Into A World Beyond The Planck Scale That Nobody Knows

Introduction to Renormalizable Quantum Gravity
with Asymptotic Background Freedom

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Preface

The various Planck scales proposed by Planck at the end of the 19th century are fundamental units that characterize a tiny world where quantization of gravity is necessary. The Planck length $l_{\rm pl} = 1.616 \times 10^{-35}\,{\rm m}$ is known as the smallest unit of length. Many physicists believe that there is no world shorter than it, or that the world is governed by a physical law different from diffeomorphism invariance. Putting a cutoff in the Planck length and thinking of it as an entity of quantized spacetime shows that. However, this way of thinking is exactly the cause of many problems in the theory of gravity. It is simply to escape from the problem of spacetime singularity. Essential problems such as renormalizability, unitarity, origin of primordial fluctuations, time, and so on ultimately come down to the problem of how to describe the trans-Planckian world as a continuum quantum field theory.

The root of the problem lies in the basic idea that gravity is distortion of spacetime caused by the presence of matter. The Einstein equation is a concrete expression of that. It is a fundamental equation of the universe that holds in a very wide range from a macroscopic scale beyond the Hubble distance, which is the largest known scale in the universe, to the vicinity of the Planck length. No other equation has such a wide range of applications. Despite being such an excellent equation, the existence of spacetime singularities cannot be denied, and the laws of physics break down there. The cause is not in diffeomorphism invariance introduced as a guiding principle, but in the Einstein equation. The reason why we have to introduce a wall at the Planck scale is that the theory has not been formulated properly so that diffeomorphism invariance holds in the world beyond the Planck scale.

In order to resolve the problem of singularities, it is necessary to reconsider the relationship between matter and gravity. They are not standing on the same ground. That is, to reconfirm that matter is defined in spacetime, whereas gravity defines spacetime itself. Matter is normally described as a particle that propagates in space, and gravity is also described as a particle, graviton, if its fluctuation is small. However, if the fluctuation becomes so large that time and distance become uncertain, the particle picture propagating in space is no longer justified. As we approach the Planck scale, you will encounter such a world in which the substance of time and space is lost. The words such as “background-metric independence”, “background independence”, or “background freedom” often used in the field of quantum gravity research represent such a state. Including this quantum state, an equation describing gravitational states in general is what is called the Hamiltonian constraint, representing that whole Hamiltonian vanishes as a consequence of diffeomorphism invariance.

Overcoming the Planck scale wall requires a quantization method of gravity that is not only renormalizable, but also achieves the background-metric independence. It can never be realized in ordinary quantum gravity theories in which graviton is considered as a basic element. Therefore, it is necessary to introduce some non-perturbative field theory method. One of the topics of this book is to introduce a method of describing the background-free world with a special conformal field theory (CFT) in which conformal invariance arises as a quantum symmetry of diffeomorphism invariance. In this theory, the roles of gravity and matter are no longer equal, and there is a purely gravitational excited state, which can act as a source of everything.

Quantum spacetime with background freedom is far from the spacetime you usually imagine. This implies that if such a quantum spacetime exists, then there has to be a transition from such a spacetime to the present classical spacetime. The moment is called the “spacetime phase transition”. A renormalizable quantum field theory of gravity discussed in this book indicates the existence of a physical energy scale where such a phase transition occurs. In the higher energy region than that, a background-free world expressed by the CFT will be realized, while below that, will settle to a classical spacetime described by Einstein’s theory of gravity in which conformal invariance is completely broken. I argue that the moment of this change, which must have occurred in the early universe, is the true picture of the Big Bang. The continuum quantum field theory with these properties is called “asymptotically background-free quantum gravity”.

This book is an easy-to-understand summary of the achievements in quantum gravity research based on field theory for half a century. Why quantization of gravity is necessary and what physical phenomena have to be explained? Why is there a non-trivial state, or entropy, even though the Hamiltonian has zero eigenvalue? Including these descriptions, I tried to explain the theory in plain language without using mathematical formulas other than symbolic ones as much as possible. Fortunately, there are so many descriptions about gravity that resistance to words may be less than in other research fields. The words themselves can be easily understood by searching online, but here I am aiming to help you understand the meaning behind the words. I would like to clarify the essence of the problem by stepping into the area of physics that has been avoided. If you want to know mathematical details of the contents, please refer to my professional book below and research papers listed at the end. This book is written for the general public, but is also a unique specialized book that discuss the world of the Planck scale. There are some technical and difficult parts, but it is written so that you can understand the whole picture even if you skip them.

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Chapter 1

What is The Planck Scale?

Every physical phenomenon has an inherent scale that describes its dynamics. First of all, let us briefly summarize the largest to smallest scales that exist in the universe.

The largest scale you can know is the Hubble distance, which is about 4000 Megaparsecs (Mpc). 1 parsec (pc) is 3.26 light-years and Mega (M) is a Million (10^6) times. Since one light-year is 9.5×10^{15} m, the Hubble distance is about 10^{26} m. This corresponds to the size of the universe that can be observed today, and the existence of a scale larger than this cannot be confirmed.

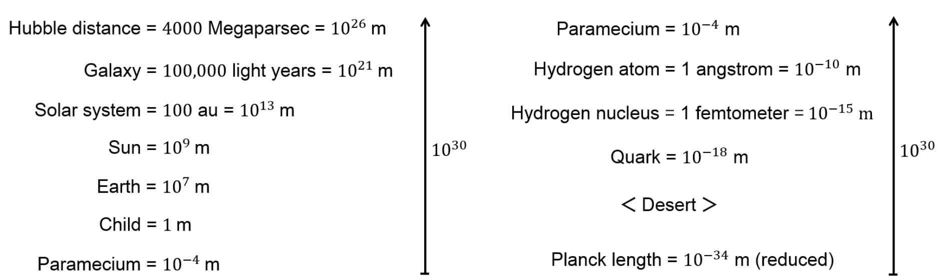

Figure 1-1: Various scales that exist in the universe from maximum to minimum.

The next largest scale is the size of a cluster of galaxies, which is $4 \sim 5$ Mpc. They form a further group called a super cluster of galaxies, which is $10 \sim 30$ Mpc. The size of the galaxy itself is about 100,000 light-years, or 10^{21} m. Continuing further, the size of the solar system is 100 astronomical units, or 10^{13} m, the sun is 10^9 m, the earth is 10^7 m, a child is 1 m, and the size of a paramecium is 10^{-4} m. In other words, the Hubble distance is about 10^{30} times the size of a paramecium.

At shorter distances, familiar scales in particle physics appear. The size of a hydrogen atom is about 1 angstrom (1Å= 0.1 nm = 10^{-10} m), and its nucleus is about 1 femtometer (1 fm = 10^{-15} m), and the size of a quark that makes up the nucleus is estimated to be 10^{-18} m. Quarks are one of the smallest elementary particles introduced by Gell-Mann that make up matter, and it is known that there are six types in total. No smaller components are known so far.

What is known as a much smaller scale than these is the Planck length $l_{\rm pl}$. It is a length scale composed of three fundamental constants as follows:

$$l_{\rm pl} = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35}\,\text{m},$$

where G is the Newton constant that appears in Einstein's theory of gravity, which represents the strength of the gravitational interaction, $\hbar$ is the Planck constant, which appears when describing a quantum world, and c is the speed of light, which is a fundamental constant of the theory of relativity. The Planck length is thought to be a scale related to a quantum theory of gravity, as can be inferred from the inclusion of $\hbar$ in the definition.

A mass scale can also be obtained by changing the combination. It is called the Planck mass, given by

$$m_{\rm pl} = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-5}\,\text{g}.$$

Furthermore, multiplying the Planck mass by the square of the speed of light gives the Planck scale with an energy dimension. Also, when describing gravitational equations of motion, the reduced Planck length $L_{\rm P} = \sqrt{8\pi}\, l_{\rm pl} = 0.81 \times 10^{-34}$ m and the reduced Planck mass $M_{\rm P} = m_{\rm pl}/\sqrt{8\pi} = 4.3 \times 10^{-6}$ g, which differ only by a factor from the above, are often used.

The size of matter is, in general, inversely proportional to its inherent mass, though it depends on whether the matter itself is a fundamental element or has an internal structure. It can be easily understood from the fact that according to Einstein's famous formula $E = mc^2$, mass m is equivalent to energy E, and multiplying the reciprocal of energy by $\hbar$ gives the dimension of length. That is, the higher the energy, the smaller the scale appears.

For example, it is known that there are two important energy scales that govern microscopic worlds where quarks appear. One is the scale of electroweak theory proposed by Glashow, Weinberg, and Salam independently that describes weak and electromagnetic interactions in a unified manner, and the other is of quantum chromodynamics (QCD) that describe strong interactions. The former is about 100 GeV, where 1 eV is energy gained when an electron is accelerated by 1 V, and G (Giga) is a symbol that denotes 1 billion (10^9) times. The length given by the reciprocal of this scale is on the order of quark size 10^{-18} m. The QCD energy scale is about 200 MeV, and the corresponding length is about 1 fm, which represents the size of the nucleus.

The Planck energy that characterizes a quantum gravity world is approximately 10^{19} GeV. This is a tremendous number of 10^{17} times the energy you can see quarks. Particle physicists call this 17-digit energy region where the existence of a special scale is unknown, "the desert." Of course, it is up to researchers whether or not the existence of such an area seems unnatural. A widely known candidate for a scale located in the desert is the grand unified theory (GUT) energy scale, which is thought to exist near 10^{16} GeV if the theory is established. Apart from that, it is indicated that quantum gravity also has a new dynamical energy scale different from the Planck scale in the desert area, which will be clarified step by step in later chapters.

There is a difference of about 10^{30} between the Planck length and the size of a paramecium, where the reduced one is used to get a nice digit. There is also a difference of about 10^{30} between a paramecium and the Hubble distance that is the largest scale in the universe. From these comparisons, you can see that the Planck length is tremendously small. And, this indicates that there is a difference of about 10^{60} between the maximum and minimum scales in the universe as follows:

$$\frac{\text{Hubble distance } c/H_0}{\text{Planck length } L_\mathrm{P}} \simeq 10^{60}, \tag{1-1}$$

whose number of digits is called "Nayuta". One of the purposes of this book is to connect these two extreme scales with the theory of gravity, including quantum gravity.

Now, let us consider the main subject, what happens when quantizing gravity. It is well known that when quantization is performed, discrete structures appear as energy levels and so on. In the case of quantum gravity, some researchers think that spacetime may become discrete in units of the Planck length, and others think that quantum excitations of gravity appear discretely in units of the Planck mass. However, such a simple view has serious problems. If spacetime is discontinuous, diffeomorphism invariance, which is the guiding principle that defines the theory of gravity, is broken. Also, as explained in the next chapter, if there is an elementary excitation with mass exceeding the Planck mass, it is nothing but a black hole in Einstein's theory of gravity.

Ignoring these difficulties for the time being, let us see what happens when trying to quantize Einstein's theory of gravity. Since quantization methods for ordinary quantum field theory are defined in flat spacetime with the Minkowski metric, first consider a small fluctuation of the gravitational field around flat spacetime, that is, a gravitational wave, and quantize it. The quantized fluctuation is regarded as one of elementary particles and is called a "graviton". However, a new problem arises here, which is that in order for physical quantities to be calculated as finite quantities, quantum field theories must be renormalizable, but this theory is not so. In other words, when quantum corrections are calculated, new types of ultraviolet divergences appear one after another and cannot be systematically removed, so it is not possible to derive physical quantities that are meaningful as quantum theory.

In the quantization employing such weak-field approximation, an ultraviolet cutoff is introduced to impose a restriction so as not to enter a high energy region above the Planck energy and to make the theory finite. That is, introduce a discrete structure that there is no distance shorter than the Planck length, and think that such a structure is an entity of quantum spacetime. However, the weak-field approximation cannot deal with the essential problems of gravity, and one of which is manifested as unrenormalizability.

This is not the only incompatibility between Einstein's theory of gravity and quantum theory. The fact that the Einstein-Hilbert action defined using the scalar curvature is not positive-definite is also a problem. If you are thinking of small fluctuations around flat spacetime, that is not a problem, but as gravitational fluctuations increase, instability comes out. Moreover, as long as using this action, the problem of spacetime singularity can never be solved even when going to quantum theory. The reason will be explained in detail in Chapter 3.

Only this theoretical aspect is not the difficulty in quantizing gravity. Complicated calculations are also a factor in the difficulty. Einstein's theory of gravity is a beautiful theory, but since the gravitational field is a tensor field, it is much more difficult to handle than vector fields or scalar fields. Furthermore, while known matter fields have momentum dimensions, the gravitational field is basically a dimensionless field. Therefore, renormalizable interactions of matter fields are represented by at most up to four-point functions of the field, whereas the gravitational interactions obtained by expanding in the weak-field approximation become multi-point functions that lasts infinitely. In the old days, many researchers were buried in calculations, exhausted, and often ended up just doing calculations. No way to solve the problems was found, and then the quantum field theory method became obsolete.

First of all, it is important to construct a systematically computable quantum theory of gravity. Most of the proposed quantum gravity theories are based on the thinking that it is sufficient only to discuss the outside avoiding invisible or difficult-to-see places. It is a way of thinking like the scattering matrix theory that was popular in the 1960s, which is an algebraic method in order to understand strong interactions using scattering matrices and current algebras without describing the inside of hadrons. The scattering matrix theory is still useful and mathematically beautiful theory using complex analysis, and attracts many researchers. However, in the end, it was replaced by QCD, in which fields are introduced and the structure of hadrons is directly revealed.

The subject of this book is to present a new method of introducing "fields" into the world beyond the Planck scale in order to overcome the problems with quantization of gravity. Here, the essence of the theory is diffeomorphism invariance, and the most important is to preserve this invariance to the last. For this purpose, it is necessary to perform quantization while maintaining continuity, or at least to be able to restore the continuity in an appropriate ultraviolet limit. It implies thinking of a renormalizable theory. As a result, the theory undergoes major modifications near the Planck scale. In the sequent chapters, the problems with singularity, renormalizability, unitarity, excitation, inflation, cosmological constant, etc. will be reconsidered in the modified theory, and how they are solved will be shown.

From the next chapter, scales will be expressed in natural units normalized as $c = \hbar = 1$. Using this unit system, mass and energy are expressed in the same unit, as can be seen by setting $c = 1$ in Einstein's famous equation $E = mc^2$. Also, length and time are in the same unit, which is the reciprocal of energy. Hereafter, the magnitude of mass and energy will be expressed mainly using GeV. In Appendix A, various fundamental constants are summarized and useful constants for converting to natural units are presented.

Chapter 2

Why Quantum Gravity is Necessary?

One of places where quantum gravity theory is needed is near the center of black holes. It is believed that quantization of gravity is necessary to solve the singularity problem. The beginning of the universe is also so. The universe is not stationary and is thought to have been expanding since its inception. In other words, going back to the past, it is inferred that the universe is in a state of high temperature and high density, and eventually reaches the region where quantum gravity is required. Here, theoretical and phenomenological reasons why quantum gravity is required in these regions will be explained more specifically.

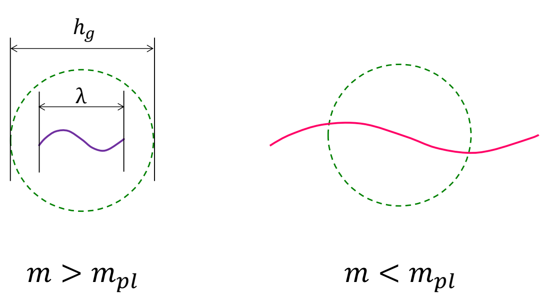

Figure 2-1: The Compton wavelength of a particle with mass m is given by $\lambda \sim 1/m$. On the other hand, the horizon size (dotted line) of the particle is given by $h_g \sim m/m_{\rm pl}^2$. If $m > m_{\rm pl}$, as shown on the right, $\lambda < h_g$, thus information of the particle is trapped inside the horizon and lost. In this way, in a world beyond the Planck scale, the normal particle picture collapses.

First of all, it should be pointed out that the elementary particle picture represented by an ideal point without spread is a concept that contradicts Einstein's theory of gravity (see Fig. 2-1). It is because such an object is a black hole in the standpoint of this theory. However, if mass of a particle is smaller than the Planck mass $m_{\rm pl} \,(= 1/\sqrt{G}) = 1.221 \times 10^{19}\,\text{GeV}$, the Compton wavelength (inverse of mass), which is a measure of position uncertainty, is longer than the horizon size created by the mass itself, hence it can be regarded as a particle approximately. On the other hand, if there is a particle with mass exceeding the Planck mass, the horizon size exceeds the Compton wavelength, and information of the particle cannot be seen from the outside. It is nothing but a black hole.

Since all known elementary particle masses are sufficiently smaller than the Planck mass, it is not necessary to consider quantum effects of gravity. However, quantum gravity is thought to cause an excitation in units of the Planck mass. How will it be described? This is not only an elementary excitation problem, but also a black hole problem.

The basic equation that defines Einstein's theory of gravity is the Einstein equation. Let the gravitational field, or the metric tensor field, be $g_{\mu\nu}$, the Ricci tensor be $R_{\mu\nu}$, the scalar curvature be R, and the energy-momentum tensor of matter parts be $T^{\rm M}_{\mu\nu}$, then it is usually expressed as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\, T^{\rm M}_{\mu\nu}. \tag{2-1}$$

This equation expresses that the presence of matter causes spacetime distortion. Conversely, it is just saying that in order to eliminate singularities where curvature becomes infinite, the matter itself has to disappear.

Turning to the early universe, this fact also affects a question of how the present matter universe was created. It is believed that matters that make up the universe were produced when the Big Bang occurred. At that time, something to be a source of matters is needed. Within the framework of Einstein's theory of gravity, it also has to be a matter. In other words, an unknown matter field that creates everything has to be introduced. Although the Einstein equation is an excellent equation that describes spectacular time evolution of the universe spanning tens of digits from the moment of the Big Bang to the present, it also shows limits of Einstein's gravity theory.

A deeper look into the early universe reveals needs for quantum gravity theory more realistically. Major progresses in cosmology today are due to the fact that fluctuations (anisotropies) of the cosmic microwave background (CMB) radiation can be measured with high accuracy by astronomical satellites, the Cosmic Background Explorer (COBE) launched from NASA's Kennedy Space Center in 1989, the Wilkinson Microwave Anisotropies Probe (WMAP) launched as its successor in 2001, and more recently the Planck satellite launched by the European Space Agency (ESA) in 2009. From these experimental results, parameters of standard cosmology have been determined with high precision, and inflation theory proposed by Guth, Sato, and Starobinsky independently, which claims that there was a period of exponentially rapid expansion of space before the Big Bang, has come to be strongly supported.

The idea of inflation was introduced primarily to solve the "horizon problem" of why long-distance correlations much longer than the horizon size exist in the early universe immediately after the Big Bang and the "flatness problem" of why curvature of space is so small. If this idea is interpreted naturally, then the universe has expanded approximately 10^{60} times from its birth to the present. It implies that the Hubble distance, which is the largest scale in the universe, and the Planck length, which is the smallest scale, are connected, as shown in (1-1). From this, it is expected that traces of quantum fluctuations of gravity at the beginning of the universe are recorded in the observed fluctuation spectra of CMB.

Only by this fact, however, it cannot be believed that the trace of quantum gravity remains. Rather, you might think that much of information in the early universe has been lost in the process of the long-term evolution, let alone the trace of quantum gravity effects. Contrary to your intuition, it does remain actually. If you go back in time to the history of the universe, you can get answer to the question why the trace remains. To show that, you first need to have a good understanding of the Friedmann solution, which is the basis of standard cosmology.

The Friedmann solution is a dynamical solution of the Einstein equation

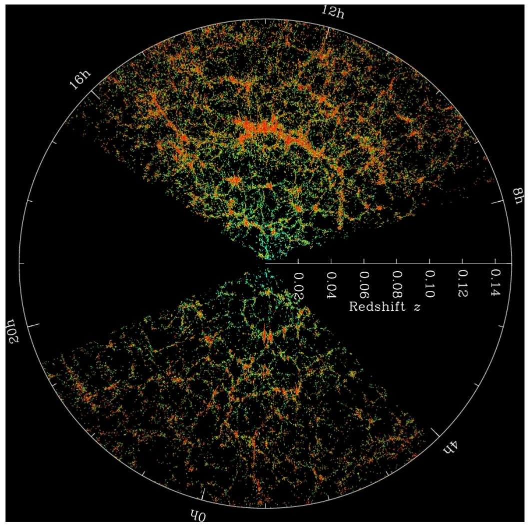

Figure 2-2: Observation results of galaxy distribution in the whole sky by SDSS [https://www.sdss.org]. Each point represents a galaxy. The part without data is the galactic plane.

solved presuming homogeneity and isotropy. It should be emphasized here that this solution is unstable. Normally, such a solution is not chosen as physics, because if you give a small fluctuation (perturbation) around this solution, it will grow with time and deviate significantly from the solution. Nevertheless, the universe can still be well approximated by the Friedmann solution. The Sloan Digital Sky Survey (SDSS) observations of galaxy distribution over a wide area where galaxies can be seen as mass points (Fig.2-2) show this fact brilliantly. The fact that the unstable Friedmann universe has lasted for more than 10 billion years means that initial amplitudes of fluctuations were unnaturally small.

