2 \geq 0$$

この条件を新たに評価する！

Crossing symmetry



$$u^d - v^d = \sum_{\Delta, l} p_{\Delta, l} [v^d g_{\Delta, l}(u, v) - u^d g_{\Delta, l}(v, u)]$$

conformal blocks

From this relation,

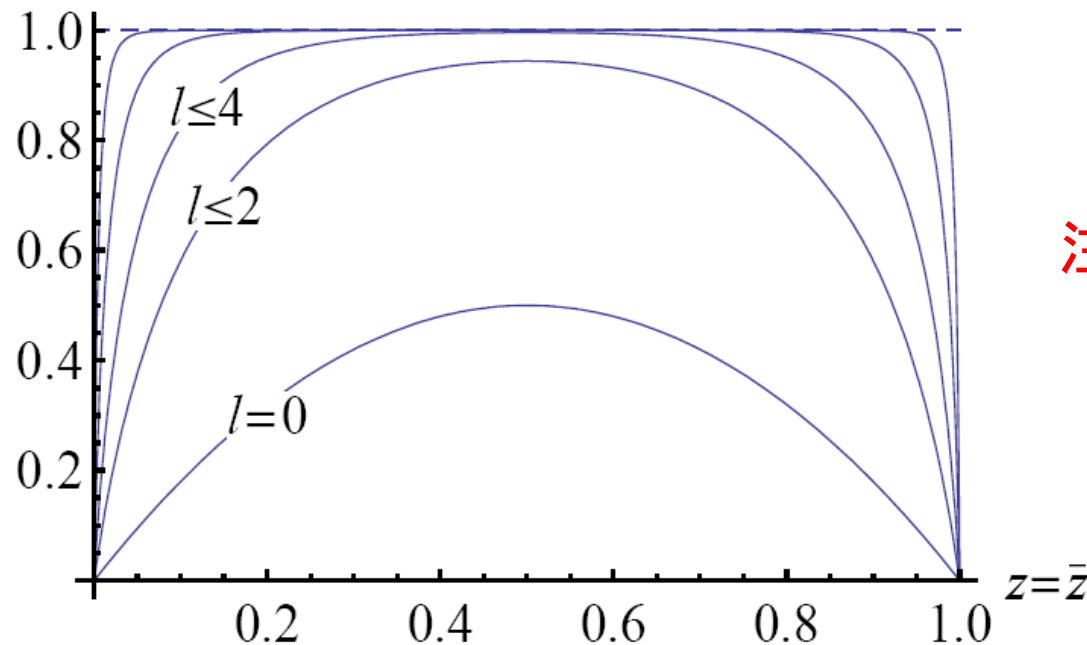
$$F_{d, \Delta, l}(z, \bar{z}) = \frac{v^d g_{\Delta, l}(u, v) - u^d g_{\Delta, l}(v, u)}{u^d - v^d}$$

$$\sum_{\Delta, l} p_{\Delta, l} F_{d, \Delta, l}(z, \bar{z}) = 1 \quad u = z\bar{z}, \quad v = (1-z)(1-\bar{z})$$

Ex: $D=4$ free scalar ϕ (conformal dim. $d = 1$)

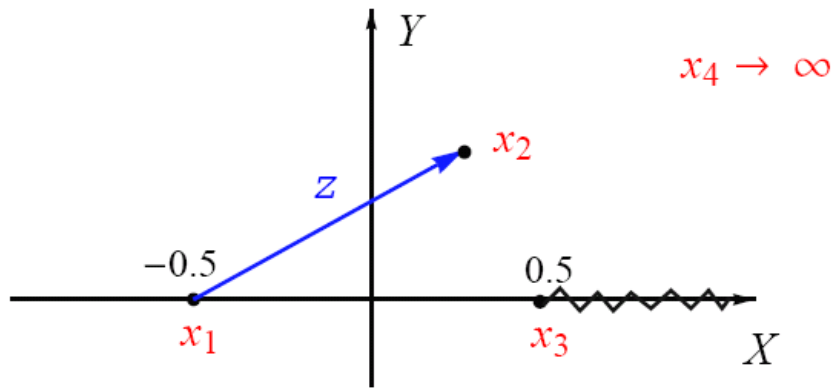
$$\phi \times \phi \Rightarrow O_{\Delta,l} \propto \phi \partial_{\mu_1} \cdots \partial_{\mu_l} \phi + \cdots \quad (\Delta = l + 2, l = \text{even})$$

Structure const.: $f_{\Delta,l} \propto \delta_{\Delta,l+2} \delta_{l,2n} \quad p_{l+2,l} = 2^{l+1} \frac{l!^2}{(2l)!}$



注) $z = \bar{z} = 1/2$
の所の収束性が
良い

check of eq. $\sum_{l=\text{even}} p_{l+2,l} F_{1,l+2,l}(z, \bar{z}) = 1$



$$z = \frac{1}{2} + X + iY, \quad \bar{z} = z^*$$

$$F(X, Y) = F(X, -Y) = F(-X, Y)$$

Consider following operation:

$$\Lambda[F] = \sum_{\substack{m,n \text{ even} \\ 2 \leq m+n \leq N}} \lambda_{m,n} \partial_X^m \partial_Y^n F|_{X=Y=0}$$

$z = \bar{z} = 1/2$ で値を評価する(収束性が良いと期待できるから)

Then, crossing symmetry means

$$\sum_{\Delta, l} p_{\Delta, l} \Lambda [F_{d, \Delta, l}] = 0$$

Primary scalar ϕ_d 同士のOPE及びそのconformal blockを考える

$$\phi_d \times \phi_d \sim 1 + \sum_{\Delta \geq f} O_{\Delta} + \sum_{\substack{l > 0 \\ l = \text{even}}} \sum_{\Delta \geq D-2+l} O_{\Delta, l}$$

ここでは中間状態に現れるlowest scalar O_{Δ} のconformal dim. Δ にunitarity boundより強い制限を与える

Strategy

d を固定して f の値を変えて、このOPEがunitarity条件

$p_{\Delta, l} \geq 0$ と矛盾するかどうかをみる

ある f 以上で矛盾するならlowest scalarの Δ の取り得る値は

$$\frac{D}{2} - 2 \leq \Delta \leq f \quad (\text{上限が求まる})$$

Lowestが決まれば次はnext lowest scalar, higher spinへ

Positivityの判定条件

(d, f は固定)

不等式で書かれた条件式の集合 B を考える

$$B = \left\{ \Lambda [F_{d,\Delta,l}] \geq 0 \quad \text{for all } \Delta \geq f \ (l = 0) \right. \\ \left. \text{and for all } \Delta \geq D - 2 + l \ (l > 0) \right\}$$

この無限個の不等式系を満たす解 ($\lambda_{n,m}$) が存在するなら

[linear programmingの応用]

有限個

$$\rightarrow \sum_{\Delta,l} p_{\Delta,l} \Lambda [F_{d,\Delta,l}] \neq 0 \quad \text{because } p_{\Delta,l} \geq 0 \quad (* \text{ trivialな場合は除く})$$

→ この d, f の組み合わせはpositivityを満たさない

→ **forbidden !**

逆に、不等式を満たす解が無ければallowed !

(* 解の値そのものは必要ない)

Linear programming(線形計画法)

線形計画法(Linear Programming)の簡単な例 (変数が2つ、制限を課す不等式が5つの場合)

```
> with(Optimization):
```

```
> F1:=x+2*y: F2:=x+3*y+1: F3:=-2*x-y+2: F4:=2*x-3*y-3: F5:=-2*x-y+2:
```

ここでは、ある関数 F を5つの拘束条件 $B = \{ F1 \geq 0, \dots, F5 \geq 0 \}$ の下で極小化すること考える。

```
> B:={F1>=0, F2>=0, F3>=0, F4>=0, F5>=0};
```

```
B := [0 ≤ x + 2 y, 0 ≤ x + 3 y + 1, 0 ≤ 2 x - 3 y - 3, 0 ≤ -2 x - y + 2]
```

```
> F:=x+y; LPSolve(F,B);
```

```
F := x + y
```

```
[0.42857142857143, [x = 0.857142857142857094, y = -0.428571428571428380]]
```

注) 極小化すべき関数がない場合は定数と置けばよい。例えば“ $F=0$ ”を代入。

この結果は不等式系を満たす変数 x, y が存在するかどうかの判定を与えていることになる。

最後の不等式を少し変形してみる。

```
> G5:=-2*x-y-2: B2:={F1>=0, F2>=0, F3>=0, F4>=0, G5>=0}; LPSolve(F,B2);
```

```
B2 := [0 ≤ x + 2 y, 0 ≤ x + 3 y + 1, 0 ≤ 2 x - 3 y - 3, 0 ≤ -2 x - y + 2, 0 ≤ -2 x - y - 2]
```

```
Error, (in Optimization:-LPSolve) no feasible solution found
```

Errorが出る。このときは不等式系を満たす解がない。

実際の計算

不等式系 B の有限化

$$B = \left\{ \Lambda[F_{d,\Delta,l}] \geq 0 \quad \text{for all } \Delta \geq f \ (l = 0) \right. \\ \left. \text{and for all } \Delta \geq D - 2 + l \ (l > 0) \right\}$$

$$\left\{ \begin{array}{l} \text{上限:} \quad l \leq l_{\max} \quad \Delta \leq \Delta_{\max} \text{ for all } l \\ \text{離散化:} \quad \Delta \rightarrow \Delta_j \ (\Delta_{j+1} = \Delta_j + \delta, \ \Delta_0 = D - 2 + l) \end{array} \right.$$
$$\delta = o(0.01) \quad l_{\max} = o(50) \quad \Delta_{\max} = o(50)$$

不等式の数 $o(10^4)$

未知数 $\lambda_{n,m}$ の数 $N(N+6)/8 = o(100)$ (N が大きいほど制限が強い)

$$\Lambda[F] = \sum_{\substack{m,n \text{ even} \\ 2 \leq m+n \leq N}} \lambda_{m,n} \partial_X^m \partial_Y^n F|_{X=Y=0}$$

第四章

計算結果

$D=4$: lowest scalarの共形次元に制限を与える(あまり面白くない)