After the Big Bang, the small fluctuations grow and structures such as stars, galaxies, and clusters of galaxies are formated. Although simple perturbation theory cannot be applied to these structure formation due to nonlinear effects, the fluctuations until the universe is neutralized are still small and so perturbation theory is applicable. The study of describing the evolution of the universe in perturbation theory based on this fact is called cosmological perturbation theory. This will be explained in Chapter 9. After neutralization, light becomes free (decoupling) from interactions with matters, so its fluctuation does not grow. Therefore, if spectra of the fluctuation at the time of decoupling are known, the current fluctuation spectra of CMB can be roughly known.

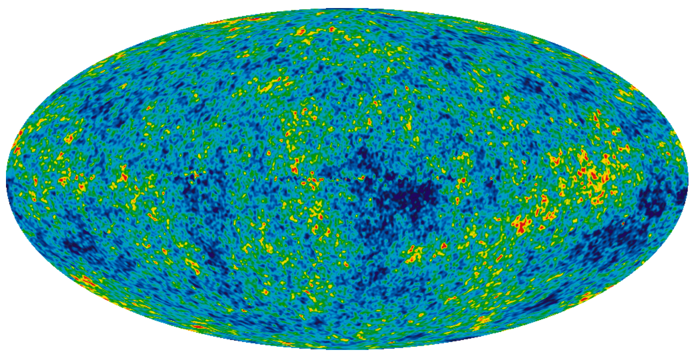

Figure 2-3: All-sky CMB temperature fluctuation distribution by WMAP [https://map.gsfc.nasa.gov].

The CMB temperature T discovered by Penzias and Wilson in 1964 has a Planck distribution of 3 degrees K. The COBE astronomical satellite was the first to observe that the temperature fluctuation amplitude $\delta T/T$ representing the magnitude of deviation from the distribution is on the order of 10^{-5}. For this discovery, Mather and Smoot, who led the experiment, were awarded the Nobel Prize in 2006. The more detailed temperature fluctuation distribution of CMB obtained by the successor WMAP is shown in Fig.2-3, and the temperature fluctuation spectrum obtained by statistically processing it is shown in Fig.2-4.

This spectrum mainly records history of the universe from the radiation-dominated era to the present. It is believed that the spectrum in the early universe was an almost featureless scale-invariant form. That is called the Harrison-Zel'dovich spectrum, which is, when writing it in Fig.2-4, given by a horizontal straight line that passes through the bottom of the recess in the area where the multipole moment l is small. In other words, the

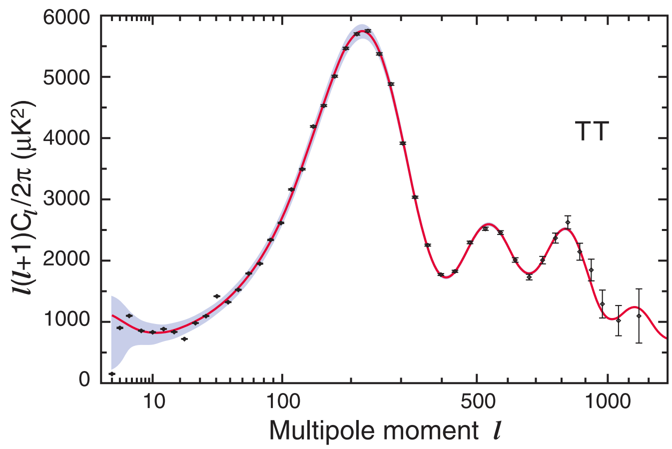

Figure 2-4: WMAP CMB temperature fluctuation spectrum [C. Bennett, et.al., Astrophys. J. Suppl. **208** (2013) 20].

deformation from the straight line represents dynamics that the fluctuations received during the evolution of the universe.

Most of these deformations occur during the period changing from the radiation-dominated to matter-dominated universe and then until neutralization. Prior to that, since the almost scale-invariant spectrum is maintained, you can go back indefinitely as long as the era of radiation-dominated universe continues. When the size of the fluctuation is larger than the horizon size, the spectrum hardly changes. In particular, the CMB multipole component of $l < 30$ is a large-size fluctuation component that has entered the inside of the horizon after the neutralization of the universe, or has not yet entered, and retains the primordial spectrum immediately after the Big Bang as it is. Therefore, as long as Einstein's theory of gravity is correct, it is possible to extract information of primordial spectra generated after the Big Bang from the current CMB spectra.

The facts that the standard cosmology brings seem to contradict our intuition, because it is natural to think that the early universe was a melting pot of high energy reactions and the fluctuation was so large. From this, rather than thinking that a small value was selected as the initial value, it

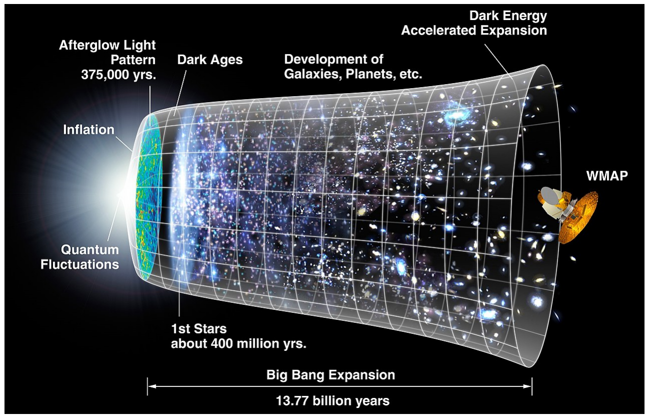

Figure 2-5: Sketch of the evolution of the universe drawn by WMAP [https://map.gsfc.nasa.gov].

is easier to accept as a scenario to think that there was some mechanism to reduce the fluctuation at the beginning at least for the fluctuation involved in the structure formation of the universe. Clarifying this thing is one of the major issues in the theory of early universe.

I believe inflation theory is the key to solving this problem. It makes sense to capture inflation as a mechanism that gives the very small fluctuations required for the Friedmann universe to continue for more than 10 billion years. It makes the story clearer to think that the force to ignite inflation is given by quantum gravity effects, and the fluctuation reduces during that period.

One of the goals of quantum gravity theory is to clarify such inflation dynamics, and to derive the primordial spectrum that gives the initial condition of the Friedmann universe. The observations indicate that it should be a nearly scale-invariant and scalar-like spectrum with very small amplitudes. How does inflation occur? Why does a scalar mode dominate and is it scale-invariant in the early universe, even though the gravitational field is a tensor field? How is the process of reducing the fluctuation described? The scenario of the evolution of the universe from a trans-Planckian world to the present, including answers to these problems in quantum gravity, will be explained mainly in the latter half of this book.

Chapter 3

Early Attempts and Setback

One of the earliest approaches combining quantum field theory and gravity was the study by Uchiyama and DeWitt in 1962 on renormalization of quantized matter fields in curved spacetime. Uchiyama is famous for completing research that links non-Abelian gauge field theory and gravity, independently of Yang and Mills, and announced the results at a conference at Kyoto University in 1954. Unfortunately, the reaction there was so negative that he hesitated to write his treatise and the publication was delayed in 1956.

In the late 1960s, DeWitt began a real attempt to quantize Einstein's theory of gravity. In those days, it was an age when only the electromagnetic field and the gravitational field were recognized as fields that mediate forces. Since the former was completed as quantum electrodynamics (QED), it is no wonder that the next step was quantizing the gravitational field. In order to construct a unified theory that Einstein dreamed of, that is, to formulate all fields in a unified manner under one theoretical system, it would have been necessary to put the gravitational field on the same quantum-mechanical ground as other fields. At the same time, it was expected to be able to solve the problem of spacetime singularities.

In the 1970s, ´t Hooft, Veltman, Deser, Nieuwenhuizen, and others began research on renormalization of gravity. At first, it was attempted to quantize the gravitational field by assuming that it is weak. That is, considering a small fluctuation (perturbation) from a certain classical background spacetime, the fluctuation is quantized. Usually, in this case, the gravitational field is expanded as

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa H_{\mu\nu}, \tag{3-1}$$

adopting flat spacetime as a background to consider a situation that the fluctuation $H_{\mu\nu}$ propagates in vacuum, where $\eta_{\mu\nu}$ is the Minkowski metric. Of course, you can choose a curved background spacetime according to your purpose. The expansion parameter κ is called coupling constant, which represents the strength of gravitational interactions, and has a dimension of the square root of the Newton constant G, that is, the Planck length $l_{\rm pl}$. The fluctuations of the gravitational field propagating in vacuum in this way are called gravitational waves. In particle physics, gravitational waves are often called "gravitons", just as electromagnetic waves are called photons. Interactions involving gravitons are represented by Feynman diagrams such as Fig.3-1 and Fig.3-2.

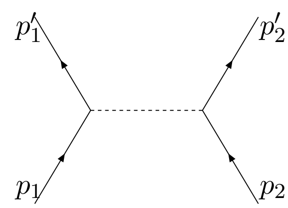

Figure 3-1: Scattering process between scalar particles (solid) exchanging a graviton (dotted). This interaction generates the Newton potential.

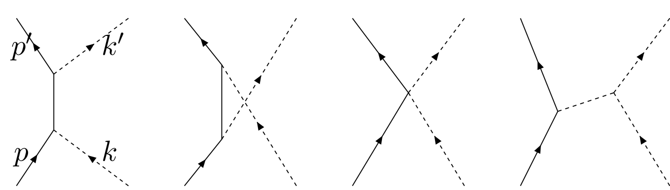

Figure 3-2: Compton scattering between gravitons and scalar particles.

However, there are many problems in formulating quantum gravity by such weak-field approximation. This approach, in the first place, cannot answer anything about singularities, which is the biggest problem of Einstein's theory of gravity. Even so, in those days, there was no other concrete method for calculating quantum corrections, so research on quantum gravity began from such an easy setting.

For the time being, let us put aside the problem of singularities and move on. Then another major problem of ultraviolet divergences arises. In quantum field theories, Feynman diagrams involving loops appear as quantum corrections, and when they are calculated, the results become infinite. Most of known quantum field theories are renormalizable and such divergences can be eliminated systematically, whereas Einstein's theory of gravity becomes unrenormalizable. The reason is that in renormalizable quantum field theories, coupling constants representing the strength of interactions are dimensionless, whereas the gravitational coupling constant κ has dimensions.

Why only the coupling constant of the gravitational field has dimensions. That is easy to see from the fact that the action must be dimensionless. Ordinary gauge fields and fermion fields have dimensions as inherent properties, and their actions, including derivatives in coordinate space, become dimensionless without introducing extra scales, thus the coupling constant also becomes dimensionless. On the other hand, the gravitational field $g_{\mu\nu}$ is a completely dimensionless field essentially. Therefore, a quantity given by a four-dimensional volume integral of the scalar curvature R with two derivatives, which constitutes the action of Einstein's theory of gravity, has a dimension of the square of the length. In order to make it dimensionless, it is necessary to introduce a scale to make up for the lack of dimensions, that is, the Newton constant G. The action obtained in this way is called the Einstein-Hilbert action.

The fact that the gravitational field is a strictly dimensionless field is important in the following discussions. In the expansion (3-1), the field is redefined so that $H_{\mu\nu}$ has the same dimensions as ordinary gauge fields, but it is for convenience, and unlike gauge fields, interaction terms resulting from this expansion become multipoint functions that last infinitely.[1] Also, when discussing scale-invariant dynamics of the gravitational field in later

[1] According to the convention in this research field, (3-1) is stopped at the first order of $H_{\mu\nu}$, but the inverse of the metric tensor defined by $g^{\mu\lambda}g_{\lambda\nu} = \delta^\mu_\nu$ is expanded by the multiple product of the field the lasts infinitely like $g^{\mu\nu} = \eta^{\mu\nu} - \kappa H^{\mu\nu} + \kappa^2 H^\mu{}_\lambda H^{\lambda\nu} + \cdots$. And also, $\sqrt{-g}$ and R are expanded in the same way.

chapters, the fact that it is dimensionless is essential.

In general, unrenormalizable quantum field theory requires an upper limit on energy to handle divergences. In other words, some kind of ultraviolet cutoff is provided so that it cannot go beyond that. It implies that it is considered as a theory that is effective only in an energy region below the cutoff. In Einstein's theory of gravity, the cutoff is usually put at the Planck energy.

However, theories with such a limitation are inappropriate as fundamental theories that describe high energy physics. In order to discuss issues such as singularities, an infinitely small area has to be able to be handled. Therefore, renormalizability is a necessary condition to describe a world beyond the Planck scale.

Early attempts to modify Einstein gravity

The problem with Einstein's theory of gravity lies in its action called the Einstein-Hilbert action. One of the characteristics that actions of renormalizable theories have is that the highest derivative kinetic terms of fields do not have a scale. Therefore, as energy is higher, the kinetic term becomes dominant, and terms with a dimensionful parameter such as mass terms can be ignored. In contrast, the Einstein-Hilbert action which is the highest derivative term in the theory, has a scale, namely, the Newton constant, thus it is not possible to consider a high energy limit in which the scale can be ignored. This indicates that the theory cannot describe a world without the scale and hence there is an upper limit on energy. The problem mentioned at the beginning of the previous chapter also arise from this.

Since the late 1970s, attempts have been made to modify Einstein's theory of gravity in order to solve the problem of renormalization and enter a trans-Planckian region. The principle that underpins Einstein's gravity theory is diffeomorphism invariance, and the so-called Einstein equation is one of equations derived from it. Modifications should only be made while preserving diffeomorphism invariance. One of them is a method of introducing dimensionless gravitational actions including fourth derivatives by Stelle, Tomboulis, Fradkin and Tseytlin and so on.

The four-derivative gravitational action is constructed as follows. First, three important quantities that characterize gravity are introduced, which are the scalar curvature R adopted as the Einstein-Hilbert action, the Ricci tensor $R_{\mu\nu}$ describing the Einstein equation, and the Riemann curvature tensor $R_{\mu\nu\lambda\sigma}$ representing the degree of spacetime distortion. Then consider three functions obtained by squaring them and contracting spacetime indices using the metric tensor as R^2, $R_{\mu\nu}R^{\mu\nu}$, and $R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$. Since these are scalar functions with fourth derivatives, their integrals over a four-dimensional spacetime volume are dimensionless. Therefore, quantum gravity theories employing them as actions have dimensionless coupling constants and are candidates for renormalizable theories.

The Riemann curvature tensor, which expresses whether spacetime is curved, is particularly important, and it can be thought of as a quantity corresponding to a field strength $F_{\mu\nu}$ of a gauge field such as a photon. In fact, in mathematics, $F_{\mu\nu}$ is called the curvature. The problem with the Einstein equation is that it does not contain the Riemann curvature tensor corresponding to the strength of the gravitational field. This is in contrast to the fact that equations of motion for gauge fields contain the field strengths, such as $\nabla_\mu F^{\mu\nu} = J^\nu$, where J^μ is a source current. Therefore, it seems reasonable to include the square of the Riemann curvature tensor in a quantum gravity action. However, unfortunately, not everything works well even if such actions are adopted.

First of all, various good points of the above actions are summarized here. They have the following properties:

1. The theory becomes renormalizable.

2. The action is bounded below.

3. There are no spacetime singularities.

As already mentioned, the property 1 comes from the fact that gravitational coupling constants used for perturbation become dimensionless.

The property 2 is guaranteed by considering the square of the curvature.[2] This is extremely important from the standpoint of stability. In contrast, the Einstein-Hilbert action defined by the first power of the scalar curvature R, which takes values from $-\infty$ to ∞, is not bounded below and has no bottom of energy. It suggests that a quantum system where fluctuations of the gravitational field are so large becomes unstable, even though it may not be a problem as long as considering small perturbations around a regular spacetime. Hence, adding a gravitational action consisting of the curvature squared as main part leads to removing the instability in Einstein's theory of gravity.

The property 3 is one of the most expected properties in quantum gravity. The Schwarzschild black hole, which is a representative of solutions of the Einstein equation with singularities, is a Ricci flat solution in which $R_{\mu\nu}$ and also R disappear. On the other hand, the Riemann curvature tensor $R_{\mu\nu\lambda\sigma}$ has a value and diverges at the center. A singularity is exactly the point where the Riemann curvature tensor diverges. This says that an action composed of the Riemann curvature tensor squared diverges for a solution with such a singularity.[3] In general, a field configuration in which an action diverges is excluded as unphysical. At that time, it is also an important condition that its action is positive-definite, because it means that the existence probability of such a field configuration in which an action diverges becomes zero. It can be understood statistical mechanically by using the path integral quantization method, which will be discussed again in Chapter 6.

These arguments reveal the problems of Einstein's theory of gravity. In general, physical objects are ones that an action becomes finite when they are assigned, as mentioned above. Solitons and instantons in gauge theories are

[2] Strictly speaking, this is listed as a property that Euclidean quantum field theory actions obtained by applying the Wick rotation should satisfy. See Chapter 6 for details.

[3] The square of the Riemann curvature tensor is often removed from the action using the Euler (Gauss-Bonnet) combination, but it can only be done when there is no singularity manifestly in spacetime, such as when considering perturbation theory around a regular spacetime. Here is discussing the general case without such a premise.

typical examples. On the other hand, not only the Einstein-Hilbert action is indefinite but also disappears for the Schwarzschild solution with vanishing scalar curvature, so that black hole singularities can exist physically. This is a big problem in Einstein's theory of gravity.

At present, there are still many attempts to formulate quantum theories of gravity based on Einstein's theory of gravity. One of the reasons for doing so is that you do not have to worry about unitarity problems. However, this is wrong, because the Einstein-Hilbert action cannot remove spacetime singularities. It should be considered that the unitarity of these theories is guaranteed only in the case of the weak-field approximation in which the scattering matrix can be defined.

What have been described above are the good points of the fourth-derivative gravitational action that the Einstein-Hilbert action does not have. Writing this way, everything seems to work, but actually the story is not so simple. First, the action cannot be uniquely determined only by the discussion so far. The square of the Riemann curvature tensor should be included to eliminate singularities, but diffeomorphism invariance alone still leaves some arbitrariness. Even more problematic is the method of quantization. Needless to say, the first attempt was carried out in a weak gravitational field approximation, that is, in a graviton picture employing a method of perturbation expansion around flat spacetime as shown in (3-1). At that time, instead of κ, a dimensionless coupling constant is newly introduced as an expansion parameter. The reason why this approach was adopted is, of course, that it was only the method that concrete calculations could be performed in the 1970s.

Quantization in the weak-field approximation is still problematic even in this case. What looking for is a theory that describes a small world that is shorter than Planck length, that is an ultra-high energy world. The weak-field approximation is based on a strange assumption that the gravitational field becomes weak near the center of the black hole and flat spacetime appears there. This says that as approaching the center, a calm world that is not much different from environment on the earth will appear, even though it is believed that everything would be crushed by gravity. Looking at the early universe, it is also believed that it is a world where all matter would be crushed by gravity, but the weak-field approximation assumes that it is not so.

As a historical background, the success of QCD, which was introduced as a theory of strong interactions that describes physics of hadrons, would have influenced the adoption of the weak-field approximation. The property "asymptotic freedom" this theory indicates is that the interaction is weakened at short distances, that is, at the high energy limit, and quarks and gluons that are constituents of hadrons behave like free particles. The early study on renormalizable quantum gravity theory was carried out under the assumption that it also has such a property, as a kind of gauge field theory.

Putting aside the physical strangeness of the setting for a while, let us see what happens as theoretical problems when the gravitational field is quantized with the weak-field approximation. The perturbation theory itself can be defined without any problem, and it becomes renormalizable. However, a big problem arises in physical quantities. That is the so-called "ghost problem". Ghost is a term that refers to being unphysical, meaning that its action has the opposite to a normal particle so that it is not bounded below. Hence, ghosts are particles that should not be observed, because unitarity is broken when they contributes to scattering as physically existing particles. In general, free field theories with fourth derivative have twice as many degrees of freedom as normal second-derivative field theories. The problem is that if one of them is physical, the other always becomes unphysical.

On the other hand, there is a work by Lee and Wick, who have suggested that ghosts do not appear due to quantum corrections in the theory with interactions that show the asymptotic freedom. The fourth derivative quantum gravity theory in the 1970s was studied based on this idea. It says that ghosts do not appear in the real world at low energy, but when the interaction disappears at high energy, it eventually becomes a world where ghost particles are actually propagating. Hence, a simple perturbation theory by the weak-field approximation did not lead to an essential solution to the problem.

This ghost problem in higher-derivative field theories is a highly specialized problem that is not found in general quantum field theory textbooks, and many researchers take it as it is as a No-Go theorem. Most of who stick to Einstein's theory of gravity seem to think so. To overcome the Planck scale wall, it is necessary to find answer to this question. The original cause of the ghost problem in fourth-derivative field theories is that a field can be decomposed into physical and unphysical modes. The problem with adopting the weak-field approximation is that these modes behave as independent degrees of freedom at the zeroth order of perturbations in which only kinetic terms remain. The essence of Lee and Wick's work is to connect these two modes through interactions so that they are not independent of one another.

Their work gives a hint for solving the problem. The story of unitarity changes if there is a gauge symmetry that connects the two modes in the ultraviolet limit. The appearance of a ghost mode in the real world means that the mode is gauge invariant by itself. If such a gauge symmetry exists, however, it can be made gauge variant, namely unphysical. As a matter of course, such a situation cannot be realized in a perturbation theory with simple weak-field approximation, which is because in the ultraviolet limit where interactions disappear, even if there is a gauge symmetry, the gauge transformation works mode-by-mode and does not mix with each other.