$D=2$: CFTの厳密解(2D Ising)と比較する

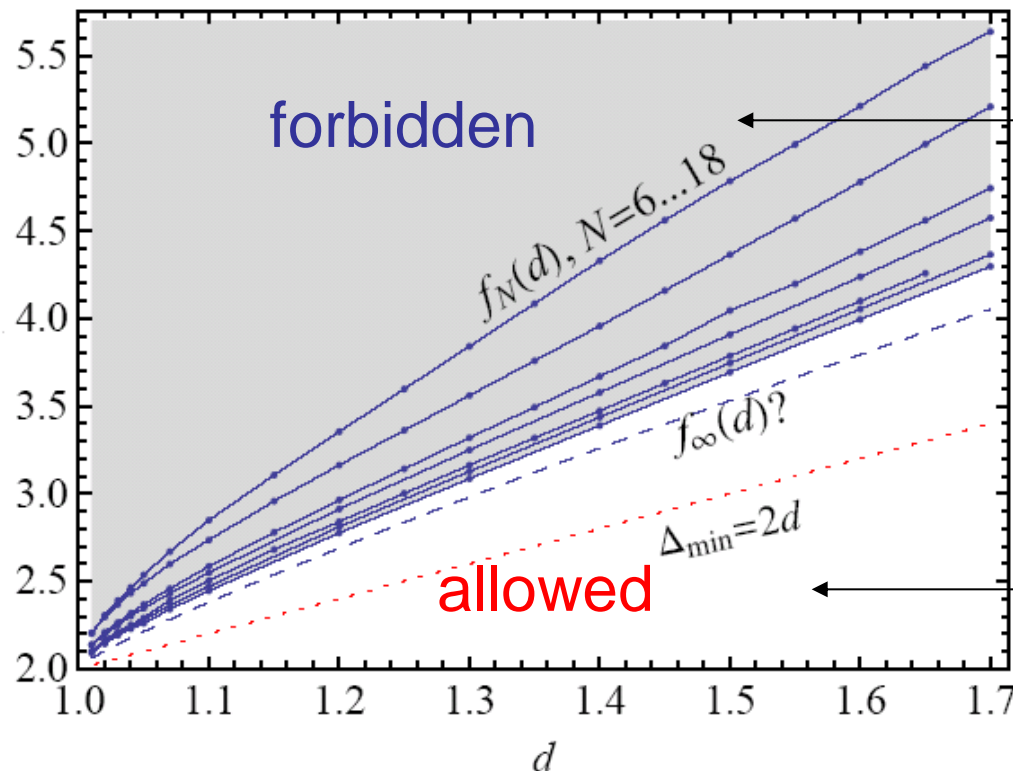
$D=3$: 3D Isingのcritical exponentsの計算

D=4 Results

Lowest scalar conformal dim. の上限

$$\phi_d \times \phi_d = \mathbb{1} + O_\Delta + \dots$$

$$\Delta \leq 2 + 0.7(d-1)^{1/2} + 2.1(d-1) + 0.43(d-1)^{3/2}$$



LP問題の解あり
→ positivityと矛盾

LP問題の解なし
→ positivityと矛盾しない

D=2 Results

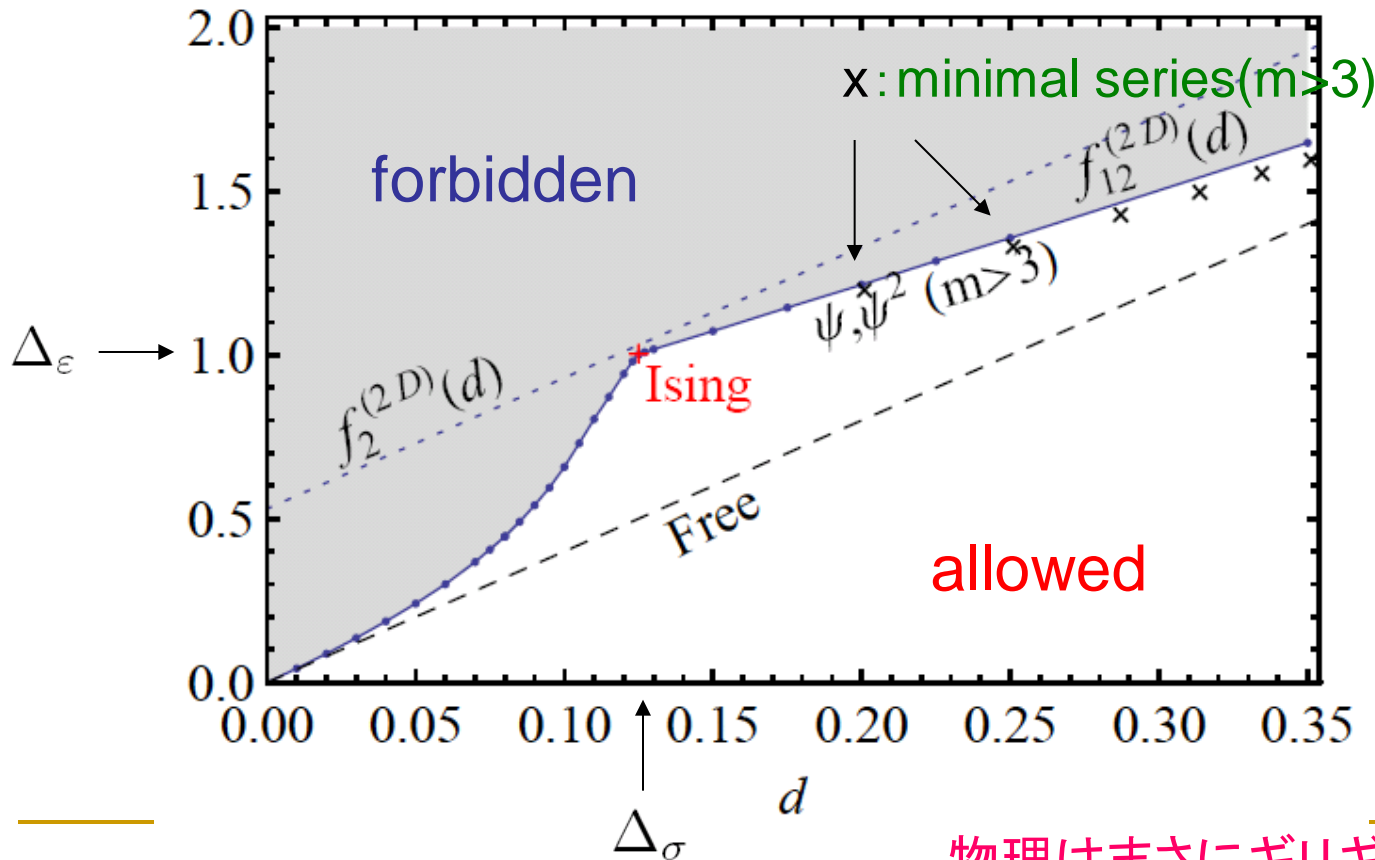
2DCFT厳密解との比較

$$\Delta_\sigma = 1/8 = 0.125, \quad \Delta_\varepsilon = 1 \quad (\text{Ising})$$

$$\sigma \times \sigma = 1 + \varepsilon$$

Belavin-Polyakov-Zamolodchikov, NPB241(1984)333

Friedan-Qiu-Shenker, PRL52(1984)1575



$$c = 1 - \frac{6}{m(m+1)}$$

$$\Delta_\sigma = \frac{1}{2} - \frac{3}{2(m+1)}$$

$$\Delta_\varepsilon = 2 - \frac{4}{m+1}$$

(m=3 Ising)

物理はまさにギリギリの所に現れる

Loest scalarの Δ をallowed valueのmaxに固定して、
 next lowest scalarの Δ' に付いてのallowed regionを探す

$$\phi_d \times \phi_d \sim 1 + O_{\Delta} + \sum_{\Delta' \geq f} O_{\Delta'} + \sum_{\substack{l > 0 \\ l = \text{even}}} \sum_{\Delta \geq D-2+l} O_{\Delta, l}$$

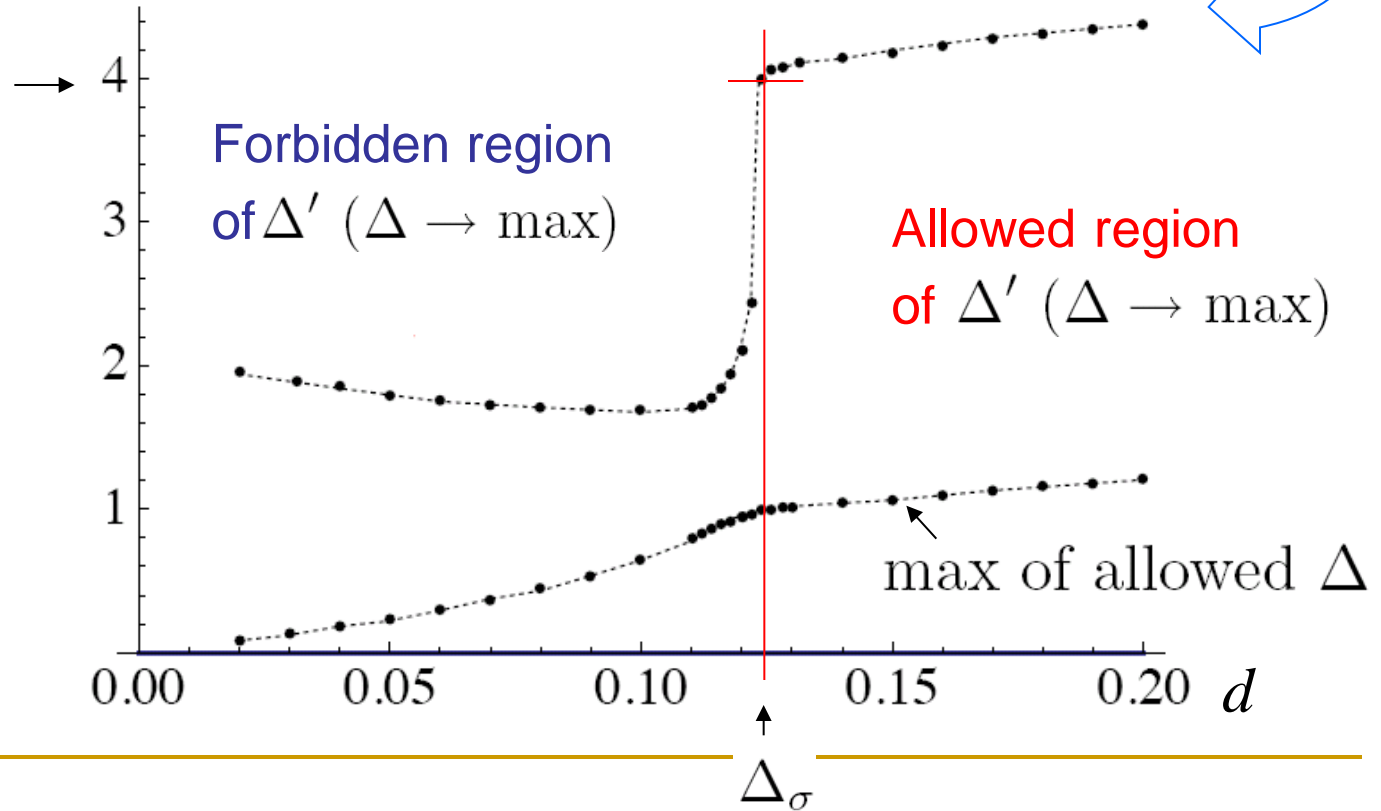
↑
fix

2D Ising

$$\Delta_{\varepsilon'} = 4$$

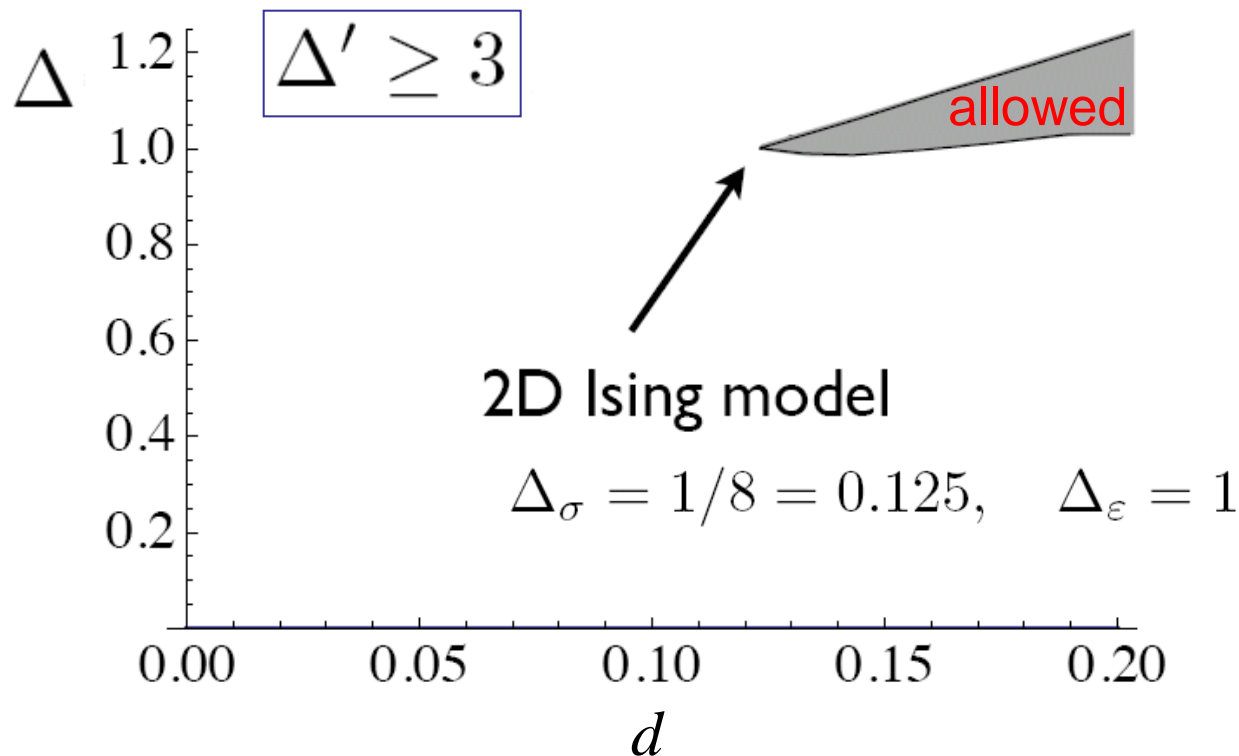
$$\varepsilon' = L_{-2} \bar{L}_{-2} \cdot 1$$

(irrelevant op.)