Something is missing to overcome the physical and theoretical problems with quantization of gravity. An image of spacetime where gravity is quantized is that it is in a state in which time and distance fluctuate greatly so that they do not make sense. By going back to the origin, it is necessary to construct a theory that expresses such a quantum state correctly.

The state in which the gravitational field fluctuates so much that time and distance become uncertain means that even if a certain background spacetime is specified, it itself no longer has physical meaning. Such a property is called "background-metric independence" or "background freedom". That is also says that the conventional picture of particles propagating in a specific background spacetime is not valid.

In order to express that, it is necessary to introduce some non-perturbative method. The core part of a new formulation introduced in the next chapter will be described by a special case of conformal field theory (CFT) that is a representative of non-perturbative field theories. It is a CFT in which conformal invariance is realized as a gauge symmetry, unlike normal ones. That is to say that all worlds with different scales connected to one another by conformal transformations become gauge equivalent. In this way, the background-metric independence is expressed algebraically, and this gauge symmetry solves the ghost problem.

In the first place, the ghost problem is not limited to quantum theory. Einstein's theory of gravity has a ghost mode with negative metric that causes the indefiniteness of the Einstein-Hilbert action. It can be locally removed and the theory can be described only by physical gravitational wave modes. However, the existence of a negative-metric mode plays an important role when considering global structures of spacetime such as the Friedmann universe. It shows that such a ghost mode is required to preserve diffeomorphism invariant, that is, to make the whole Hamiltonian vanishing. In the next few chapters, I will describe that ghosts play an important role to preserve diffeomorphism invariance even in quantum gravity. It can be found that ghost modes are indispensable elements for describing quantum spacetime in which the Hamiltonian vanishes, but they never appear locally as physical quantities.

On approaches without using fields

Since the 1980s when quantum field theory method was stalled, new methods that did not based on it became mainstream. String theory and loop quantum gravity are typical examples. Before introducing a quantization method of the gravitational field that realizes background-metric independence, here is a brief summary of these methods that do not use the field.

String theory was originally proposed by Nambu to describe strong interactions that governs hadron physics, and is a theory that is descended from the scattering matrix theory developed in the 1960s. It is a formulation in which scattering amplitudes for interactions between particles can be calculated without accompanied ultraviolet divergences.

The essence of this theory is to consider stringy objects with one-dimensional spread rather than point-like objects as basic elements of matters. Such objects propagate in flat spacetime while repeating joining and separating. An oscillation mode of the string corresponds to a particle, and eigenenergy of the oscillation becomes its mass. The modes in which mass becomes zero are particularly important, which become particles that make up a model of elementary particles. Among them, not only scalar and vector modes, but also graviton, which is a tensor mode, are include, and therefore string theory has been studied as a candidate for unified theory. Moreover, superstring theory with fermion degrees of freedom has come to be called the theory of everything or the ultimate theory.

Another major feature of the theory is that spacetime must be 10 or 26 dimensions in order for Lorentz invariance to be preserved, and superstring theory is defined in 10 dimensions. Therefore, in order to describe the real world, which is 4 dimensions, it is necessary to round and reduce the extra 6-dimensional space so that it cannot be observed, which is called compactification.

String theory provides a theoretical system that can calculate scattering amplitudes based on a weak-field approximation, which is defined as a perturbation expansion by the coupling constant α' (the inverse of the string tension) that has a dimension squared in length, corresponding to the Newton constant. Therefore, an effective action of string theory configured to reproduce the scattering amplitude has a form expanded in derivatives expressed by powers of curvature whose lowest term is the Einstein-Hilbert action. Another feature is that each term of it becomes a local action that can be treated classically, unlike that in renomalizable quantum field theories.[4]

The only parameter of the theory is α' with dimensions and it is a theory

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[4] The quadratic curvature term of the string effective action is only the Gauss-Bonnet type, like the action of Lovelock's theory of gravity. When the Gauss-Bonnet term is expanded by (3-1), in any dimension, kinetic terms given by two-point functions of the field vanish and only interaction terms of three or more points appear. All higher-derivative terms also produce only interaction terms. That is, since the kinetic term is derived only from the Einstein-Hilbert term, the only modes that freely propagate as asymptotic fields are physical gravitons, thus the perturbative unitarity of the scattering matrix holds.

in which there is nowhere this dimensionful scale can be ignored, like Einstein's theory of gravity. Therefore, the square root of its reciprocal becomes an upper limit of energy and the perturbation expansion is effective only below that. In any case, it would be difficult to break through the upper limit as long as objects are described within the standpoint of propagating in a specific background spacetime, whether it is point- or string-like.

In general, there is an inherent energy scale for each physical phenomenon. In string theory, since the scale is originally only α', scales for describing phenomena that occur at lower energies than this scale are given by radii of space rounded variously. There are innumerable ways to do this in principle, but geometry of compactified space is usually determined so that required symmetries remain. The problem with compactification is that it is basically arbitrary.

Since the beginning of the 1990s, a high-dimensional object called D-brane has appeared as a new degree of freedom. It was originally introduced to describe a strong coupling state of string theory. It has been known that there are two methods for describing it: one is to express it as a theory of world volume (trajectory of motion of object) and the other is to express it as a soliton solution of supergravity theory that appears as an effective theory of string theory. From the relationship between them, the AdS/CFT correspondence was proposed by Maldacena at the end of the last century, which is a conjecture that a conformal field theory (CFT) will appear at the boundary of the AdS (= Anti de Sitter) space. This idea , which was born from string theory, is now also called the holographic principle and is developing almost independently of string theory.

The research field where the AdS/CFT correspondence is utilized has extended to areas of QCD, quark-gluon plasma, and condensed matter physics, which began when Witten applied it to analysis of spectra of gluon excited states (glueballs). However, although gravitational words are used in these studies, most of them are stories that are not directly related to dynamics of gravity. The essence is to introduce a scale for describing dynamics while maintaining gauge symmetry. By the way, it is well known that the method of simply adding mass terms to gauge fields breaks gauge symmetry, while in renormalizable gauge theories, a mass scale is introduced through nonlocal quantum corrections without breaking the symmetry. Here, such scale introduction is done by curving a space. Furthermore, by intersecting several D-branes in extra dimensions, it is possible to realize global symmetry that research subjects should have. The equation of motion in curved spacetime derived in this way is a local field equation that is relatively easy to solve. The success of this approach shows that gauge symmetry and global symmetry determine most of physical phenomena.

It is argued that if CFT is examined inversely through this correspondence, you should be able to describe quantum gravity in AdS, which is called AdS quantum gravity. However, it is a story that depends on the background spacetime, and it is essentially different from the method described in the next chapter, in which CFT directly comes out as a field theoretical representation of quantum gravity that shows background freedom.

String theory as a unified theory is in a situation that it includes everything but cannot explain anything. However, knowledge of geometry and group theory is required when rounding space, and many researchers are fascinated by its mathematical beauty. It seems that there are more such researchers nowadays. Although the interest in mathematical science seems endless, you can hardly explain dynamics of gravity.

Another representative example, loop quantum gravity proposed by Rovelli and Smolin, is a theory that has gained support by emphasizing background freedom that string theory does not have. It is one of attempts to realize background freedom by defining spacetime discretely and performing a sum over all possible spacetime configurations. A state which satisfies what is called the constraint conditions that energy-momentum tensor becomes zero, that is, conditions that diffeomorphism invariance holds, is represented by a graph consisting of points (nodes) and lines (links) connecting them. It is called a spin network, while the one to which discrete time is added is called a spin form.

Unlike lattice gauge theories, this theory does not take a continuum limit. Rather, it is an idea that discretizing area and volume is quantization of gravity. Depending on the number of links that penetrate surface and the number of nodes contained in space, area and volume of there are given in units of the Planck scale. However, the discretization breaks diffeomorphism invariance that is a continuous symmetry, at that time. As for spatial components, you may devise to satisfy the constraint condition by introducing a variable like Wilson loops, but it cannot satisfy the Hamiltonian constraint condition related to time component. In the first place, it is doubtful whether states that satisfy the constraint conditions derived from the continuous symmetry can be expressed in terms of the spin network without excess or deficiency.

This originally began with an attempt to canonically quantize Einstein's theory of gravity employing the Arnowitt-Deser-Misner (ADM) form, and is related to an attempt to solve the Wheeler-DeWitt equation. In order to construct a state that satisfies the constraint conditions, the gravitational field is rewritten to Ashtekar's new variable and the loop variable mentioned above is introduced. At this point, it is still described as a classical theory using the Poisson brackets. Replacing the Poisson brackets with commutators is a standard procedure to move to quantum theory, but problems arise here, which will be touched upon in the next two chapters. Loop quantum gravity says that quantization was performed by expressing the state with a spin network without doing such a procedure. It should be considered that this is a new approach that is different from normal methods such as the path integral.

Chapter 4

How to Realize Background Freedom

In order to build a unified theory that includes gravity, the gravitational field has to be treated like any other matter field. It may be said that the picture of gravitons propagating in a specific spacetime is premised. However, such a position will be not taken here. Because, matter fields are defined in the presence of spacetime, but the gravitational field defines spacetime itself. Unlike other fields, the gravitational field is essentially dimensionless and is coupled to all fields directly. More attention should be paid to this fact. The roles of the gravitational field and matter fields are naturally different, and the difference becomes prominent in the world beyond the Planck scale. Gravitational quantum fluctuations are so great there that the picture in which particles propagate is no longer appropriate.

What can be said in the discussion so far is that, after all, perturbation methods based on a simple weak-field approximation have to be given up. In the first place, perturbing around flat spacetime where the Riemann curvature tensor vanishes is inappropriate for describing strong gravitational fields near the center of black holes. Even if black hole spacetime is adopted as a background, it cannot be said that quantum gravity theory could be constructed because the neighborhood of singularity cannot be treated at all.

Then what should you do? First, in order to make singularities unphysical, as mentioned in the previous chapter, a fourth-derivative gravitational action containing the square of the Riemann curvature tensor should be employed. Moreover, some idea that does not rely on conventional weak-field approximation is required. The property that determines quantum gravity called "background-metric independence" or "background freedom" is exactly diffeomorphism invariance, which is called by another name to emphasize that it is a new properties acquired by quantization. This word describes a state where spacetime is fluctuating so much that the concept of distance and time is lost, and so says that classical spacetime becomes unclear as a reference. That is, even if it is chosen as a reference, it no longer makes sense because spacetime is fluctuating greatly around it. Hence, the background-metric independence cannot be described at all within a weak-field approximation which assumes that all perturbations around a specific spacetime are small.

Some kind of nonperturbative treatment is required. A hint is in the early universe. Results derived from CMB observations by the WMAP astronomical satellite and others suggest that fluctuations in the early universe were scale-invariant and scalar-like. It seems natural to find its origin in conformal invariance. If a phase transition in spacetime occurs near the center of black holes and a conformally invariant world is realized, it also leads to a solution of the singularity problem.

Now, as a renormalizable quantum field theory with such properties, I present a quantum gravity theory in which the background-metric independence is expressed as a special conformal invariance (see also Appendix B). In this theory, as a quantity that expresses the field strength of gravity, not the Riemann curvature tensor itself, but the Weyl tensor

$$C_{\mu\nu\lambda\sigma} = R_{\mu\nu\lambda\sigma} - \frac{1}{2}\left(g_{\mu\lambda}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\lambda} - g_{\nu\lambda}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\lambda}\right) + \frac{1}{6}\left(g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda}\right)R \qquad (4\text{-}1)$$

plays an important role. In addition, the Euler density (Gauss-Bonnet combination), $G_4 = R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$, is necessary, which becomes topological when integrated in four-dimensional spacetime.

The action is composed mainly of the square of the Weyl tensor called the Weyl action and the topological Euler term, which are conformally invariant fourth-derivative quantities. Furthermore, lower derivative terms such as the Einstein-Hilbert action, the cosmological term, and matter field actions are added to these. At high energies beyond the Planck scale, the fourth-derivative terms predominate and conformal invariance appears. The Einstein-Hilbert action becomes effective in lower energy regions than the

Planck scale. In this way, you will be able to describe a trans-Planckian world that can never be reached by Einstein's theory of gravity.

Recall the discussion in the previous chapter here. It was stated that quantization with a simple weak-field approximation such as (3-1) does not work, because the setting not only is unnatural physically, but also causes the problem of ghosts. Such a perturbation expansion corresponds to carrying out around flat spacetime with $R_{\mu\nu\lambda\sigma} = 0$. On the other hand, adopting the Weyl action means perturbing around a conformally flat spacetime with $C_{\mu\nu\lambda\sigma} = 0$. This is also in line with inflation theory that many cosmologists believe as an idea to solve the horizon and flatness problems.

The point of non-perturbative quantization, which is a main subject here, is that when expanding the gravitational field, the conformal factor which is the most important factor determining distance is specially treated by extracting in an exponential form so as to be positive manifestly like

$$g_{\mu\nu} = e^{2\phi} \bar{g}_{\mu\nu}. \tag{4-2}$$

The scalar-like field ϕ is called the conformal-factor field. The remaining degrees of freedom of the gravitational field, which represent deviations from the conformally flat spacetime, are expanded with a weak-field approximation as

$$\bar{g}_{\mu\nu} = \eta_{\mu\lambda} \left(e^h\right)^\lambda{}_\nu = \eta_{\mu\lambda} \left(\delta^\lambda{}_\nu + h^\lambda{}_\nu + \frac{1}{2} h^\lambda{}_\sigma h^\sigma{}_\nu + \cdots \right), \tag{4-3}$$

where $h_{\mu\nu}$ is a 9-component tensor field that satisfies the traceless condition ($h^\mu{}_\mu = \eta^{\mu\nu} h_{\mu\nu} = 0$).

Of course, it is best to be able to handle all degrees of freedom non-perturbatively, but that is never easy. Therefore, I have proposed a perturbation theory that well captures the characteristics of the world beyond the Planck scale. See Chapter 13 for a completely non-perturbative approach.

The coupling constant t is introduced as a dimensionless parameter that controls the expansion (4-3). Since the field strength of the traceless tensor field is given by the Weyl tensor, the perturbation theory is formulated by introducing the reciprocal of t^2 in front of the Weyl action defined by its square.[1] When performing perturbation calculations concretely, it is usually done after redefining the field like $h^\mu{}_\nu \to t h^\mu{}_\nu$. However, when describing its dynamics, it is easier to understand if the coupling constant is left before the Weyl action, so the discussion proceeds as it is without redefining it.

The conformal-factor field ϕ has to be handled exactly without introducing a coupling constant for it. Because, the fourth-derivative actions are conformally invariant and thus they do not depend on the conformal-factor field. Hence, this field will be completely fluctuating without being controlled by their actions, that is, the conformal-factor field remains non-perturbative. However, if the theory does not depend on it at all, its dynamics will be the same as if it does not exist, resulting in a theory that lacks an important part of the gravitational field. Where the dynamics of the conformal-factor field comes from and how it is treated, that is the key of quantization described below.

Methods of quantization

As widely-known quantization methods, there are the canonical quantization method and the path integral method. The issue of quantization in gravity is to see whether the theory can be actually formulated according to these quantization methods. Einstein's theory of gravity could not do it because of many problems mentioned before.

Here, quantization of the above-mentioned gravity will be performed employing the path integral method that manifestly preserves diffeomorphism invariance. It should be noted that the canonical quantization method, on the other hand, premises that actions are defined in a specific background spacetime, usually flat spacetime, and canonical commutation relations are set in that spacetime. Since time is treated specially at that time, it is hard to say that diffeomorphism invariance is manifestly maintained, which will be revisited in the next chapter.

[1] How to insert the coupling constant in this way is the same as in gauge field theory. Letting $\mathfrak{g}$ be a coupling constant, then introducing it as the reciprocal of $\mathfrak{g}^2$ before an action given by the square of the gauge field strength $F_{\mu\nu}$ implies a perturbation expansion around $F_{\mu\nu} = 0$.

The path integral of the quantum gravity is defined by putting the action, denoted as I, on the exponent of exponential function and performing functional integrals over the gravitational field $g_{\mu\nu}$. The partition function is then symbolically written as

$$Z = \int [dg]_g \, e^{iI(g)}, \tag{4-4}$$

where $[dg]_g$ denotes a path integral measure of $g_{\mu\nu}$, and the subscript g indicates that it is defined so as to be diffeomorphism invariant using $g_{\mu\nu}$ itself which is a dynamical variable. Contributions from matter fields are omitted here for simplicity.

A new obstacle on quantization arises here. This integral measure shows that if you try to perform the integral with respect to $g_{\mu\nu}$, you have also to integrate the $g_{\mu\nu}$-dependence involved in the measure. Hence, first of all, this nested structure has to be resolved.

Usually, for practical reasons, the path integral is performed by replacing the invariant measure with a measure defined in a specific spacetime. In flat spacetime, it should be written as $[dg]_\eta$ with the subscript of $\eta_{\mu\nu}$ that is the Minkowski metric. The use of this measure may be justified if a weak-field approximation such as (3-1) is applied, then you can perform the path integral for the field $H_{\mu\nu}$ according to the textbook. But, in general, this simple replacement breaks diffeomorphism invariance.

In fact, when using the expansion (4-2), in order to resolve the nested structure while preserving diffeomorphism invariance, it is necessary to rewrite the path integral measure as

$$[dg]_g = [d\phi]_\eta [dh]_\eta \, e^{iS(\phi,\bar{g})}, \tag{4-5}$$

where $[dg]_\eta$ is orthogonally decomposed and represented by the product of $[d\phi]_\eta$ and $[dh]_\eta$. The exponential part e^{iS} represents a Jacobian needed to ensure diffeomorphism invariance, and the function S of the conformal-factor field ϕ in the exponent is a quantity satisfying the Wess-Zumino consistency condition, called the Wess-Zumino action.[2]

[2] If you consider a simultaneous change of $\eta_{\mu\nu} \to e^{2\omega}\eta_{\mu\nu}$ and $\phi \to \phi - \omega$, then the measure

The Wess-Zumino action that appears here is a quantity whose conformal variation is called "conformal anomaly", and has been systematically well studied. Conformal anomaly is a term derived from the study of quantum field theory in curved spacetime, and was investigated by Capper, Duff, Deser, Isham and others in the 1970s. It says that even if you start with a conformally invariant classical action, the invariance is broken when it is quantized. Historically, the word anomaly has been used, but it is not physically anomalous. The conformal anomalies generally arise with the advent of a new mass scale upon quantization, which are necessary for quantum correction terms, containing the scale, to be diffeomorphism invariant.

Hence, the quantum gravity theory (4-4) under the expansion (4-2) can be redefined as a quantum field theory in flat spacetime with an action

$$I_{\rm QG}(\phi, \bar{g}) = S(\phi, \bar{g}) + I(g) \tag{4-6}$$

so that the partition function is expressed as

$$Z = \int [d\phi]_\eta [dh]_\eta \, e^{iI_{\rm QG}(\phi, \bar{g})}. \tag{4-7}$$

Once the Wess-Zumino action S is determined, it can be calculated using normal quantum field theory methods.

This expression shows that although the action (4-6) has no longer a diffeomorphism invariant form, the effective action that is given by the logarithm of the partition function involving quantum corrections becomes an invariant form. Here, it should be noted again that the Minkowski metric $\eta_{\mu\nu}$ was introduced for convenience only, while the metric tensor with physical meaning is (4-2). The strategy adopted here is to first define the theory

(4-5) is invariant because $g_{\mu\nu}$ (4-2) is invariant under this change. On the other hand, the right-hand side becomes $[d\phi]_{e^{2\omega}\eta}[dh]_{e^{2\omega}\eta}\, e^{S(\phi-\omega, e^{2\omega}\bar{g})} = [d\phi]_\eta [dh]_\eta \, e^{iS(\omega,\bar{g})}\, e^{S(\phi-\omega,e^{2\omega}\bar{g})}$, where $[d\phi]_\eta$ is invariant under the shift of ϕ for any background because its integration region is $(-\infty, \infty)$. In order for this to return to the original form, it has to satisfy $S(\omega, \bar{g}) + S(\phi - \omega, e^{2\omega}\bar{g}) = S(\phi, \bar{g})$. This is called the Wess-Zumino consistency condition. This condition was originally shown in curved spacetime. Quantizing gravity, the above property of $[d\phi]_{\hat{g}}$ ultimately results in the background-metric independence for the conformal change of $\hat{g}_{\mu\nu}$.

in a specific regular background, here in flat one, and then show that it does not depend on how to choose the background.

The Wess-Zumino action S, which arises to preserve diffeomorphism invariance, gives dynamics of the conformal-factor field, namely, kinetic and interaction terms of ϕ in fourth derivative. In particular, S exists even in the zeroth order of the coupling constant t, that is, the ultraviolet limit of $t \to 0$. This S is called the Riegert action after the discoverer, and quantization of this action was first performed by Antoniadis, Mazur, and Mottola.

Although above writing that the Wess-Zumino action occurs accompanied with a mass scale, the Riegert action is special. The fact that it appears at the zeroth order of t means that it does not involve the scale. This is an essential point in constructing a particular conformal field theory without the scale described below. Here, somewhat confusing is that even though the Riegert action has originally been derived as a conformal anomaly, it works to restore the conformal invariance exactly as a new appearance of the background-metric independence when performing the path integral over the conformal-factor field ϕ. This will become clearer in the next section.

Finally, so far proceeding without paying attention to $\hbar$, but here will specify clearly where it appears when revived. Considering the action I by dividing it into the fourth derivative gravitational part $I^{(4)}$ and a lower derivative term $I_{\rm EG}$ consisting of the Einstein-Hilbert action and matter actions, the reciprocal of $\hbar$ is entered only in front of $I_{\rm EG}$. Since the gravitational field is a dimensionless field essentially, $I^{(4)}$ is completely dimensionless, thus it does not contain $\hbar$. Similarly, $\hbar$ is not included in the Wess-Zumino action S derived from the path integral measure. This means that all fourth-derivative gravitational actions are quantities that describe purely quantum dynamics, and the whole including them as weights, except $I_{\rm EG}$, could be regarded as a whole measure of the path integral. Hence, the path integral of quantum gravity is often symbolically expressed using only $I_{\rm EG}$ as $\int \mathcal{D}g_{\mu\nu}\, e^{iI_{\rm EG}/\hbar}$, but here it is shown that the measure is expressed exactly as $\mathcal{D}g_{\mu\nu} = [d\phi dh]_\eta\, e^{iS+iI^{(4)}}$. This is also the reason why the Weyl action and the Wess-Zumino action resulting from the measure are treated on the same footing in the following discussion.

From the consideration of $\hbar$, it turns out that the ghost mode is essentially a quantum entity and does not appear as a classical one like a particle. Moreover, as described below, it can be shown that the ghost mode is not gauge invariant, namely unphysical.

Background freedom as conformal invariance

The key point of this quantum gravity theory is in a special property of the core part of the perturbation theory realized at the zeroth order of the expansion by the coupling constant t. Normally in a weak-field approximation, the zeroth order of perturbation is represented by free fields, or gravitons, propagating in flat spacetime, and the perturbation theory describes how they interact and scatter. On the other hand, in this theory, such free fields do not appear. The core part is described by a special conformal field theory, and the coupling constant t represents the degree of deviation from it. Therefore, the scattering matrix, which is a physical quantity premised on the existence of free fields asymptotically, is not defined.

Now, let us see what exactly the special conformal field theory is and what its physical meaning is. The difference from normal conformal field theories is that conformal invariance is a gauge symmetry or BRST symmetry, that is part of diffeomorphism invariance.[3] Hence, although normally only vacuum is conformally invariant, physical quantities must also be conformally invariant in this conformal field theory. That is, all theories with different backgrounds connected to each other by conformal transformations are gauge equivalent, and thus they become physically indistinguishable. This is an algebraic representation of background-metric independence. This symmetry is called "BRST conformal invariance" to emphasize that it is a gauge symmetry.

In the following, describe it specifically using mathematical formulas. First of all, diffeomorphism is a transformation that under a coordinate

[3] Becchi, Rouet, Stora, and Tyutin discovered in the 1970s that gauge invariance could be extended to include gauge fixing term and associated ghost term. BRST stands for four acronyms. The BRST quantization method was sistematically formulated by Kugo and Kojima soon after its discovery.

transformation $x^\mu \to x'^\mu = x^\mu - \xi^\mu(x)$, a line element defined by $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$ using the metric tensor field becomes invariant as $ds^2 = ds'^2$. If the gauge parameter ξ^μ is infinitesimal, then diffeomorphism is expressed as $\delta_\xi g_{\mu\nu} = g_{\mu\lambda}\nabla_\nu \xi^\lambda + g_{\nu\lambda}\nabla_\mu \xi^\lambda$, where the variation is defined by $\delta_\xi g_{\mu\nu}(x) = g'_{\mu\nu}(x) - g_{\mu\nu}(x)$ and ∇_μ is a covariant derivative.

Expand diffeomorphism under (4-2) and (4-3) and consider 15 gauge degrees of freedom ζ^μ that satisfy the conformal Killing equation as the gauge parameter ξ^μ.[4] The BRST conformal transformation then arises at the lowest of the perturbation as

$$\delta_\zeta \phi = \zeta^\lambda \partial_\lambda \phi + \frac{1}{4}\partial_\lambda \zeta^\lambda,$$
$$\delta_\zeta h_{\mu\nu} = \zeta^\lambda \partial_\lambda h_{\mu\nu} + \frac{1}{2}h_{\mu\lambda}\Big(\partial_\nu \zeta^\lambda - \partial^\lambda \zeta_\nu\Big) + \frac{1}{2}h_{\nu\lambda}\Big(\partial_\mu \zeta^\lambda - \partial^\lambda \zeta_\mu\Big). \quad (4\text{-}8)$$

This transformation says that the gravitational field is a field whose conformal dimension is zero. In the transformation of the conformal-factor field ϕ, there is a shift term that does not contain the field, which is not found in normal conformal transformations and shows that this transformation is derived from diffeomorphism. The generators of each transformation are constructed using energy-momentum tensors derived from the Riegert and Weyl actions, respectively.

Physical quantities must be invariant under the conformal transformation (4-8). One of the features of this transformation is that it involves the field. In ordinary gauge transformations, the right side at the zeroth order of perturbation depends only on gauge parameters, not on fields. Under such transformations, modes that compose fields do not mix with each other so that ghost modes become gauge invariant as they are. On the other hand, under the transformation (4-8), modes in the field are mixed with each other, and it can be shown that all ghost modes do not be gauge invariant in their own right. Hence, the presence of the BRST conformal invariance restricts physical quantities greatly, even though the gauge parameters have

[4] All local gauge degrees of freedom ξ^μ other than the conformal Killing vector ζ^μ are not considered here as they were used to eliminate field degrees of freedom as much as possible, such as by imposing a radiation gauge-fixing condition.

only 15 degrees of freedom, and solves the ghost problem from a different perspective than before.

The ghost modes are not physical quantities in themselves, but they are indispensable elements for diffeomorphism invariance to hold. Physical states of quantum gravity are expressed as states in which whole Hamiltonian vanishes exactly, as described in the next chapter. The BRST conformal invariance truly defines such diffeomorphism invariant states that appear in the ultraviolet limit far beyond the Planck scale. Also, it is due to the ghost modes that such states can exist as non-trivial states with entropy, not a vacuum state with nothing.

The unitarity issue from the standpoint of conformal field theory will be explained in Chapter 6 as well. The property that the BRST conformal invariance is realized in the high energy limit is called "asymptotic background freedom",[5] and the physical implications it suggests will be discussed again in Chapter 7. The quantum theory of gravity with such a property is called "asymptotically background-free quantum gravity".

On Historical Background

Here is mentioning a historical background of the BRST conformal invariance briefly. This quantization method was started by Polyakov in the first half of the 1980s for the case that spacetime is two-dimensional, and then led to the discovery of an exact solution of two-dimensional quantum gravity by Polyakov, Knizhnik, and Zamolodchikov in the latter half of the the same decade. Following that discovery, Distler and Kawai, David and others formulated two-dimensional quantum gravity as a quantum Liouville field theory. Its action is the Liouville action, which is a two-dimensional Wess-Zumino action originally discovered by Polyalov, and the Riegert action was found as a four-dimensional counterpart of that.

In two dimensions, the traceless tensor field has two components, which

[5] This is a concept similar to "asymptotic safety" proposed by Weinberg in 1979 in the sense that the existence of ultraviolet fixed points is premised, though the approach is different. It will be discussed in the last section of Chapter 10.

can be eliminated using the gauge degrees of freedom of diffeomorphism, thus quantum gravity can be expressed as a theory of the conformal-factor field ϕ only. In addition, the Einstein-Hilbert action, which is the action I in this case, becomes a topological invariant (Euler characteristic), and quantization is represented as a path integral that performs over all possible gravitational field-configurations in each topology and then sums up for topology.

Chapter 5

What is Hamiltonian Constraint?

It is widely believed that diffeomorphism invariance, as well as gauge invariance in gauge fields, is a fundamental principle that holds from classical theory to quantum theory, from an infinitesimal world to a cosmic-scale world. There are some researchers who do not allow essential modifications for the Einstein equation as a divine equation, but what should hold is diffeomorphism invariance. Although the Einstein equation is incompatible with quantum theory, diffeomorphism invariance and quantization of gravity are never in conflict. However, it should be noted that there are some sensuous differences from ordinary quantization. One of them is what is commonly called the Hamiltonian constraint. When considering quantum gravity, you should pay more attention to the existence of this condition. The background-metric independence shown in the previous chapter exactly expresses that.

Normally, gravity is thought to be generated where matter is present. To emphasize this view, the Einstein equation is expressed in the form of (2-1). However, this traditional expression hides an essence of diffeomorphism invariance. The Einstein equation is the equation of motion for the gravitational field, that is derived as an equation in which a gravitational variation of the whole action including the Einstein-Hilbert action vanishes. On the other hand, gravity is coupled to all fields, and quantities obtained by varying the action with respect to the gravitational field are nothing but energy-momentum tensors. In other words, letting I be the whole action and denoting its energy-momentum tensor by $T_{\mu\nu}\,(= g_{\mu\lambda}g_{\nu\sigma}T^{\lambda\sigma})$, then the equation of motion is given by the variational principle as $\delta I/\delta g_{\mu\nu} = \sqrt{-g}\,T^{\mu\nu}/2 = 0$, thus expressing the Einstein equation as

$$M_{\mathrm{P}}^2\left(-R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R\right) + T^{\mathrm{M}}_{\mu\nu} = 0 \qquad (5\text{-}1)$$

is essentially correct, where $M_{\mathrm{P}} = 1/\sqrt{8\pi G}$ is the reduced Planck mass, defined by dividing the Planck mass m_{pl} by $\sqrt{8\pi}$. In particular, the time-time component of the energy-momentum tensor is the Hamiltonian density, and the Hamiltonian is given by integrating it over space volume. That is, the Hamiltonian disappears.

In quantum field theory, equations of motion can be expressed by an identity called the Schwinger-Dyson equation in the path integral method. Letting φ be a field variable in a certain spacetime and I be its action, it is written as

$$\int [d\varphi] \frac{\delta}{\delta\varphi(x)} \left(e^{iI(\varphi)} \cdots \right) = i \left\langle \frac{\delta I(\varphi)}{\delta\varphi(x)} \cdots \right\rangle = 0,$$

where $\langle \cdots \rangle = \int [d\varphi] \cdots e^{iI(\varphi)}$ represents a vacuum expectation value, or correlation function, and the dots denote fields that are in a different location than x here. Taking a free scalar field as an example, this equation shows that the equation of motion, called the Klein-Gordon equation, holds in the form of the expectation value. If there are interactions, then the equation of motion involves terms from quantum corrections.

The identity is a generalization to functional integrals of the fact that $\int dv\, \partial f(v)/\partial v = 0$ holds in ordinary integrals when $f(v)$ is a function that disappears at infinity or is defined in a closed space. That is, letting the integral variable be the field φ instead of v, and replacing the ordinary integral and derivative with those of functionals. At this time, it is assumed that the field disappears at infinity or is defined in a compact spacetime without boundaries.

Let us apply the Schwinger-Dyson equation to the gravitational field. Putting aside the issue with path integral measures for a moment, the identity can be intuitively expressed as

$$\int [dg] \frac{\delta}{\delta g_{\mu\nu}(x)} \left(e^{iI(g)} \cdots \right) = \frac{i}{2} \langle \sqrt{-g}\, T^{\mu\nu}(x) \cdots \rangle = 0. \qquad (5\text{-}2)$$

That is, the energy-momentum tensor including quantum corrections exactly vanishes. This identity shows that the theory is invariant under a change of the metric tensor field, which is alternative expression of the background-metric independence. In order to define it concretely, it is necessary to correctly formulate the path integral in the diffeomorphism invariant manner, as shown in the previous chapter.

Early attempts to formulate a quantum theory of gravity in canonical quantization result in equations such as the Hamiltonian and momentum constraints and also the Wheeler-DeWitt equation, but they are essentially the same as Eq. (5-2). However, it should be noted that many of equations called so have been discussed within the framework of Einstein's theory of gravity using the ADM formalism initiated by Arnowitt, Deser, and Misner. Therefore, they still have many of the problems mentioned in Chapter 3. In addition, since canonical quantization itself treats time specially when setting commutation relations, there is also a problem that diffeomorphism invariance becomes obscure.[1] Moreover, from the standpoint of path integral, there is the fact that these equations have not been derived with care for the measure. Therefore, in (5-2), the path integral measure of the gravitational field is written by omitting the subscript g like $[dg]$, instead of $[dg]_g$ introduced in the previous chapter. This measure indeterminacy is also the main cause of difficulties in canonical quantization known as the "operator ordering problem".

Now, rewrite equation (5-2) to a correct one and proceed the argument. The main part of the action I is given by the conformally invariant fourth-derivative action consisting of the Weyl action and the Euler term, and lower-derivative actions such as the Einstein-Hilbert action and matter actions are added to it. Rewrite the diffeomorphism invariant measure into the practical one defined in flat background as in (4-5), and then denote the whole action as $I_{\rm QG}$, including the Wess-Zumino action S as a Jacobian required for its rewriting, as in (4-6). The partition function can be written as (4-7), thus

[1] Two-dimensional cases where time and space are of the same dimension are special, so diffeomorphism invariance may hold even with canonical quantization.

the Schwinger-Dyson equation is expressed as

$$\int [d\phi dh]_\eta \frac{\delta}{\delta\phi(x)} e^{iI_{\rm QG}} = i\big\langle \sqrt{-g}\, T^\lambda{}_\lambda(x)\big\rangle = 0,$$
$$\int [d\phi dh]_\eta \frac{\delta}{\delta h^\nu{}_\mu(x)} e^{iI_{\rm QG}} = \frac{i}{2}\Big\langle \sqrt{-g}\left(T^\mu{}_\nu(x) - \frac{1}{4}\delta^\mu{}_\nu T^\lambda{}_\lambda(x)\right)\Big\rangle = 0. \quad (5\text{-}3)$$

These are collectively written as $\langle\sqrt{-g}\, T^\mu{}_\nu(x)\rangle = 0$. From the way of perturbation expansion (4-2) and (4-3), this equation holds exactly for the conformal-factor field, while perturbatively for the traceless tensor field.

The equation of motion (5-3) is an expression of the background-metric independence, and the expectation value involves quantum corrections. In other words, the equation of motion can be expressed as an equation that variation of the effective action with respect to the background metric vanishes. That has the following structure:

$$T^{(4)}_{\mu\nu} + M_{\rm P}^2\left(-R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R\right) + T^{\rm M}_{\mu\nu} = 0, \quad (5\text{-}4)$$

where $T^{(4)}_{\mu\nu}$ is a fourth derivative gravitational term in which contributions from the Wess-Zumino action, the Weyl action, and quantum corrections generated by quantizing them are incorporated.

Physical meanings of diffeomorphism invariance

Now, what does the vanishing of the whole energy-momentum tensor mean as physics? Two important things are addressed here:

I. One is that since the whole Hamiltonian vanishes, there is no time globally defined in the whole system as assumed in ordinary quantum theory. That is, there is no longer usual notion of time evolution like the Schrödinger equation.

This is also true in classical theory, as can be seen by rewriting the Einstein equation as in (5-1). Time progresses differently depending on strength of the gravitational field. In places where the gravitational field is weak, such as on the earth, changes in gravitational potentials are monotonic, and thus it is possible to measure differences in the progress of time depending on locations, but time does not proceed in the same way throughout the whole system at all.

In quantum spacetime where gravity is completely fluctuating, the concept of time itself disappears. The property called the background-metric independence states that even if the Minkowski spacetime is selected as a reference background, the reference no longer makes sense if the conformal factor fluctuates greatly around it. In other words, physical time and distance defined by the metric tensor (4-2) fluctuate, making them practically impossible to measure. In the previous chapter, it has been shown that this property will be realized as the BRST conformal invariance in the high-energy limit far beyond the Planck scale.

Approximate time is given as a dynamical change of the gravitational field that results from solving the equation of motion (5-4) where the energy-momentum tensor vanishes. As the conformal factor, the scale factor in cosmology, begins to increase monotonically, it becomes time in the entire universe, which is a process where the conformal invariance gradually breaks due to physical scales such as the Planck mass. The flow of time will begin with inflation. As will be shown later, during that period, the fluctuations with sizes that are involved in determining structures of the universe reduce, thus time becomes an entity as a uniform flow. The reduced fluctuations are then inherited by the Friedmann solution, which is the solution of the Einstein equation, and the time also continues.

It should be noted here that there is a non-trivial spacetime solution despite thinking of a solution where the Hamiltonian vanishes. In other words, the state with zero eigenvalue of the Hamiltonian is not uniquely determined, but it changes, so that time is created. Why such a change occurs is from the fact that the Einstein-Hilbert action is not bounded below. If this action were positive-definite, like that of matter fields, then there would be no dynamical solution and no time would occur. This is also linked with the statement in Chapters 3 and 4 that ghost modes are necessary to make quantum gravity states with zero Hamiltonian.

More notably, the equation (5-4) holds even if the $T^{\mathrm{M}}_{\mu\nu}$ term which is a source of mass disappears, due to the presence of the $T^{(4)}_{\mu\nu}$ term. In other words, it is possible to describe a world in which there is no matter and only gravitational fluctuations exist. In this way, you can discuss spacetime with a completely different perspective from that of the Einstein equation where matter creates gravity. Furthermore, this equation can also describe a transition process from such a world of only gravity to a world with matters. An excited state and inflation model of quantum gravity, in which such a process plays an important role, will be described in detail in later chapters.

Since there is no absolute time, the normal meaning of "conservation" does not hold. Conservation means that it does not change during the evolution of the universe, and will be expressed here as renormalization group invariance (see Chapter 10). The energy-momentum tensor is exactly a renormalization group invariant. Entropy of the universe, given by the effective action because the whole Hamiltonian vanishes, is also conserved as a renormalization group invariant. This indicates that the entropy of the universe is originated from quantum fluctuations of gravity, and since it is preserved constant, adiabatic conditions premised in Gamow's Hot Big Bang model are strictly correct.

II. Another significant fact indicated by the equation of motion (5-4) is that there is no zero-point energy.

Before understanding this fact, it should be pointed out that the energy-momentum tensor is a "normal product", which is a composite field that behaves as a finite operator in correlation functions. For free fields, it refers to normal ordered operators. In ordinary quantum field theory, it can be generally shown as follows. Consider a finite correlation function $\langle \cdots \rangle = \int [d\varphi]_g \cdots e^{iI(\varphi,g)}$ in curved spacetime. Then, since a quantity obtained by applying variation with respect to the metric tensor to the correlation function is apparently finite, it can be shown that the energy-momentum tensor $T^{\mu\nu}$ including quantum corrections is a finite operator as[2]

$$\frac{\delta}{\delta g_{\mu\nu}(x)}\langle\cdots\rangle = \frac{i}{2}\langle\sqrt{-g}T^{\mu\nu}(x)\cdots\rangle = \text{finite}. \tag{5-5}$$

This is a general conclusion drawn from diffeomorphism invariance, and can be clearly shown by performing quantization with the path integral method that preserves the invariance strictly.

Adler, Collins, and Duncan have shown in 1977 that $T^{\mu\nu}$ is actually represented by a normal product, as part of the study of conformal anomalies. Brown and Collins, and then Hathrell further developed their work and derived peculiar couped renormalization group equations using the fact that the energy-momentum tensor is not only a normal product but also a renormalization group invariant. Recently, it has been shown that solving their renormalization group equations determines the form of the gravitational action. This work will be covered in Chapter 10.

In ordinary quantum field theory, the energy-momentum tensor is only shown to be finite as in (5-5), but even stronger in quantum gravity, it vanishes exactly as shown in (5-3), or (5-4). This means that there is no zero-point energy.

In general, it may be said that when canonical quantization is employed in quantum field theory, applying normal ordering is a task to recover diffeomorphism invariance. In fact, it can be shown that when the quantum gravity theory (4-7) is quantized using the canonical quantization method to obtain generators of the BRST conformal symmetry, normal ordering has to be applied to them in order for the conformal algebra to close properly, that is, for diffeomorphism invariance to hold.

The fact that there is no zero-point energy is related to problems with the origin of fluctuations in the early universe. In Einstein's theory of gravity, the source of matters that makes up the present universe must still be some

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[2] An important point to note as a renormalizable quantum field theory is that the $T^{\mu\nu}$ defined by this variation is written in terms of bare quantities before renormalization, that is, is a renormalization group invariant. The expression (5-5) says that $T^{\mu\nu}$ can be rewritten to a normal product defined by renormalized quantities, in which all of contributions concerning conformal anomalies are contained.

kind of matter, and when considering inflation theory, an unknown scalar field called inflaton is introduced for that purpose. At that time, the origin of fluctuations is explained as zero-point energy of the inflaton. However, in quantum theory of gravity, such zero-point energy does not exist, instead excited states of quantum gravity will be the source of everything.

What is called the cosmological constant problem is also related to this fact. This problem comes from the fact that calculating zero-point energy by introducing an ultraviolet cutoff at the Planck scale, a large cosmological constant of the fourth power of the Planck mass is obtained, which does not agree with observations at all. In the quantum gravity theory describing a world beyond the Planck scale, however, the root cause of the problem disappears.