この計算はLowest Δ とnext Δ' の間にgapがあることを想定している

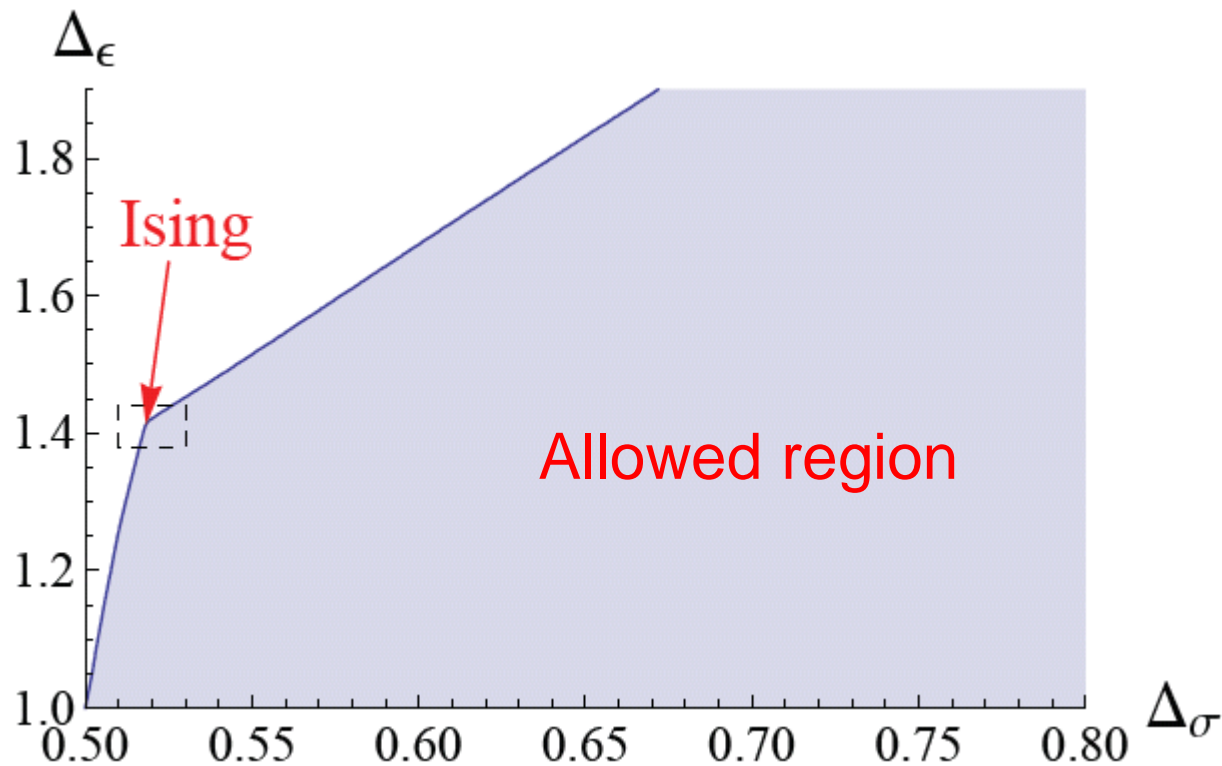
そこで、 $\Delta' \geq 3$ の制限を課して再度lowest scalar Δ の allowed regionを探す



$D=3$ Results

同様の計算を $D=3$ で行うと

3D Ising $\Delta_\sigma = 0.5182(3)$, $\Delta_\epsilon = 1.413(1)$
(Lattice Monte-Carlo simulation)



次に、lowest scalar Δ_ε とnext lowest scalar $\Delta_{\varepsilon'}$ の間に gapがあることを考慮に入れて計算する

Operator	Spin l	\mathbb{Z}_2	Δ	Exponent
σ	0	-	0.5182(3)	$\Delta = 1/2 + \eta/2$
σ'	0	-	$\gtrsim 4.5$	$\Delta = 3 + \omega_A$
ε	0	+	1.413(1)	$\Delta = 3 - 1/\nu$
ε'	0	+	3.84(4)	$\Delta = 3 + \omega$
ε''	0	+	4.67(11)	$\Delta = 3 + \omega_2$
$T_{\mu\nu}$	2	+	3	n/a
$C_{\mu\nu\kappa\lambda}$	4	+	5.0208(12)	$\Delta = 3 + \omega_{NR}$