In the first place, the appearance of a large cosmological constant of the cutoff to the fourth power depends on regularization methods. For example, it does not arise if you use dimensional regularization that preserves diffeomorphism invariance. Nevertheless, if you dare to introduce a cutoff at the Planck scale, it means that such a setting has a physical meaning. In other words, it is the same as assuming that there is no world beyond the Planck scale and there is no field there. From this, it can be seen that it is necessary to describe the world beyond the Planck scale in order to discuss the cosmological constant problem. This issue, which is also related to renormalizability of quantum gravity, will be revisited again in Chapter 11.

Chapter 6

Conformal Field Theory and Unitarity

Quantum field theory in which vacuum states are invariant under conformal transformations is called conformal field theory (CFT). It is a representative of non-perturbative quantum field theories. Conformal invariance has been studied as an important symmetries involved in physical phenomena such as phase transitions. As nontrivial places where it is realized, critical points in statistical systems that show critical phenomena and fixed-points in quantum field theory where beta functions disappear are well-known. As one of them, the ultraviolet fixed-point of quantum gravity has been added.

In quantum field theory, even if an action is conformally invariant, conformal invariance is normally broken by quantization. The conformal anomaly already mentioned in Chapter 4 stands for this fact. It represents the existence of a physical scale, which can be used to describe physical phenomena (see Chapter 7). From this, it can be seen that it is quite nontrivial that conformal invariance appears at the quantum level.

The conformal transformation is a transformation in which when coordinates are transformed as $x^\mu \to x'^\mu$, a line element ds representing distance changes by a conformal factor so that

$$ds^2 \to ds'^2 = \Omega^2(x) ds^2. \qquad (6\text{-}1)$$

At this time, the metric tensor that determines distance remains fixed. In constarct, it can also be defined as the Weyl transformation that extends the metric tensor by the conformal factor without changing the coordinates, but it is originally defined as the coordinate transformation.

Conformal invariance has come to be discussed in physics at the beginning of the 20th century. First, it was shown that the Maxwell equation is conformally invariant. Later, the Dirac equation for massless fermions was shown to be conformally invariant and was actively studied by physicists and mathematicians. The Weyl curvature tensor (4-1) was also discovered around the same time. In those days, it was still discussed as a classical symmetry.

Entering the second half of the last century, conformal invariance came to be recognized as an essential nature in understanding quantum field theory and critical phenomena in statistical systems. Although I am not sure when the concept of "universality" in critical phenomena was born, the conformal bootstrap method proposed by Polyakov, Ferrara, Gatto, Grillo, Mack and others in the 1970s is a method that presupposes the existence of the universality. This method was originally conceived to understand hadron physics based on scaling hypotheses predicted from the short-distance behavior of strong interactions, and was an idea that be descended from an algebraic approach using current algebras and scattering matrices in the 1960s. It has then developed as a method of understanding critical phenomena from conformal invariance and unitarity without using actions or Hamiltonians.

In the mid-1980s, Belavin, Polyakov, and Zamolodchikov discovered degenerate representations in two-dimensional conformal field theory, demonstrating that the conformal bootstrap method is a quite useful method in two dimensions. Furthermore, Friedan, Qiu, and Shenker showed that physical critical exponents can be determined by imposing conditions of unitarity (positive-definiteness) further, so that two-dimensional critical phenomena have been able to be classified. The results are in perfect agreement with critical exponents of a series of exactly solvable lattice models, including the Ising model, solved by Andrews, Baxter and Forrester at the same time.

Entering this century, the conformal bootstrap method has made great strides and it has become clear that it works even in three or more dimensions. It is significant as there is no exactly solved lattice model, including the Ising model, in three or more dimensions. Until then, two dimensions have been considered special because conformal algebra becomes infinite dimensional. However, this study suggests that even in higher dimensional conformal field theory, where conformal algebra is finite dimensional, critical exponents can be determined by imposing conformal invariance and unitarity. The following is a brief summary of recent developments in conformal field theory, and then argue unitarity in quantum gravity.

Critical phenomena and conformal field theory

First, briefly touch on the relationship between conformal field theory and critical phenomena. Here, conformal field theory defined in Euclidean space is considered, which can be returned to the theory in Minkowski spacetime by making an analytic continuation, called the Wick rotation, (Osterwalder-Schrader reconstruction theorem). In general, it is believed that a D-dimensional classical lattice-statistical model such as the Ising model provides a conformal field theory on D-dimensional Euclidean space when a continuum limit is taken just above critical points.

Let T be a variable such as temperature that controls critical phenomena of a statistical system, and T_c be a critical point. In general, denoting a field involved in critical phenomena to be O, correlation functions far from the critical point decay exponentially like

$$\langle O(x)O(0)\rangle \sim e^{-|x|/\xi}$$

where $|x| = \sqrt{x^2}$ and ξ is a correlation length. At the critical point $T = T_c$, the correlation length becomes infinite, and the correlation function behaves in a power-law like

$$\langle O(x)O(0)\rangle = \frac{1}{|x|^{2\Delta}}.$$

This indicates that conformal invariance has appeared. Here, Δ is called conformal dimension of the field O, which must be non-negative so that the correlation function does not diverge at long distance.

In the following, let $I_{\rm CFT}$ be an action of conformal field theory that appears just above the critical point. Usually, that is unknown in most cases except in free fields. Therefore, it can be said that conformal field theory is a discipline for understanding critical phenomena from conformal invariance and unitarity without relying on actions.

Critical phenomena are classified by exponents that indicates how to approach the critical point, when considering a small perturbations from it.

For example, consider perturbation by a field O with a conformal dimension Δ smaller than D, that is, a relevant field. If deviation from the critical point is expressed by a dimensionless parameter t ($\ll 1$), then the action is deformed incorporating the perturbation as

$$I_{\mathrm{CFT}} \to I_{\mathrm{CFT}} - ta^{\Delta-D} \int d^D x \, O(x),$$

where a is an arbitrary length scale introduced to make up for missing dimensions, which corresponds to a lattice spacing in statistical models. Here, the only physical scale is the correlation length ξ, and so identifying $ta^{\Delta-D}$ with $\xi^{\Delta-D}$ from dimensional analysis yields[1]

$$\xi \sim at^{-1/(D-\Delta)}. \tag{6-2}$$

For example, considering energy operator ϵ as O represents a perturbation by temperature $t = |T - T_c|/T_c$. Letting Δ_ϵ be its conformal dimension, since a corresponding famous critical exponent ν is defined by $\xi \sim at^{-\nu}$, a relation $\nu = 1/(D - \Delta_\epsilon)$ is obtained. In addition, critical exponents of specific heat and magnetization are also determined from conformal dimensions and structure constants (operator product expansion coefficients) of fields defining conformal field theory.

Unitarity and critical exponents

In quantum field theory for elementary particles, unitarity means that scattering matrices connecting in- and out-going free-particle states in interactive processes are given by unitary matrices. On the other hand, since conformal field theory does not consider such asymptotically-free particle states,

[1] A procedure such as (6-2) that defines physical scales by associating dimensionless parameters with an arbitrary scale is called "dimensional transmutation". In the standpoint of renormalizable quantum field theory, the correlation length is a physical scale, that is, a renormalization group invariant independent of the scale a introduced arbitrarily as $d\xi/da = 0$. Substituting (6-2) into this condition and solving it, it can be seen that a beta function of t becomes negative as $\beta_t = -adt/da = -(D - \Delta)t$ from the relevant condition, and thus the system approaches the critical point of $t \to 0$ at the continuum (ultraviolet) limit of $a \to 0$. Renormalization group invariants will be touched upon later when defining physical constants in QCD and quantum gravity theory.

scattering matrices are not defined. Therefore, unitarity in conformal field theory will be expressed such as reality of fields and positive-definiteness of actions.

Quantum field theory with an action I formulated in Euclidean space can be regarded as a statistical system with e^{-I} as the Boltzmann weight. In order that the weight does not diverge, the action has to be bounded below. Also, in order for the weight to be positive-definite so that it can be interpreted as existence probability, the action is required to be real. In general, unitarity in a system in which the action contains a complex number is broken. Hence, the action that gives such a physical weight has to be real and bounded below. Furthermore, field configurations in which the action positively diverges become unphysical because their Boltzmann weights vanish so that they have zero existence probability. Therefore, soliton and instanton solutions that have field configurations where the action becomes infinite are not taken into account. It is also the same logic that solutions with spacetime singularities can be eliminated by introducing the fourth derivative gravitational action including the Riemann curvature tensor described in Chapter 3.

However, in many cases, the action is not known in conformal field theory. Therefore, unitarity conditions that do not depend on the action are required. First, as already mentioned, there is a condition that conformal dimensions have to be non-negative. The most important are conditions that express reality of fields. One of them is that two-point correlation functions of real fields become positive-definite. Since the square of a real number is positive, this is a condition that should naturally hold unless physically suspicious operations are made. On the contrary, if two-point correlation functions become negative when considering real fields, it suggests that the action is not positive-definite. In this case, the path integral will diverge, thus in order to regularize it the reality of fields is sacrificed.

By imposing this positive-definiteness condition that real fields should satisfy, lower limits of physically allowed conformal dimensions are determined, so that stronger conditions than the non-negativity are given. This conditions are called unitarity bounds. For example, in 4 dimensions, the conformal dimension of a scalar field is 1 or more. The lower limit 1 is the dimension of a free scalar field. That of a vector field is 3 or more, and the lower limit 3 is the dimension that conserved currents should have. For a second-order symmetric tensor field, it is 4 or more, and the lower limit 4 represents the dimension of energy-momentum tensors to be conserved. Here note that a free electromagnetic vector field does not satisfy this unitarity condition because its dimension is 1. This is due to the fact that the vector field itself is not gauge invariant. Hence, consider its field strength $F_{\mu\nu}$ that is gauge invariant. It is an antisymmetric tensor field and the unitarity bound in this case is 2 or more, so that the condition is satisfied. In this way, the condition must be imposed on physical quantities.

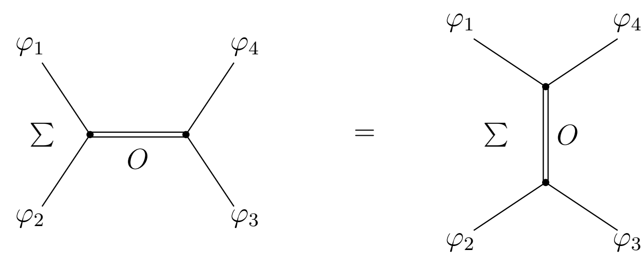

Figure 6-1: Crossing symmetry.

Since the positive-definiteness of two-point functions gives only lower limits of conformal dimensions, this condition cannot determine anything concrete. Therefore, reality on three-point correlation functions has come to be taken into account as a new condition. Three-point functions are almost determined from conformal invariance except overall coefficients. The coefficients are called structure constants because they determine structures of operator product expansions expanding products of fields by other fields. If reality of fields holds, structure constants should be real numbers. In fact, all of structure constants in a unitary series of two-dimensional conformal field theory are given by real numbers. On the other hand, it is known that structure constants of the Lee-Yang model, which is a representative of non-unitary models, become complex numbers. Recent studies have shown that upper limits of conformal dimensions can be found by imposing this condition even in three or more dimensional conformal field theory.

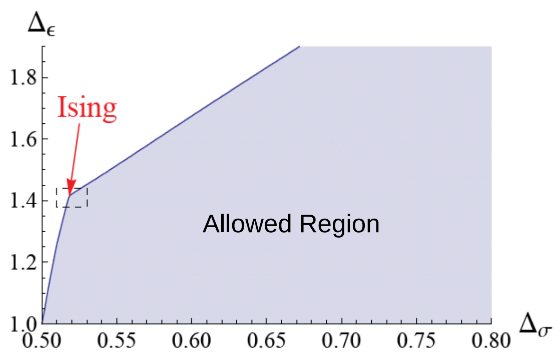

Figure 6-2: The result of numerical calculations by the conformal bootstrap method in three dimensions, where $\Delta = \Delta_\epsilon$ and $d = \Delta_\sigma$. The pale blue part is the region representing a pair of conformal dimensions allowed from unitarity. The critical exponents of the three-dimensional Ising model appear at the bent point on the boundary that is the upper limit of the allowed region. [S. El-Showk, M. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, Phys. Rev. D **86** (2012) 025022.]

A structure constant is a real number, that is, its square is positive. The recent results of finding critical exponents of the three-dimensional Ising model by imposing this condition will be briefly described below. This is the first successful application of the conformal bootstrap method developed by Polyakov and others to other than two dimensions.

Let us consider a scalar field φ_d with conformal dimension d, and assume that its operator product expansion is given by

$$\varphi_d(x)\,\varphi_d(0) = \frac{1}{|x|^{2d}} + \frac{f_{dd\Delta}}{|x|^{2d-\Delta}}\,O_\Delta(0) + \cdots .$$

Later, φ_d is identified with a spin operator σ in the Ising model. The first term on the right-hand side represents the two-point correlation function of φ_d, and its coefficient is a positive number due to unitarity, normalized to 1 here. The field O_Δ is a scalar with conformal dimension Δ, which is

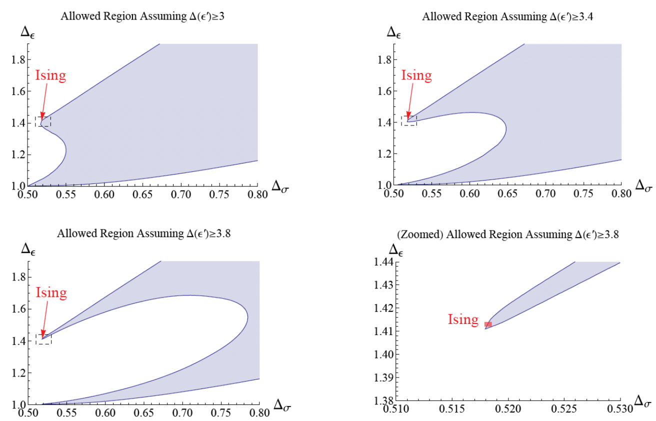

Figure 6-3: More detailed analysis on the allowed region in three dimensions. Here, $\Delta' = \Delta(\epsilon')$. From the top, the conditions are strengthened as $\Delta' \geq 3$, $\Delta' \geq 3.4$, and $\Delta' \geq 3.8$. The last figure is an enlargement of the vicinity of the Ising model point in the third figure. [S. El-Showk, M. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, Phys. Rev. D **86** (2012) 025022.]

later identified with the energy operator ϵ, and has the smallest conformal dimension among scalar fields that appears in the right-hand side satisfying the unitarity bound. The dots represent contributions of other fields, and in order for the operator product expansion to be complete, not only scalar fields but also completely symmetric tensor fields with integer spin are required. The expansion coefficient $f_{dd\Delta}$ is the structure constant between O_Δ and two φ_d. Similarly, structure constants between completely symmetric tensor fields and two φ_d are defined. The unitarity condition is that all of these structure constants are real numbers.

In order to set this condition in a form that can be handled numerically, consider four-point correlation functions of the scalar field φ_d. The discovery by Dolan, Osborn and Petkou that functions called conformal blocks can be written in a product of Gauss hypergeometric series gives the basis for this achievement. Using them, you can determine the four-point correlation function completely except for information on the structure constants. The four-point correlation function can be decomposed into products of three-point correlation functions, then a crossing symmetry shown in Fig. 6-1 holds because there is an arbitrariness on how to decompose it. Using this property, information of the structure constant $f_{dd\Delta}$ can be successfully extracted. Imposing a condition that its squared is non-negative restricts values of Δ allowed for dimension d. Fig. 6-2 shows the result as a graph of $(d, \Delta) = (\Delta_\sigma, \Delta_\epsilon)$.

At this stage, all values in the pale blue region are still allowed continuously. Therefore, let us add more restrictions by assuming that conformal dimensions of fields appearing in the operator product expansion have a discrete structure as in the case of two-dimensional conformal field theory. Fig. 6-3 shows results of the same analysis under an assumption that there is a discontinuous gap between the conformal dimension Δ of the scalar field O_Δ and a conformal dimension $\Delta' (> \Delta)$ of next scalar field. Increasing the restriction on Δ', a pair of values $d\,(= \Delta_\sigma) = 0.5182(3)$ and $\Delta\,(= \Delta_\epsilon) = 1.413(1)$ emerges as a peculiar point. These values are in agreement with the numerical calculation results of the three-dimensional Ising model by the Monte Carlo method. Further, imposing such restrictions to conformal dimensions of higher scalar fields and tensor fields, the allowed region of pale blue will eventually become like an isolated island, and will be narrowed down to this Ising point.

This was a surprising result. It has been thought that why two-dimensional conformal field theory can be solved is that conformal algebra is infinite dimensional. In higher dimensions, conformal invariance were not thought to be strong enough to determine conformal dimensions, because the algebra becomes finite dimensional, but it was not so.

Conformal invariance and unitarity in quantum gravity

What discussing in this book is a special conformal invariance that appears in the high-energy limit of quantum gravity. Whereas normal conformal invariance refers to vacua being conformally invariant, that of quantum gravity demands that since it arises as part of diffeomorphism invariance, that is, a gauge symmetry, all quantities have to be conformally invariant, as mentioned in Chapter 4. Therefore, it is called the BRST conformal invariance. Normal conformal invariance indicates that the system is independent of a particular scale, but the metric that defines distance exists as certain. On the other hand, that of quantum gravity is a conformal invariance caused by fluctuations of distance itself, so there is no scale in the true sense.

Here, the algebraic difference between the BRST conformal invariance and normal conformal invariance is summarized, once again. Unlike the conformal transformation (6-1), diffeomorphism is a transformation in which the metric tensor $g_{\mu\nu}$ is also transformed together with the coordinate transformation to preserve the line element squared $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ invariant like

$$ds'^2 = ds^2.$$

Therefore, it is difficult to intuitively understand that diffeomorphism invariance includes the conformal invariance. The BRST conformal invariance is a hidden symmetry that can be seen by rewriting the quantum gravity theory into a quantum field theory defined on a background spacetime as in (4-7). At this time, the conformal transformation can be expressed as the Weyl rescaling $d\hat{s}^2 \to d\hat{s}'^2 = e^{2\omega} d\hat{s}^2$ for the background line-element squared $d\hat{s}^2 = \eta_{\mu\nu} dx^\mu dx^\nu$. On the other hand, since the original metric tensor $g_{\mu\nu}$ is given by (4-2), the Wey rescaling of the background metric can be absorbed by changing the conformal-factor field as $\phi \to \phi' = \phi - \omega$ (see also footnote 2 in Chapter 4). In quantum gravity, this shift change is just a rewrite of the integral variable. Since the integral region of ϕ is defined by $(-\infty, \infty)$, the path integral measure is invariant under the shift change as $[d\phi']_\eta = [d\phi]_\eta$. Consequently, it shows that the theory does not change even if the background spacetime is transformed conformally.

Thus, the key point for achieving the BRST conformal invariance is that the path integral over the conformal-factor field ϕ is performed by using a practical measure defined on the background which is invariant under the shift change of ϕ. Note that the conformal transformation is here described by using the Weyl rescaling, while (4-8) given in Chapter 4 is expressed as a coordinate transformation with the background $d\hat{s}^2$ fixed.

The property that all of theories with different backgrounds connected to each other by conformal transformations are gauge equivalent comes from the fact that the original theory is written by the full metric tensor $g_{\mu\nu}$. That is, diffeomorphism invariance is nothing but expressed in an another form. This is an algebraic representation of the background-metric independence, showing that the conformal factor of spacetime fluctuates greatly in a nonperturbative manner, so that physical distance cannot be measured substantially.

In a world beyond the Planck scale, such background-metric independence will be realized, thus there is no particle picture propagating in a particular background. Therefore, scattering matrices cannot be defined either. Such quantities can be defined only in a classical spacetime after a spacetime phase transition described in the next chapter.

The unitarity in the quantum gravity theory is expressed by whether or not reality of physical fields is maintained, following that of conformal field theory. The ghost modes are required to achieve the BRST conformal invariance, but they are not physical entities because they are not invariant under the BRST conformal transformation. In fact, all BRST conformal invariants are given as real composite scalar fields. Its reality will be ensured to hold because the fourth-derivative quantum gravity action given by $I_{\rm QG}$ (4-6) is positive-definite.

It should be noted here that the ghost mode containing in the Einstein-Hilbert action is the cause of its indefiniteness, but the ghost modes in the fourth-derivative gravitational action is not so. When performing quantization, the gravitational field is decomposed into various modes, but the action is positive-definite when viewed in the original field. This indicates that the mode decomposition is for convenience only. In fact, the individual modes are not independent, connected to each other by the gauge transformation.

Finally, describe how physical states corresponding to the real composite scalar physical fields are expressed in terms of conformal algebra. Let D be a generator of the dilatation when viewed as a conformal field theory, $M_{\mu\nu}$ and K_μ be generators of the Lorentz transformation and the special conformal transformation, respectively. The physical state $|\psi\rangle$ is then defined by the following condition:

$$D|\psi\rangle = 0, \quad M_{\mu\nu}|\psi\rangle = 0, \quad K_\mu|\psi\rangle = 0.$$

Further, descendant states obtained by applying generator of the translation P_μ to this state becomes unphysical (BRST-trivial). These conditions are nothing but the Hamiltonian and momentum constraints that represent diffeomorphism invariance. It has been shown that there are an infinite number of the physical states. As emphasized before, despite the vanishing Hamiltonian, the fact that such states exist infinitely and so there is entropy is due to the ghost mode which never appears as a physical state by itself.

In terms of conformal field theory, the physical state is a real primary scalar state with spin zero, whereas there is no tensor state. This is consistent with observations of CMB suggesting that scalar fluctuations were predominant in the early universe.

Chapter 7

Phase Transition in Quantum Gravity

It has been shown in the previous chapter that critical phenomena of the three-dimensional Ising model can be understood from the conditions of conformal invariance and unitarity without knowing its action or Hamiltonian. In two dimensions, this problem can be solved exactly, and a series of critical exponents including the Ising model have been classified. It is believed that there is universality for critical exponents, and a behavior near some critical point are thought to be represented by one of the critical exponents classified using conformal invariance. This is a very strong conclusion, but you can still determine which exponent corresponds to the Ising model only if the result is known in advance by another method. Also, not all physics can be understood even if the vicinity of the critical point is known.

Actually, in hadron physics, mass spectra cannot be understood unless you can describe strong interactions until regions where conformal invariance is broken. Therefore, it is necessary to introduce a field in order to describe such regions. The field that mediates strong interactions called gluon is represented by a non-Abelian gauge field (Yang-Mills field). It was not until the 1970s that its theory was formulated as quantum chromodynamics (QCD) which is one of renormalizable quantum field theories.

When ´t Hooft, one of the founders of QCD, was working on the problem of renormalizability for non-Abelian gauge field theory, many researchers were negatively thinking about renormalization. Although successful in quantum electrodynamics (QED), the method of making the theory finite by renormalizing divergences seems terribly artificial, and in those days many people rather aimed to build a finite theory without relying on quantum field theory. The widespread acceptance of the renormalization method has, after all, been due to the success of both QED and QCD. At the same time, it is also significant that the theory was developed based on the idea of renormalization group by Wilson and others. That is a tool for systematically investigating how parameters in the theory respond to a change of energy scale. The asymptotically free property that quarks and gluons show can be brilliantly explained using the renormalization group.

QCD has come to be considered applicable to hadron physics only after it was shown to be able to describe the asymptotically-free behavior verified experimentally for strong interactions. That is a property that strong interactions become relatively weak and quarks and gluons making up nucleus behave like free particles at the high-energy limit. Theoretically, it means that its beta function, which represents response of a coupling constant to a change in energy scale, becomes negative. For the achievement discovering that QCD has this property, Politzer, Gross, and Wilczek were awarded the Nobel Prize in 2004.

However, before their discovery, 't Hooft also allegedly showed that the beta function is negative and understood its physical implications, though he did not published it as a treatise. Since the Nobel Prize can be won by only up to three persons, in order to settle this trouble, first, the Nobel Prize was awarded in 1999 to 't Hooft and his supervisor Veltman for an achievement that elucidates quantum structures of the electroweak interaction described by another non-Abelian gauge field, and then the above three persons received the prize. This may be one of the reasons why the Nobel Prize was not awarded until the beginning of this century, even though the discovery of the asymptotic freedom was in the early 1970s.

Now, let us return to the topic of quantum gravity. It has been shown that the beta function of the coupling constant t in the renormalizable quantum gravity is also negative. It indicates that the coupling constant becomes smaller in the high-energy limit, similar to non-Abelian gauge theories. In the quantum gravity, the high-energy limit refers to entering a region beyond the Planck scale. Since the coupling constant t is a parameter introduced to control the traceless tensor mode $h^{\mu}{}_{\nu}$ of the gravitational field, the smaller t means that tensor fluctuations are smaller.

On the other hand, fluctuations of the conformal-factor field ϕ that controls physical distance remain large. Spacetime that appears in the high-energy limit is described by a special conformal field theory that expresses the background-metric independence, not by a simple free field theory. Therefore, this property is referred to as the asymptotic background freedom to distinguish it from the asymptotic freedom. The physical meanings of each will be explained below.

QCD phase transition

The shielding effect, which is well known in QED, refers to a phenomenon in which when an electric charge is present, electron-positron pairs are virtually generated around it, so that the charge is shielded and looks small when viewed from a distance.

The QCD confinement is interpreted as an opposite phenomenon of this charge shielding effect. The difference from QED is that not only quarks but also gluons have “color charges”. The electromagnetic fields do not interact with themselves directly, but gluons, which are non-Abelian gauge fields, interact with each other. Therefore, when a colored particle is present, not only quark-antiquark pairs but also gluon pairs are virtually generated around it. It is considered that the gluon pairs cause the opposite phenomenon of QED, the “anti-shielding effect”. That is, at short distance the color effect is diminished, so that strong interactions become weak, while at long distance it increases and the confinement occurs.

The strength of the interaction between colors, obtained by adding the above quantum corrections, is expressed by what is called the running coupling constant (or the effective coupling constant). Letting Q be an energy or momentum scale you are thinking of, its square is expressed as

$$\bar{\mathfrak{g}}^2(Q) \propto \frac{1}{\log(Q^2/\Lambda^2_{\mathrm{QCD}})}. \tag{7-1}$$

This expression is derived from the fact that the beta function is negative, and the proportionality coefficient is a positive number determined from that. The Λ_{QCD} is a physical energy scale peculiar to QCD, which is a free parameter that cannot be determined theoretically.

This represents the effective strength of the strong interaction at the

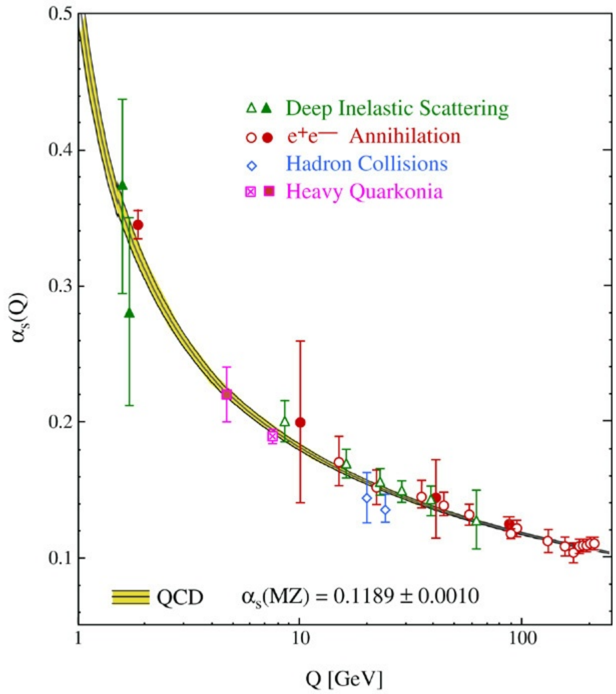

Figure 7-1: Behavior of the running coupling constant in QCD and summary of the measurement results, where $\alpha_s(Q) = \bar{\mathfrak{g}}^2(Q)/4\pi$. [G. Prosperi, M. Raciti, and C. Simolo, Progress in Particle and Nuclear Physics **58** (2007) 387.]

energy $Q\,(\geq \Lambda_{\rm QCD})$, and the higher the Q, the smaller the $\bar{\mathfrak{g}}(Q)$, indicating that it becomes weak (see Fig. 7-1). When the energy becomes low, it grows slowly in logarithm and diverges rapidly at $\Lambda_{\rm QCD}$. In other words, this QCD energy scale shows that the strong interaction becomes infinitely strong and the confinement occurs. The value is determined from experiments as $\Lambda_{\rm QCD} \simeq 210\,{\rm MeV}$. Its reciprocal $\xi_{\rm QCD}\,(=1/\Lambda_{\rm QCD})$ represents a correlation length of the strong interaction, which is $1\,{\rm fm} \simeq 10^{-15}\,{\rm m}$, a reference unit for nucleon size. A world lower than this energy can no longer be described using QCD. Therefore, $\Lambda_{\rm QCD}$ is called an infrared cutoff of QCD.

It is thought that such a phase transition occurs in the process of cosmic evolution. When temperature of the universe is higher than the QCD energy scale, quarks and gluons are moving around freely, but when temperature drops below that, they are trapped in nucleus and cannot move freely. That is, they move together as a hadron, a composite field of quarks that are tightly bound by gluons.

Technically, such dynamics can be expressed in the form that effective actions for gluons including quantum corrections disappear in proportion to the reciprocal of the running coupling constant (7-1) as it diverges. Therefore, there is no gluon kinetic term in the "low energy effective theory of QCD" that is described by introducing new field variables representing hadrons.

Spacetime phase transition

The same is true for the gravitational coupling constant t because its beta function becomes negative. As similar to (7-1), the corresponding running coupling constant squared, which represents the strength of effective gravitational interactions, is given like

$$\bar{t}^2(Q) \propto \frac{1}{\log(Q^2/\Lambda_{\rm QG}^2)}. \tag{7-2}$$

This expression predicts the existence of a new physical energy scale $\Lambda_{\rm QG}$ that determines dynamics of quantum gravity. The proportionality coefficient is determined from that of the beta function, given in Appendix B.

Here note that the physical energy or momentum Q being considered contains the gravitational field. Expressing the square of Q as the square of momentum, since the metric tensor is (4-2), it is given by

$$Q^2 = g^{\mu\nu} q_\mu q_\nu = \frac{q^2}{e^{2\phi}}, \tag{7-3}$$

where q^2 is the momentum squared defined by the metric tensor $\bar{g}_{\mu\nu}$ excluding the conformal factor. This is demanded from diffeomorphism invariance. Effective action of the quantum gravity is then expressed in terms of the running coupling constant (see (B-4) in Appendix B). Parts of the effective action that depend on the conformal-factor field ϕ through the physical momentum (7-3) are exactly the Wess-Zumino actions, S, yielding conformal anomalies, mentioned in Chapter 4.

This also apply to the QCD running coupling constant (7-1), then Q is replaced by (7-3) when considering in curved spacetime of (4-2). Hence, conformal anomalies are indispensable quantities that arise in order to preserve diffeomorphism invariance.

Now consider what the existence of the new gravitational energy scale $\Lambda_{\rm QG}$, different from the Planck mass and the cosmological constant, represent. The coupling constant t is a parameter introduced to control conformal dynamics of the fourth-derivative Weyl action. Considering in the same way as in the case of QCD, divergence of the running coupling constant (7-2) at $\Lambda_{\rm QG}$ indicates that the conformal dynamics disappear completely at that energy scale. More technically, it means that an effective action to the Weyl action with quantum corrections added disappears in proportion to the reciprocal of the running coupling constant squared.

The dynamics of the conformal-factor field is a part that does not exist in QCD. The conformally invariant fourth-derivative action does not contain the conformal-factor field, but its dynamics appear as the Wess-Zumino action for conformal anomaly. In particular, the Riegert action that appears at the zeroth order of the coupling constant t is an indispensable action for realizing the BRST conformal invariance. An effective action to the Riegert action with higher-order corrections of t added can also be written using the running coupling constant (7-2), and when it diverges, the effective action will also disappear. In this way, all of the fourth-derivative gravitational actions responsible for the conformal dynamics disappear at the energy scale $\Lambda_{\rm QG}$ and spacetime transitions to a new phase. In terms of the gravitational equation of motion (5-4), it is expressed by the disappearance of the $T^{(4)}_{\mu\nu}$ part.

Let us here see how the low-energy spacetime phase is described. In QCD, when the gluon field action, which is the second derivative, vanishes, dynamics of gluons completely disappears. On the other hand, in quantum gravity, even if the fourth-derivative gravitational action disappears, the second-derivative Einstein-Hilbert action remains, which gives the kinetic term of the gravitational field at low energy. That is, Einstein's theory of gravity appears as a low-energy effective theory. The gravitational field can then no longer be considered separately as the conformal-factor field and the traceless tensor field as in (4-2), and the gravitational field integrated as a composite field of these becomes a field variable.

Well then, what is the value of the new gravitational scale? This has to be decided from observations. The quantum gravity inflation scenario described in Chapter 9 suggests that $\Lambda_{\rm QG}$ is about 10^{17} GeV as a scale of the spacetime phase transition. Since this value is about 1/100 of the Planck mass $m_{\rm pl}$, the correlation length of the quantum gravity, $\xi_\Lambda = 1/\Lambda_{\rm QG}$, is about 100 times the Planck length. That is, if you go higher and higher in energy and explore shorter distances, Einstein's theory of gravity breaks down before you reach the Planck scale, and spacetime enters a new phase. That is why many problems with Einstein's theory of gravity can be solved.

The correlation length ξ_Λ gives a minimum distance that is measurable. Since physics at longer distance than this length is described by Einstein's theory of gravity, distance can be measured classically, but if distance is shorter than this, physics enters the region of quantum gravity, and so making the measurement becomes impossible because the gravitational field will begin to fluctuate greatly. In a microscopic world where the background-metric independence is realized as the BRST conformal invariance, physics does not depend on how to choose the metric tensor. It means that all of worlds with different scales become gauge equivalent, and thus physical distances are no longer defined there. In other word, it is a world where the concept of distance is lost. Therefore, distances shorter than ξ_Λ are virtually unmeasurable. In this way, in the renormalizable quantum gravity theory, spacetime substantially quantized by the distance ξ_Λ shall appear even though it is treated continuously without introducing an ultraviolet cutoff.

Chapter 8

Localized Excitations of Quantum Gravity

In the framework of Einstein's theory of gravity, a point-like particle with mass beyond the Planck mass becomes a black hole, because information of such a particle is hidden inside the horizon created by the mass itself and lost, as already pointed out in Chapter 2. On the other hand, the quantum gravity theory shows that there is a localized excited state with large mass exceeding the Planck mass.

It has already been mentioned that there is a new dynamical energy scale Λ_{QG} of about 10^{17} GeV in the quantum gravity theory. This value, two orders of magnitude smaller than the Planck energy scale, has been determined so that a scenario of inflation caused by quantum gravity effects described in the next chapter matches results of CMB observations. Here also, this value is employed.

Spacetime is then substantially quantized by the correlation length ξ_Λ ($= 1/\Lambda_{QG}$), about 100 times the Planck length. At distances longer than ξ_Λ, Einstein's theory of gravity can be applied to measure physical distance, but inside it, quantum fluctuations of the gravitational field become large, so that a background-free world where distance has no physical meaning will be realized. This suggests that if there is an excited state of quantum gravity, it will appear as a localized state whose diameter is given by the correlation length ξ_Λ. This picture is quite similar to a glueball state in QCD, but a big difference is that what is fluctuating inside is spacetime itself, which is hard to imagine it. From the outside, however, it seems to be a particle existing in classical spacetime.

As such an excited state, here consider a static spherical object whose mass m is about the Planck mass. Its classical horizon size, the Schwarzschild radius r_g ($= 2Gm$), becomes about the Planck length. It implies that when approaching the center of the object from a distance, quantum gravity effects begin to work from 100 times farther than the horizon. Hence, the horizon disappears behind quantum fluctuations.

The outside of such an object is expressed by the Schwarzschild solution, and the inside is described by the quantum gravity theory. Let us actually find such a state as a solution to the equation of motion (5-4). At that time, the matter energy-momentum tensor $T^{\mathrm{M}}_{\mu\nu}$ is set to zero, that is, a purely gravitational excited state is considered.

In order to find a concrete solution, the $T^{(4)}_{\mu\nu}$ part responsible for quantum gravity dynamics is boldly modeled by applying a kind of mean-field approximation as follows. As already mentioned before, the effective action incorporating quantum corrections can be written using the running coupling constant $\bar{t}\,(Q)$ (7-2). In coordinate space, the running coupling constant is a complicated differential operator acting on fields, which is a manifestation of nonlocality and nonlinearity caused by quantization. Here, in order to simplified it, replace its expectation value with a mean field depending on the radial coordinate r. That is, replace Q in (7-2) with the reciprocal of twice r and express it as a function $\bar{t}^2(r) \propto 1/\log(R_h^2/r^2)$, which diverges at $R_h\,(=\xi_\Lambda/2)$ corresponding to the radius of the object and disappears at the origin.

The equation of motion modeled by applying such mean-field approximation has the following structure (see end of Appendix B). As the running coupling constant disappears near the origin, it reduces to an equation that mainly consists of fluctuations of the conformal-factor field. Inversely, when moving away from the center of the excited state, the running coupling constant gradually increases and diverges when reaching the edge. Along with this, the $T^{(4)}_{\mu\nu}$ term disappears at the edge in proportion to the reciprocal of the running coupling constant squared. That is, the outside is described by the Einstein equation in vacuum.

The mass of the excited state is defined as $m = \int_{|\mathbf{x}|\leq R_h} d^3\mathbf{x}\, T^{(4)}_{00}(\mathbf{x})$ as the sum of the internal gravitational energies, where $\mathbf{x}$ is the spatial coordinates in the background with $r = |\mathbf{x}|$. The solution with the mass m of twice the Planck mass m_{pl} is presented in Fig. 8-1. In the distance, the Einstein

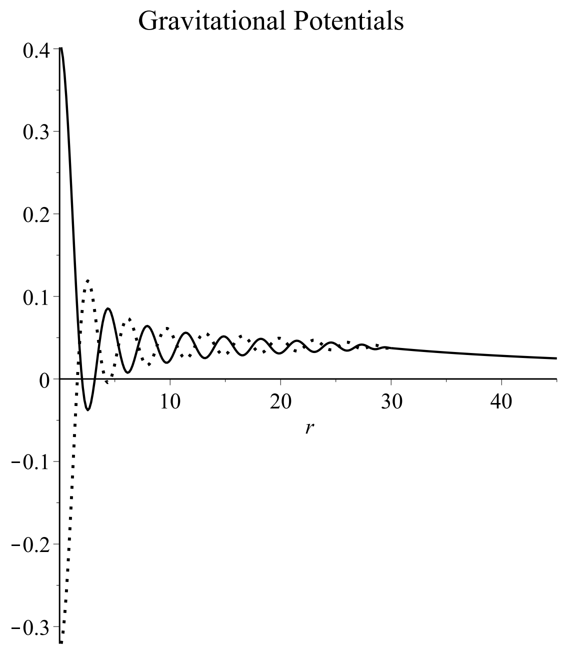

Figure 8-1: A localized excited state of quantum gravity with mass of $2m_{\rm pl}$. The gravitational potential Φ is represented by a solid line, while $-\Psi$ is represented by a dotted line. The radius of excitation, R_h $(= \xi_\Lambda/2)$, is normalized to be 30, and the outside is the Schwarzschild solution. [K. Hamada, Phys. Rev. D **102** (2020) 026024.]

equation can be written in the form of the Poisson equation, and its solution can be approximated by the Newton potential. It is given by $\Psi = -r_g/2r$ and $\Phi = -\Psi$ when introducing the gravitational potentials Ψ and Φ defined by a line element $ds^2 = -(1 + 2\Psi)d\eta^2 + (1 + 2\Phi)d\mathbf{x}^2$. Here, r_g is the Schwarzschild radius mentioned above, which is four times the Planck length because the mass is set to twice the Planck mass.

The gravitational field gradually becomes stronger as you approach the excited state from the outside. However, the radius R_h of the state under consideration is more than 20 times larger than the Schwarzschild radius r_g, and thus the magnitude of the gravitational potential at the edge, $r_g/2R_h$, is small enough for the approximation to work. Entering the inside, the gravitational potential begins to oscillate due to quantum effects and no longer monotonically increases and diverges as in a classical solution. At this time, the two gravitational potentials satisfying $\Phi = -\Psi$ outside gradually deviate from each other, and a configuration of $\Phi = \Psi$ representing fluctuations of the conformal-factor field, becomes predominant near the center. The oscillation of the internal gravitational potentials indicates that physical distance is fluctuating.

This solution employing the gravitational potentials has been derived by assuming these facts in advance. As described above, the equation of motion has been formulated by replacing the running coupling constant with the mean field, and the solution has been obtained while confirming that the calculation result does not contradict with the approximation. The mass of twice the Planck mass is not only a physically interesting mass, but fortunately a mass for which the approximation is valid. If the mass becomes larger than this, it becomes necessary to consider nonlinear effects other than the running coupling constant, which makes the calculation difficult. Anyway, this solution is still incomplete, but it will capture characteristics of the excited state.

This object can be called a particle rather than a black hole. Such purely gravitational particles with only mass are called “dark particles”. If the mass is lower than Λ_{QG}, the quantum gravity effect will not be activated, therefore such excitations will not occur. In addition, it is expected that mass spectra of the quantum gravity will be quantized by Λ_{QG}, but unfortunately it cannot be confirmed by the mean-field approximation.

The excited state probably has a lifetime of about the reciprocal of Λ_{QG} and is thought to collapse soon. The fact that the running coupling constant increases at the edge indicates that the gravitational interactions with matter fields open there. Hence, it may collapse as energy of the gravitational field gradually shifts to that of matter fields from the edge.

On the other hand, for objects whose mass is more than 100 times larger than the Planck mass, its classical horizon size becomes longer than the correlation length ξ_Λ. Such an object is nothing but a black hole when viewed from the outside. At this point, its dynamics can only be imagined, but it is thought that the gravitational field fluctuates greatly near the center, as shown here.