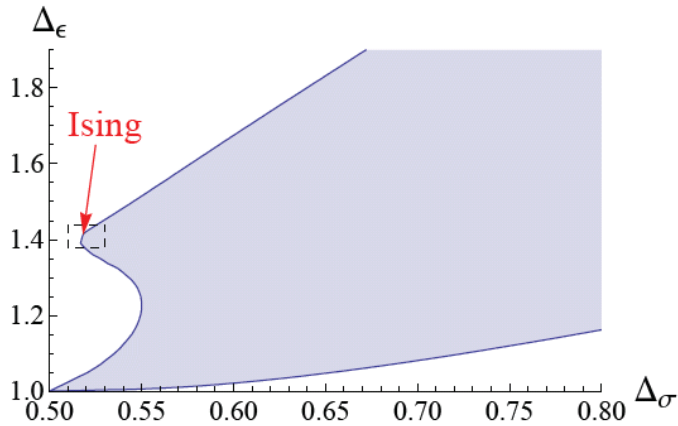
これに注目

Lattice Monte-Carlo simulation

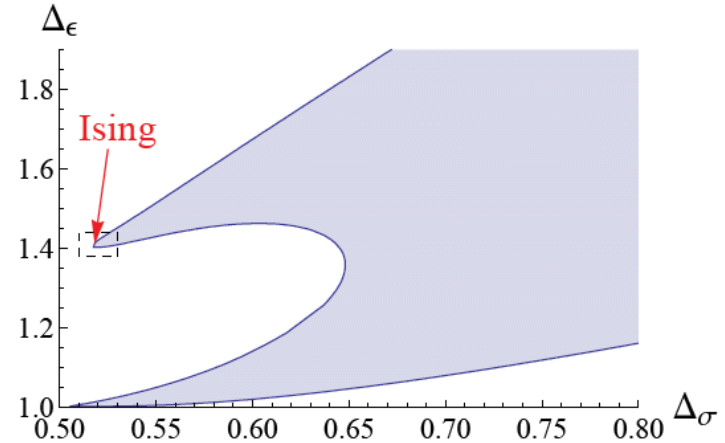
$$\begin{aligned}
 \sigma \times \sigma &= 1 + (\varepsilon + \varepsilon' + \dots) \\
 &\quad + (T_{\mu\nu} + \dots) \\
 &\quad + (C_{\mu\nu\lambda\sigma} + \dots) \\
 &\quad + \dots
 \end{aligned}
 \qquad
 \sigma \times \varepsilon = \sigma + \sigma' + \dots$$

Next lowestに制限 ($\Delta_{\epsilon'} \geq 3, 3.4, 3.8$) を加えて ($\Delta_{\sigma}, \Delta_{\epsilon}$) の allowed region を計算

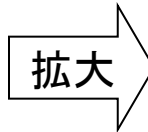
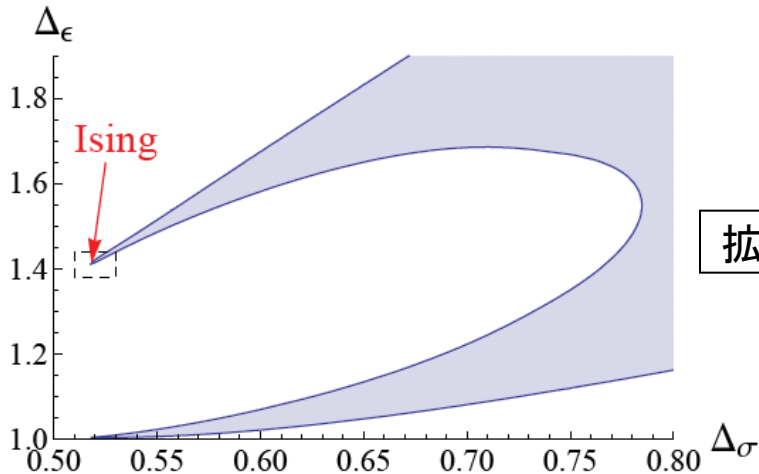
Allowed Region Assuming $\Delta(\epsilon') \geq 3$



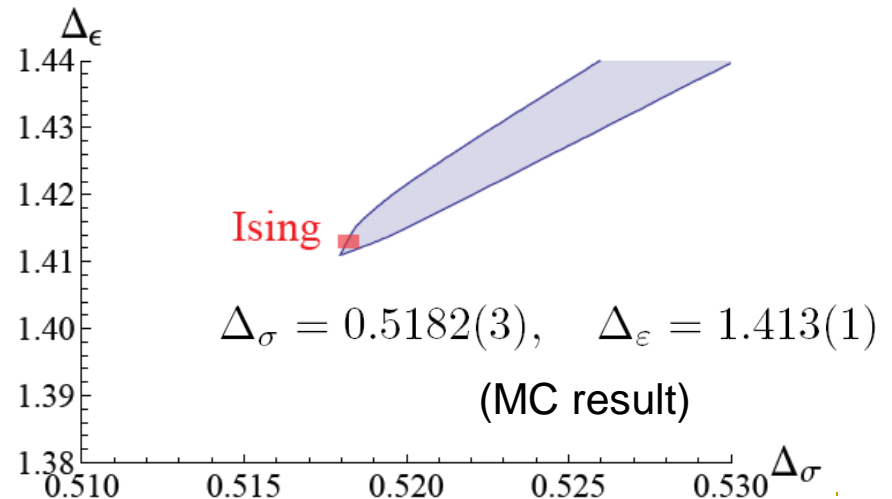
Allowed Region Assuming $\Delta(\epsilon') \geq 3.4$



Allowed Region Assuming $\Delta(\epsilon') \geq 3.8$



(Zoomed) Allowed Region Assuming $\Delta(\epsilon') \geq 3.8$



最終章

まとめ

- ◆ $D > 2$ のconformal blocksの解析的な一般式の発見(2001)
- ◆ Unitarity条件(structure constantが実数)を課すと共形次元に制限が付く
- ◆ (準)解析的な方法で3D Isingモデルの臨界指数を調べた

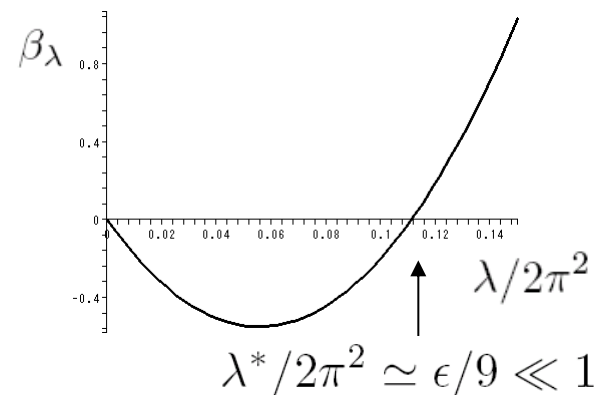
$$\Delta_\sigma = 0.5182(3), \quad \Delta_\epsilon = 1.413(1)$$