In such a large black hole, gravity can be treated classically except near the center. Hawking showed in the 1970s that under such semi-classical approximation, so-called evaporation phenomena occur in which black holes gradually become smaller while emitting radiation. Thus, when the horizon disappears at the final stage, it will change to a dark particle state as shown above. Eventually it will decay and it is expected that the black hole completely disappears.

It is thought that dark particles were produced in large amounts and densely in the early universe. In order for them to remain as so-called "dark matters" to the present, they have to be long-lived, namely stable. Since dark particles are initially expected to have no angular momentum, even if they have a short lifetime individually, they may coalesce before decaying. As they grow and form the horizon, it can be expected that a large number of primordial black holes will be formed using them as seeds. Then, since their lifetime will be long, they become candidates for dark matter.

Chapter 9

Quantum Gravity Inflation

Quantum theory of gravity is believed to be necessary also in the epoch of the creation of the universe. As inflation theory, which are proposed by Guth, Sato, and Starobinsky around 1980, has become popular, the necessity of the theory has increased. Inflation claimes that there was a period of exponentially rapid expansion of the universe, and a natural interpretation of this idea indicates that the vast area of the universe that you can currently see began with a region narrower than the Planck length when tracing back before inflation.

One of major reasons for thinking of inflation is to solve the "horizon problem". The rapid expansion of space can well explain why there were larger correlations than the horizon size in the early universe. The other is to explain the "flatness problem". If the space in the early universe had a curvature, depending on its sign, the universe would rapidly either expands or collapses, so that no stars or galaxies would form. Nevertheless, observations show that the curvature remains near zero even after more than 10 billion years. This problem can be explained by considering that the space was extremely stretched and flattened during the period of inflation.

Another important role of inflation that should be emphasized here is to generate primordial fluctuations that provide initial values for the evolution of the universe after the Big Bang. To explain the present universe, their spectra have to be almost scale-invariant, at least for the fluctuations with sizes involved in the structure formation. Moreover, the magnitude of their amplitude has to be so small. Inflation theory needs to explain these things.

Well then, why should the primordial fluctuations be very small? As already mentioned in Chapter 2, it comes from the fact that the Friedmann solution derived presuming homogeneity and isotropy in space is non-static and unstable. That is, the solution changes monotonically in time, and when considering a small fluctuation (perturbation) around it, the fluctuation grows with time and thus spacetime deviates significantly from the solution. Usually, such a solution is not chosen as physics. It is a famous story that Einstein introduced the cosmological constant to obtain a static and stable solution. However, the universe is expanding, and there is no doubt that the spectrum is "red-shifted". Moreover, even though the universe has been around for more than 10 billion years, it can still be approximated well by the unstable Friedmann solution. This fact indicates that the initial fluctuations were unnaturally small.

It may happen that such fluctuations with small amplitudes are selected as the initial values, but that looks artificial. Rather, it is more natural to think that there was some mechanism before that to make fluctuations small. Thinking that inflation played such a role simplifies the story. However, it is hard to say that inflation models constructed within the framework of Einstein's theory of gravity describe it well.

Here, a model of quantum gravity inflation derived as a solution of the equation of motion (5-4) is presented, which provides a scenario from the background-free world beyond the Planck scale to the present Friedmann universe. The history of the evolution can be understood as a process in which conformal invariance is gradually broken and derivative rank of gravitational actions contributing to dynamics becomes lower.

Inflationary solution

In the asymptotically background-free quantum gravity, there are three "physical constants" that have to be determined by observations. They are true constants whose values do not change during the evolution of the universe, and are defined as renormalization group invariants, as mentioned in Chapter 5 and also in the next chapter. Two of them, the Planck mass $m_{\rm pl}$ and the dynamical energy scale $\Lambda_{\rm QG}$, describe dynamics of the early universe. On the other hand, the third cosmological constant is not considered here as it is negligibly small.

Inflation is generally represented by a de Sitter solution in which space expands exponentially. All solutions whose space expands rapidly can be said to be inflationary ones, but here it refers to the de Sitter one. Also, the solution explained below is similar to Starobinsky's one, but there is a big difference in how inflation terminates.

The inflationary solution is one of spacetime solutions in which the Weyl curvature tensor vanishes. The asymptotic background freedom shows that such conformally-flat spacetime configurations becomes dominant in the high energy region where the running coupling constant (7-2) is small. In other words, inflation can be said to occur in the energy region higher than the dynamical scale $\Lambda_{\rm QG}$. Since the only scale that can exist in such a region is the Planck mass, it should be the scale that determines the inflationary solution. Therefore,

$$m_{\rm pl} > \Lambda_{\rm QG} \tag{9-1}$$

is required as a condition for the solution to exist.

Let us premise that in the very early days before the evolution of the universe begins, the conformal factor fluctuates so much that time and distance make no sense. At that time, assume that matters do not exist yet and the energy-momentum tensor $T^{\rm M}_{\mu\nu}$ vanishes initially. As already stated, such a world can be described by an exact conformal field theory with no scale. However, conformal invariance is not always perfect, and if there is a scale breaking it, the universe will eventually begin to evolve. The magnitude relation (9-1) indicates that the first scale the universe feels in the evolution process is the Planck scale.

The equation of motion (5-4) of the quantum gravity shows that the energy-momentum tensor of the whole system disappears. The inflationary solution in which the conformal factor expands exponentially is derived from the balance between the Einstein tensor term $T^{\rm EH}_{\mu\nu}$ and the fourth derivative term $T^{(4)}_{\mu\nu}$ responsible for the conformal gravity dynamics. Specifically, assuming a spatially homogeneous and isotropic spacetime given by a line element $e^{2\phi(\eta)}(-d\eta^2 + d\mathbf{x}^2)$, the solution will be obtained by solving the equation of motion (5-4) for the conformal-factor field ϕ.

In cosmology, the conformal factor $a(\eta) = e^{\phi(\eta)}$, which depends only on the time component, is called a scale factor. Introduce τ defined by $d\tau = a(\eta)d\eta$ as a time variable, then the time component of the line element can be written as that of flat spacetime like $-d\tau^2+a^2(\tau)d\mathbf{x}^2$, thus this variable is called physical time.[1] In the early stages when the running coupling constant is still small, the inflationary solution is given as a stable one that converges to

$$H(\tau) = H_{\rm D}, \qquad H_{\rm D} = m_{\rm pl}\sqrt{\frac{\pi}{b_c}} = M_{\rm P}\sqrt{\frac{8\pi^2}{b_c}}, \tag{9-2}$$

where $H(\tau) = \partial_\tau a(\tau)/a(\tau)$ is the Hubble variable, thus it turns out that the scale factor expands with an exponential of the time τ like $a(\tau) \propto e^{H_{\rm D}\tau}$ (See Figs. 9-1 and 9-2). Here, b_c is a coefficient in front of the Riegert action that is one of the Wess-Zumino actions necessary for diffeomorphism invariance, and is a number that depends on a model of matter fields coupled with gravity. Since a typical value of b_c is about 10 (see Appendix B), it can be seen that $H_{\rm D}$ has a value between the reduced Planck mass $M_{\rm P}$ and the Planck mass $m_{\rm pl}$. Therefore, $H_{\rm D}$ is referred to as a Planck scale. Also, a time $\tau_{\rm P}$ when the space starts to expand, given by the reciprocal of $H_{\rm D}$, is called a Planck time.

Furthermore, with the inflationary solution (9-2) as a background, consider fluctuations representing deviation from this solution. Examining their time evolutions with the equation of motion (5-4) shows that the fluctuations decrease during the inflation period. The details will be explained at the end of this section, which states that the inflationary solution is indeed stable, and even if the fluctuations are large at first, the universe eventually converges to this solution. At the same time, it implies that the time variable τ has an entity as a uniform time. Hence, such monotonically increasing scale factor signifies the birth of the concept of time in the entire universe.

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[1] As the name of the coordinate variables, η is called comoving time. Similarly, $\mathbf{x}$ is called comoving coordinates, and a coordinate $\mathbf{r}(\tau) = a(\tau)\mathbf{x}$ that represents actual distance is called physical coordinate.

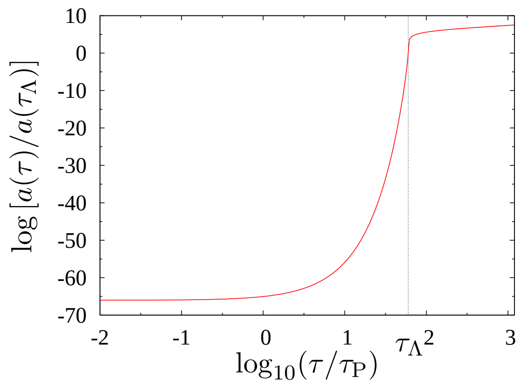

Figure 9-1: Time evolution of the scale factor $a(\tau)$. Inflation will begin at the Planck time $\tau_{\rm P}$ $(= 1/H_{\rm D})$ and terminates at a dynamical time scale τ_Λ $(= 1/\Lambda_{\rm QG})$, which is $60\,\tau_{\rm P}$ here. Spacetime then transitions to the Friedmann universe. [K. Hamada, S. Horata, and T. Yukawa, Phys. Rev. D **74** (2006) 123502.]

Big Bang as spacetime phase transition

When does inflation end? As spacetime begins to expand and the value of physical momentum Q (7-3) decreases, the running coupling constant squared $\bar{t}^2(Q)$ (7-2) given by the reciprocal of its logarithm slowly increases, and rapidly grows when Q goes down close to the dynamical scale $\Lambda_{\rm QG}$. As long as the running coupling constant is small, the inflationary spacetime with a configuration where the Weyl tensor vanishes continues stably, but as it becomes larger, the universe gradually deviates from such a spacetime configuration. Eventually, inflation is over and the universe shifts to the Friedmann spacetime described by Einstein's theory of gravity. This change is the spacetime phase transition.

In order to describe this transition process with the equation of motion (5-4) incorporating quantum corrections of gravity, the running coupling constant is expressed with a time-dependent mean field, as a variant of the approximation applied in the previous chapter. That is, replace Q with the reciprocal of the physical time τ and express it as $\bar{t}^2(\tau) \propto 1/\log(\tau_\Lambda^2/\tau^2)$, where τ_Λ is a dynamical time defined by the reciprocal of the energy scale

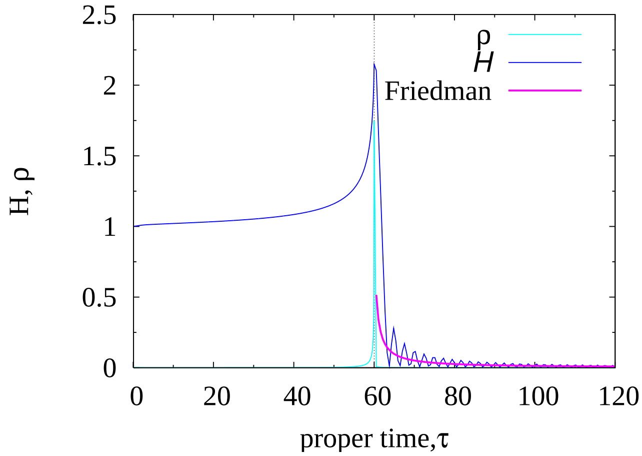

Figure 9-2: Time evolution of the Hubble variable H and matter energy density ρ, where it is normalized with $H_{\rm D} = 1$. After the phase transition point $\tau = \tau_\Lambda$ $(= 60)$, it approaches the Friedmann solution with time asymptotically. The matter is rapidly generated near the phase transition point. [K. Hamada, S. Horata, and T. Yukawa, Phys. Rev. D **74** (2006) 123502.]

$\Lambda_{\rm QG}$. The spacetime phase transition is expressed as a process in which the fourth-derivative gravitational term $T^{(4)}_{\mu\nu}$ responsible for conformal gravity dynamics disappears in proportion to the reciprocal of $\bar{t}^2(\tau)$ at the time τ_Λ.

If a phenomenon in which matters that make up the universe are generated is called the Big Bang, the spacetime phase transition is exactly the Big Bang, and one of significant points of this phase transition is that there was no matter of any kind before that. This is explained by the fact that since the whole energy-momentum tensor (5-4) is preserved to zero, the $T^{\rm M}_{\mu\nu}$ of matter fields, which was zero initially, gets a value as $T^{(4)}_{\mu\nu}$ disappears. That is, the Big Bang occurs when all energy of quantum gravity is converted into that of matters at the time of the phase transition. Interactions between matter fields and the conformal-factor field that causes the transition are given by the Wess-Zumino action S. The interactions open with increasing the running coupling constant, and become stronger rapidly near the phase transition. Therefore, the change is occurred violently.

The Big Bang was originally a word for representing the beginning of everything. The fact that the universe is expanding indicates that it started from a certain point in the past with high temperature and high density, thus the word Big Bang was born as a term to refer to that point. Actually, going back in time, the Friedmann universe reaches a singular point where the scale factor disappears. After the idea of inflation has been accepted, the Big Bang has become a term that refers to the moment when matter was created after inflation, that is, the prototype of the present universe was created. Moreover, in the inflationary universe, the scale factor never disappears, eliminating the singularity problem.

On the other hand, the term reheating is often used as an alternative to the Big Bang. This is just a word when inflation is discussed within the framework of Einstein's theory of gravity. The meaning of reheating is that in the middle of the evolution of the Friedmann universe, inflation occurred and at that point the past universe was reset once and the present universe was born. Therefore, it is characteristic to set the beginning of the inflation late so that it does not reach the Planck scale when going back to the past.

Determination of dynamical energy scale

The scenario of quantum gravity inflation depends on the ratio of the two mass scales, the Planck mass $H_{\rm D}\,(= m_{\rm pl}\sqrt{\pi/b_c})$ and the dynamical scale $\Lambda_{\rm QG}$. Since they are physical constants, there is no choice but to decide them from observations. From the measurement of the Newton constant G, the Planck mass $m_{\rm pl}\,(= 1/\sqrt{G})$ is determined to be $1.2 \times 10^{19}\,\text{GeV}$. As mentioned before, the coefficient b_c of the Riegert action that determines the newly introduced Planck scale $H_{\rm D}$ is a number that depends on a model of matter fields coupled with gravity, and $b_c = 10$ is adopted here.

One of the indicators that characterizes inflation is the expansion rate called the number of e-foldings. It is here defined by $\mathcal{N}_e = \log\left[a(\tau_\Lambda)/a(\tau_{\rm P})\right]$ which expresses how much the universe has expanded from the Planck time $\tau_{\rm P}$ to the dynamical time τ_Λ where inflation ends. Its approximate value is

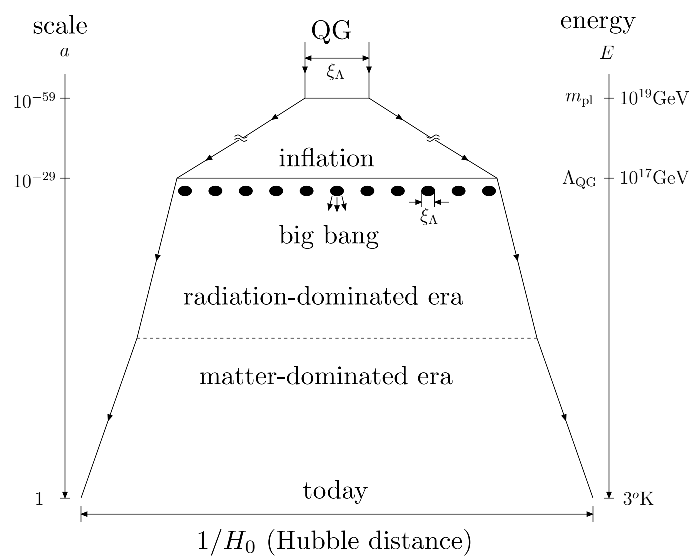

Figure 9-3: Cosmic evolution scenario with quantum gravity inflation. A fluctuation which was the size of the correlation length $\xi_\Lambda = 1/\Lambda_{\rm QG}$ ($\gg l_{\rm pl}$) before the Planck time expands about 10^{59} times until today to the Hubble distance $1/H_0$ ($\simeq$ 5000 Mpc) comparable to the size of the universe, namely $1/H_0 \simeq 10^{59} \times \xi_\Lambda$.

given by the ratio of the two scales as

$$\mathcal{N}_e \simeq \frac{H_{\rm D}}{\Lambda_{\rm QG}}. \tag{9-3}$$

The number of e-foldings required by a typical inflation scenario is between about 60 and 70.

In addition, the magnitude of fluctuations of the scalar curvature just before the end of inflation is roughly estimated to be about $\delta R \sim \Lambda_{\rm QG}^2$ from dimensional analysis. Considering a quantity divided by the scalar curvature $R = 12H_{\rm D}^2$ of the inflationary solution as a dimensionless index, it is also given using the ratio of the two scales as

$$\left.\frac{\delta R}{R}\right|_{\tau_\Lambda} \sim \frac{\Lambda_{\rm QG}^2}{12H_{\rm D}^2}. \tag{9-4}$$

This gives the magnitude of the square root of the amplitude of the primordial power spectrum, which is the initial value of the Friedmann universe.

The magnitude of (9-4) required to explain the CMB observation results has to be a very small value of about 10^{-5}. From this, the ratio of the Planck scale to the dynamical scale can be estimated as $H_\mathrm{D}/\Lambda_\mathrm{QG} = 60$, so that

$$\Lambda_\mathrm{QG} \simeq 1.1 \times 10^{17}\,\mathrm{GeV} \tag{9-5}$$

is obtained. Then the number of e-foldings also becomes an appropriate value of about 60 from (9-3). As a matter of fact, the Hubble variable H does not remain a constant value H_D all the time, and even after the phase transition, spacetime is somewhat in an accelerated expansion until it asymptotics to the Friedmann universe, so the number of e-foldings $\mathcal{N}_e$ reaches close to 70 when they are included.

Expressing the number of e-foldings of 70 in digits, the universe expanded about 10^{30} times during the inflation period. Furthermore, it expands 10^{29} times from the energy scale Λ_QG transferred to the Friedmann universe to the current 3^oK. In total, the universe has expanded 10^{59} times. This number of digits implies that the Hubble distance, which is the largest scale in the universe, and the correlation length of quantum gravity are related as

$$\frac{1}{H_0} \sim \frac{a_0}{a(\tau_\mathrm{P})}\,\xi_\Lambda \simeq 10^{59} \times \xi_\Lambda, \tag{9-6}$$

as shown in Fig. 9-3, where a_0 is the current scale factor.

Origin of primordial power spectrum

From the discussion so far, it can be considered that the origin of the primordial power spectrum, giving the initial value of the Friedmann universe, is in the world of quantum gravity before inflation.

Well then, what is the difference from the conventional way of thinking? Inflationary cosmology, which is discussed within the framework of Einstein's theory of gravity, usually has to introduce an unknown phenomenological scalar field called inflaton, which serves as the source of matters that make up the present universe. Furthermore, the origin of all fluctuations is considered to be given by zero-point energy of the field and its value is calculated by introducing an ultraviolet cutoff. On the other hand, in the asymptotically background-free quantum gravity in which diffeomorphism invariance holds everywhere even beyond the Planck scale, zero-point energy disappears due to this invariance, as stated before. Instead, the excited state of quantum gravity that gives a value to $T^{(4)}_{\mu\nu}$ is the origin of fluctuations. This is also related to the cosmological constant problem, which will be discussed in Chapter 11.

The asymptotic background freedom indicates that before inflation, where the running coupling constant $\bar{t}\,(Q)$ is still small, fluctuations of the conformal factor are dominant, while tensor fluctuations are small in proportion to the coupling constant. This is a cosmologically favorable property and can explain well why the tensor-to-scalar ratio of the CMB fluctuations shown by observations is small.[2]

A spectrum of fluctuations is given by their two-point correlation function. More precisely, a mean square of difference between fluctuations at two points is a measurable quantity.[3] The initial value of fluctuations should be set before the Planck time when the universe begins to expand exponentially. The scalar spectrum at this point is given by a two-point function of the conformal-factor field ϕ derived from the Riegert action. It is expressed as a logarithmic function that does not lose its correlation even over long distances, never seen in ordinary quantum field theories. The logarithm is a reflection of that the field is dimensionless, which results in true scale invariance.

A power spectrum obtained by Fourier transforming the equal-time log-

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[2] The BRST conformal invariance suggests that there are no tensor fluctuations of the CMB scales originated before inflation. Therefore, the tensor-to-scalar ratio does not give limit on the inflation scale so that we can go over the Planck scale wall in quantum gravity inflation.

[3] Note that the mean square vanishes at the same point. The observable is a relative value of fluctuations of two distant points, not an absolute value provided by zero point energy.

arithmic correlation function from coordinate space to three-dimensional momentum space is a scale-invariant one, commonly known as the Harrison-Zel'dovich spectrum. As a notation of power spectra, those that are made dimensionless by multiplying them by an appropriate power of an absolute value of momentum (wave number) are often used. Using this convention, the scale-invariant spectra are expressed by positive constants that do not depend on wavenumber. That of the conformal-factor field is exactly a positive constant with an amplitude of $1/2b_c$.

Scale invariance indicates that there is a correlation in all wavelengths. On the other hand, its invariance is rapidly lost on the dynamical scale $\Lambda_{\rm QG}$. It shows that when physical distance exceeds the correlation length ξ_Λ ($= 1/\Lambda_{\rm QG}$), the correlation rapidly decays and disappears. The relationship (9-6) shows that the distance ξ_Λ in the Planck time is now extended to the Hubble distance. The low multipole component of $l = 2$ in the observed CMB fluctuation spectrum represents the presence or absence of such a large size correlation. Hence, the fact that its component is falling sharply can be explained by the new dynamical length scale ξ_Λ.