これまでの場の理論的方法との比較

Wilson-Fisher 1972 (ϵ expansion)

$$S = \int d^D x \left[\frac{1}{2} (\partial\phi)^2 + \lambda\phi^4 \right] \quad D = 4 - \epsilon$$

$$\beta_\lambda = -\epsilon\lambda + 9\lambda^2/2\pi^2$$



At fixed point $\left\{ \begin{array}{l} \Delta_\phi = (1 - \epsilon/2) + \epsilon^2/108 \rightarrow 0.51 \\ \Delta_{\phi^2} = (2 - \epsilon) + \epsilon/3 \rightarrow 1.33 \end{array} \right.$

$\epsilon \rightarrow 1$
($D \rightarrow 3$)

良く合うけれど、正しさに疑問がある！

$$\phi \times \phi \sim \phi^2$$

$\nwarrow \quad \nearrow$
 $\sigma \quad \epsilon$

$$(\Delta_\phi = 0.5180, \Delta_{\phi^2} = 1.4102 \text{ at } o(\epsilon^5))$$

CFTの発展

1. 1909 Cunningham, Bateman: conformal inv. を物理に導入
2. 1970 Polyakov: conformal inv. が critical point で現れることを指摘
3. 1973 Ferrara, Grillo, Gatto: conformal bootstrap の始まり
4. 1984 Belavin, Polyakov, Zamolodchikov: $D=2$ CFT の厳密解
5. 2001 Dolan, Osborn: conformal blocks の一般形の発見

今日紹介した仕事:

$D > 2$ の臨界指数が CFT から推察された最初の例 !

付録

(cf. $O_{\mu_1 \dots \mu_l}^\dagger(x) = O_{\mu_1 \dots \mu_l}(x)$ on Minkowski)

Hermite性 (Euclid CFT)

$$O_{\mu_1 \dots \mu_l}^\dagger(x) = \frac{1}{(x^2)^\Delta} I_{\mu_1 \nu_1} \cdots I_{\mu_l \nu_l} O_{\nu_1 \dots \nu_l}(Rx)$$

Inversion: $x_\mu \rightarrow Rx_\mu = \frac{x_\mu}{x^2}$ $I_{\mu\nu} = \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}$

Out-state

$$\begin{aligned} \langle \{\mu_1 \cdots \mu_l\}, \Delta | &= \langle 0 | O_{\mu_1 \dots \mu_l}^\dagger(0) \\ &= \lim_{y^2 \rightarrow \infty} (y^2)^\Delta I_{\mu_1 \nu_1} \cdots I_{\mu_l \nu_l} \langle 0 | O_{\nu_1 \dots \nu_l}(y) \end{aligned}$$

$$\langle \{\mu_1 \cdots \mu_l\}, \Delta | P_\mu = 0$$

$$K_\mu | \{\mu_1 \cdots \mu_l\}, \Delta \rangle = 0$$

$$P_\mu^\dagger = K_\mu$$

2-point function

$$\langle O(x)O(x') \rangle = \frac{1}{(x^2)^\Delta} \langle O^\dagger(Rx)O(x') \rangle = \frac{1}{(x^2)^\Delta} \langle \Delta | e^{-iK_\mu(Rx)_\mu} e^{iP_\nu x'_\nu} | \Delta \rangle$$

where

$$O(x) = e^{iP_\mu x_\mu} O(0) e^{-iP_\mu x_\mu} \quad O^\dagger(x) = e^{iK_\mu x_\mu} O(\infty) e^{-iK_\mu x_\mu}$$

$$P_\mu^\dagger = K_\mu \quad \langle \Delta | = \langle 0 | O(\infty) \quad | \Delta \rangle = O(0) | 0 \rangle$$

From this,

$$\langle O(x)O(x') \rangle = \frac{1}{(x^2)^\Delta} \sum_{n=0}^{\infty} C_n^\Delta \left(\frac{x \cdot x'}{\sqrt{x^2 x'^2}} \right) \left(\frac{x'^2}{x^2} \right)^{n/2} = \frac{1}{(x - x')^{2\Delta}}$$

Gegenbauer polynomial

$$C_n^\Delta = \frac{1}{(n!)^2} \frac{x_{\mu_1} \cdots x_{\mu_n} x'_{\nu_1} \cdots x'_{\nu_n}}{(x^2 x'^2)^{n/2}} \langle \Delta | K_{\mu_1} \cdots K_{\mu_n} P_{\nu_1} \cdots P_{\nu_n} | \Delta \rangle$$

$$nC_n^\Delta = 2(\Delta + n - 1)zC_{n-1}^\Delta - (2\Delta + n - 2)C_{n-2}^\Delta$$

$$z = x \cdot x' / \sqrt{x^2 x'^2} \quad \frac{1}{(1 - 2zt + t^2)^\Delta} = \sum_{n=0}^{\infty} C_n^\Delta(z) t^n$$