Reduction of fluctuations and stability of inflation

The spectrum of quantum gravity fluctuations before inflation begins has been described above. However, it is not passed down as it is to the Friedmann universe. Here, let us see how the fluctuation evolves during the inflation period, especially how the amplitude reduces. Specifically, consider fluctuations (perturbations) around the inflationary solution and show results of examining their time evolution by solving the equation of motion (5-4).

To describe spectra of the expanding universe, it is convenient to use an absolute value of spatial momentum in comoving coordinates called comoving wavenumber. An actual wavenumber called physical wavenumber is the one divided it by the scale factor $a(\tau)$, like the square root of (7-3). If the current scale factor value a_0 is normalized to 1, comoving wavenumber represents magnitude of the current wavenumber. In the following, discussions will proceed with $a_0 = 1$. The inflation model introduced here indicates that physical wavenumber at the Planck time when the expansion begins was about 10^{59} times the current wavenumber.

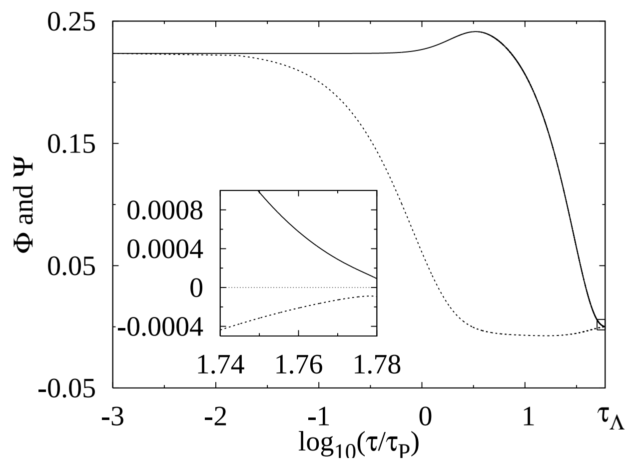

Figure 9-4: Solutions of linear evolution equations for the gravitational potentials Φ (solid) and Ψ (dotted) in the inflationary background. The initial value is set to $\Phi = \Psi = 1/\sqrt{2b_c} = 0.224$, and the comoving wavenumber is $k = 0.01\,\mathrm{Mpc}^{-1}$. Normalize with $H_\mathrm{D} = 1$ so that $\tau_\mathrm{P} = 1$, and set $\tau_\Lambda = 60$. The two gravitational potentials change while decreasing the amplitudes, respectively, and become $\Phi = -\Psi$ at the phase transition point τ_Λ. [K. Hamada, S. Horata and T. Yukawa, Phys. Rev. D **81** (2010) 083533.]

The cosmic evolution scenario shown in Fig. 9-3 implies that the current largest-size fluctuations was the one with a physical wavelength of about the correlation length ξ_Λ when going back before inflation, which was Λ_QG in terms of physical wavenumber. Here, consider time evolution of fluctuations with physical wavenumbers from Λ_QG to several orders of magnitude higher than that, which is thought to have been involved in the structure formation of the universe after the Big Bang.

The equation of motion is expressed in terms of comoving wavenumber and solved for each wavenumber. As in the previous chapter, fluctuations

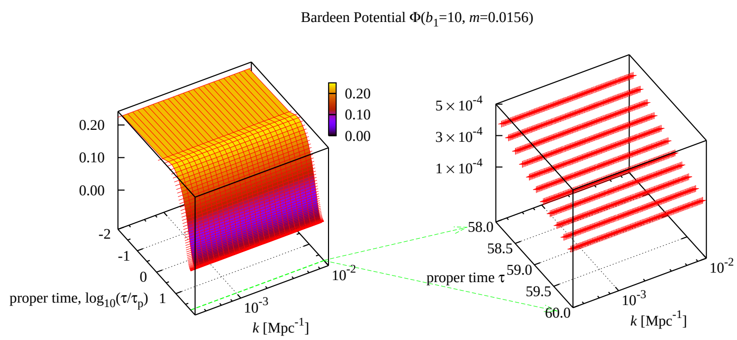

Figure 9-5: Time evolution of the gravitational potential Φ. The line at the phase transition point $\tau = \tau_\Lambda$ gives the primordial power spectrum.

dealt with here can be investigated using the two scalar-like gravitational potentials defined by the line element $ds^2 = a^2[-(1+2\Psi)d\eta^2 + (1+2\Phi)d\mathbf{x}^2]$. The difference from the previous one is that the square of the scale factor a of the inflationary solution is multiplied to the whole. Calculation methods are basically the same as before, except that a time-dependent mean-field approximation is employed for the running coupling constant that represents nonlinear and nonlocal effects near the phase transition point. On the other hand, as another nonlinear effect that appears in the early stage when the fluctuations are still large, there are contributions from the exponential function of the conformal-factor field contained in the Einstein tensor part. It cannot be ignored when the physical wavenumber exceeds the Planck scale $H_{\rm D}$, thus lower wavenumbers than $H_{\rm D}$ are considered here to avoid such nonlinearity. Unlike the previous chapter in which static fluctuations was examined, linear equations of motion derived in this way become complicated coupled ones because the fluctuations are treated around the time-dependent background, but they can be solved numerically.

The initial spectrum before inflation begins is almost scale-invariant, but as already stated above, there is a lower limit on possible wavenumbers because correlations longer than the correlation length ξ_Λ decay rapidly and disappear. Now let us see that it represents the falloff of the low multipole component of $l = 2$ mentioned before. Since the comoving wavenumber of this component is about $\lambda_0 = 0.00026\,\mathrm{Mpc}^{-1}$, you can find that the ratio between this value and the physical scale (9-5) is $\lambda_0/\Lambda_{\mathrm{QG}} \sim 10^{-59}$. The right-hand side is exactly the scale factor before the expansion of the universe begins, and this relation is nothing but (9-6) derived from the inflation scenario because the reciprocal of λ_0 corresponds to the Hubble distance. Thus, the sharp falloff at $l = 2$ can be explained well using the correlation length.

The comoving wavenumber is written as k, and calculation results of the coupled linear equations of motion for the fluctuations are displayed in the region of $k > \lambda_0$. Also, the upper limit of k is taken to be $m_0 \sim 10^{-59} \times H_{\mathrm{D}}$, as mentioned above, which represents the comoving wavenumber whose physical one is H_{D} in the Planck time. Since the inflationary solution is calculated with $H_{\mathrm{D}}/\Lambda_{\mathrm{QG}} = m_0/\lambda_0 = 60$, it is given by $m_0 = 0.0156\,\mathrm{Mpc}^{-1}$.

Fig. 9-4 shows results of calculating the behavior of the gravitational potentials Φ and Ψ when $k = 0.01\,\mathrm{Mpc}^{-1}$. Fig. 9-5 shows the results within the above wavenumber range, displaying only Φ. Since the initial stage is filled with the fluctuation of the conformal-factor field that satisfies $\Phi = \Psi$, the initial value of the gravitational potential is given by the square root of the amplitude $1/2b_c$ of the power spectrum. In this way, it can be found that the amplitude gradually decreases while maintaining the scale-invariant form represented by a horizontal straight line, at least within the range shown. In addition, it can be seen from Fig. 9-4 that at the phase transition point, the fluctuation changes so as to satisfy $\Psi = -\Phi$.

The exponential conformal factor being in the Einstein tensor part, which makes calculations difficult in the higher frequency region, is a nonlinear term that remains even in the limit where the coupling constant t disappears. When viewed as conformal field theory, it is a necessary factor for the Einstein-Hilbert action to work as a conformally invariant operator which can be regarded as a potential term, not an ordinary mass term. From this, it is believed that this nonlinear term has an effect of weakening the form of the spectrum to a power of the mass scale. Therefore, this effect may cause the scale-invariant spectrum to slowly red-tilt at $k > m_0$.

Numerical calculations incorporating nonlinear terms of this type are

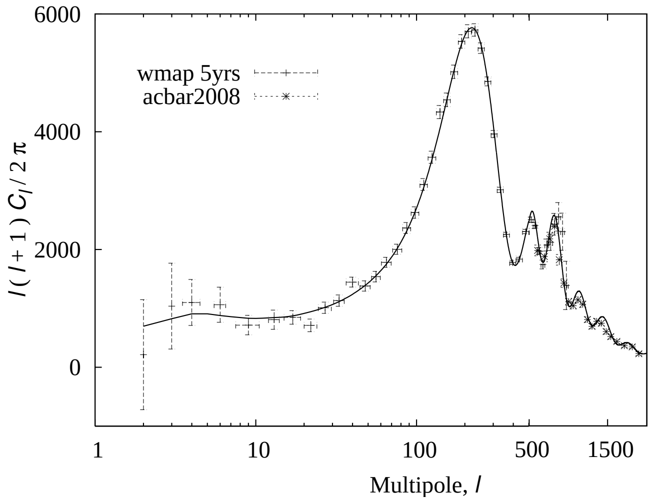

Figure 9-6: CMB temperature fluctuation (TT) power spectrum. The calculation result (solid) is displayed together with the data of WMAP5 and ACBAR2008. The cosmological parameters are set to $\tau_e = 0.08$, $\Omega_b = 0.043$, $\Omega_c = 0.20$, $\Omega_{\rm vac} = 0.757$, $H_0 = 73.1$, $T_{\rm cmb} = 2.726$, $Y_{\rm He} = 0.24$, and the tensor-to-scalar ratio is $r = 0.06$. The falloff of low multipole components is explained by a dynamical factor that attenuates at $\lambda_0 = 0.00026\,{\rm Mpc}^{-1}$. [K. Hamada, S. Horata and T. Yukawa, Phys. Rev. D **81** (2010) 083533.]

difficult and have not yet been performed. The calculated range, $\lambda_0 < k < m_0$, corresponds to the observed CMB multipoles from $l = 2$ to beyond the first acoustic peak. Here, assuming that the scale-invariant form is maintained even in the region of $k > m_0$, the CMB spectrum is calculated and is compared with the experimental results. Fig. 9-6 shows the results obtained in this way.

From after the Big Bang to the present

During the inflation period, the quantum fluctuations of gravity are stretched. Although fluctuations of various sizes are generated along the way, the amplitude of the specific fluctuations expanding up to the size involved in determining the later large-scale structure of the universe is greatly reduced. Hence, the basis of the classical spacetime structure leading to the present is formed, in which matters and dark excitations generated as local fluctuations are flying around as particles. The reduced gravitational quantum fluctuations that have expanded to a sufficiently large size are inherited in the Friedmann universe without being involved in local disturbances at the Big Bang. In order to obtain the temperature fluctuation spectrum of CMB shown at the end of the previous section, it is necessary to know how they have been transmitted through the spacetime until today. This section briefly describes the history of their evolution. The fluctuations represent deviation from the spatially homogeneous and isotropic Friedmann solution, and their time evolution is described by the Einstein equation and conservation equations of energy and momentum for each matter generated.

The universe immediately after the Big Bang can be described as a perfect fluid in which various matters frequently interact with each other and thermal equilibrium is achieved. At first, the radiation-dominated era with high energy density of particles moving at the speed of light mainly on photons and neutrinos continues. As the universe expands and temperature drops by the inverse of the scale factor, particles with mass such as quarks gradually slow down, and they bind to each other due to strong interactions and change into stable protons and neutrons. Further, heavier atomic nuclei such as helium nucleus are sequentially generated, and the world of matters is composed of them together with photons, neutrinos, and electrons.

When temperature drops to about $1\,\mathrm{eV}$, the universe enters the matter-dominated era, where energy density of massive matters that have become non-relativistic exceeds that of radiations. This is because energy density of non-relativistic matters decreases with the cube of temperature, whereas that of radiations decreases faster with the forth power according to the Stefan-Boltzmann law, thus they reverse on the way of the evolution.

Shortly after entering the matter-dominated era, atomic nuclei and electrons combine to form atoms, and the universe rapidly neutralizes. After that, since electric charges disappear, photons can propagate freely, and the universe become clear. This phenomenon is called recombination, and the moment when it occurs is cut out and called the “last scattering surface”.

Matters also contain still unknown objects not yet observed directly that interact almost only with gravity. Among them, a stable one that remains until today is called “dark matter”. In particular, the one that is treated non-relativistically assuming that mass is relatively heavy, is called “cold dark matter”. Its existence has been predicted to explain the observed CMB spectrum, formation of large scale structures in the universe, anomalies in the galaxy rotation curve, and gravitational lens effects suggesting the existence of invisible mass.

In the world where the Einstein equation holds, fluctuations of the gravitational potentials are reflected in density fluctuations of each matter. As they grow, the unevenness in mass distribution becomes large, and structures such as stars, galaxies, clusters of galaxies, and superclusters are formed. The formation of these structures begins mainly after the universe has been neutralized. Stars and galaxies are very large concentrations of mass when viewed individually, but when looking at the universe so wide that even galaxies can be regarded as points, the magnitude of fluctuations indicating the existence of large-scale structures such as clusters of galaxies and superclusters, which are deviations from the homogeneous and isotropic distribution of galaxies, is about 1/10. It shows that the entire universe is still at a level that can be sufficiently approximated by the Friedmann solution.

Large-size fluctuations that affect the large-scale structures begin to grow after the universe has been neutralized. Until then they do not change much and the small amplitude immediately after the Big Bang is maintained. Figs. 9-7 and 9-8 show that such changes occur by examining how various fluctuations evolve from the past (upper right) to the last scattering surface where the universe is neutralized. The comoving wavenumber k represents a current wavenumber of fluctuations, and the reciprocal of Mpc is used as a unit. The time axis is represented by the redshift z, which is defined as $z+1 = a_0/a$ using the scale factor. The reciprocal of k is a wavelength of fluctuations, which is large enough to regard a galaxy as a point. As a side note, a cluster of galaxies has a size of $4 \sim 6\,\mathrm{Mpc}$, and a supercluster has a size of 10 $\sim$ 30 Mpc. Going back in time, the physical wavelength at the last scattering surface is about one-thousandth of the current length, because it has to be multiplied by the scale factor for that time. If you go back to before inflation begins, it falls within the Planck length.

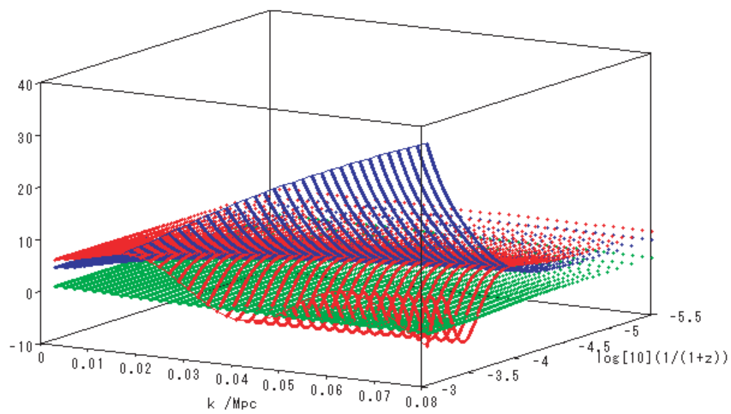

Figure 9-7: Time evolution of various fluctuations from the radiation-dominated era to the decoupling time ($z \simeq 10^3$). From the top, the density fluctuation of cold dark matter (blue), the density fluctuation of photon (red), and the gravitational potential Φ (green) are displayed using the logarithm of the redshift z as time. The calculation is performed assuming the Harrison-Zel'dovich spectrum in which the initial value of Φ is unity, independent of the wavenumber k. The cold dark matter fluctuation increases monotonically from short-wavelength fluctuation that enter the inside of the horizon after shifting to the matter-dominated era. On the other hand, the photon fluctuation begins to oscillate greatly.

Most of the deformations of the various fluctuation spectra occur during the period until the universe shifts from the radiation-dominated era to the matter-dominated era and then becomes neutral. Before that, since the scale-invariant spectra are preserved, you can go back to the past indefinitely as long as the radiation-dominated era continues.[4] Therefore, as long as Einstein's theory of gravity is correct, information immediately after the Big Bang when the primordial spectrum was generated can be extracted from the current CMB spectrum.

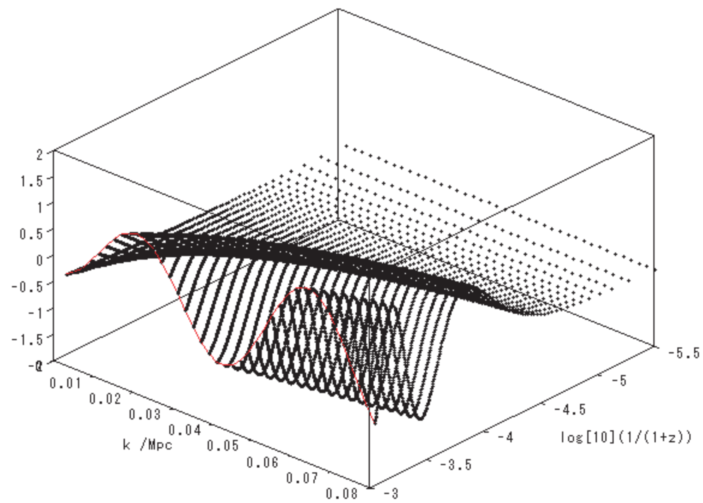

Figure 9-8: Time evolution of a perturbation variable for the CMB temperature fluctuation (a variable that appears in the Sachs-Wolfe effect). The calculation is performed under the same conditions as in Fig. 9-7. The last solid line is a spectrum at the decoupling time ($z \simeq 10^3$). The first extreme value near the wavenumber 0.02Mpc^{-1} corresponds to the first acoustic peak of the CMB spectrum.

After the neutralization of the universe, photon becomes free (decoupling) from interactions with matters, so its fluctuation does not grow. This is expressed as the so-called Sachs-Wolfe effect, which gives a relationship between the gravitational potential at the decoupling time and the current CMB temperature fluctuation. Therefore, if the spectrum of the gravita-

[4] Fuctuations in sizes larger than the horizon size hardly change in any era. In particular, the CMB multipole component of $l < 30$ corresponds to a fluctuation with a size that has entered the inside of the horizon after the neutralization or has not yet entered, and thus it is thought that it retains the primordial spectrum immediately after the Big Bang as it is.

tional potential at the last scattering surface representing the moment of decoupling is known, the current CMB spectrum can be roughly known through the Sachs-Wolfe effect. Fig. 9-8 shows this fact, and the structure of a trigonometric function appearing on the last scattering surface ($z \simeq 10^3$) determines rough structure of the current CMB spectrum.

In order to understand the growth of the matter fluctuations involved in the structure formation, it is necessary to consider nonlinear effects after the neutralization. On the other hand, the photon fluctuation of the size considered here can be handled by almost linear approximation from after the Big Bang to the present. Therefore, it is possible to make a prediction with high accuracy. The study of the evolution of the universe using perturbation theory based on this fact is called cosmological perturbation theory.

The CMB spectrum can be roughly divided into three regions: the low multipole component region $l < 30$ in which the scale-invariant spectrum of the early universe is believed to be maintained almost as it is, the region $30 < l < 700$ where a plasma fluid oscillation of photons and baryons generated before the universe is neutralized appears, and the Silk damping region $l > 700$ where the amplitude of the photon fluctuation decreases exponentially during the neutralization process. In this damping region, the perfect fluid approximation does not hold and an anisotropic stress appears. This effect is not taken into account in the figures shown in this section.

The Silk damping occurs because thermal equilibrium cannot be maintained gradually as the mean free path of light becomes longer during the period from the start of the neutralization (recombination) process until the light is completely free (decoupling) from matters. If the wavelength is longer than the mean free path, a perfect fluid approximation holds, but if it becomes shorter, photon diffusions occur, so that the fluctuation is averaged and the amplitude decreases.[5] This effect begins to appear beyond the first acoustic peak, and becomes significant where l exceeds 700. Therefore, the

[5] The perfect fluid is a fluid with zero viscosity ($\neq$ ideal gas). Viscosity is a quantity proportional to the mean free path, and having zero viscosity means that the system is in a tightly coupled state with zero mean free path. In such a frequently interacting system, heat exchange is closed in the system and thermal equilibrium can be achieved.

cosmological perturbation theory assuming the perfect fluid is applicable in the long wavelength region up to at most $l < 700$. In order to handle the Silk damping, it is necessary to solve the Boltzmann equation that takes the Thomson scattering into account.

The calculation of the CMB spectrum has already been programmed, and existing calculation codes such as CMBFAST have been published. One of the important role in early universe cosmology is to give its initial condition, the primordial spectrum. The goal of the quantum gravity theory is exactly to provide that.

Chapter 10

About Renormalizablility

In this chapter, some topics on renormalization will be discussed in a little more detail. In particular, the part involved with gravity will introduce a recent topic that the combination of the four-derivative gravitational actions decided from physical considerations in Chapter 4 can be determined from conditions of renormalizability. The story will be more specialized, so if you are not familiar with terms, you may skip it.

What is renormalization?

The fundamental theory of elementary particles is described by a renormalizable quantum field theory. A typical example is quantum electrodynamics (QED), which describes interactions between electrons and photons. This is one of the most successful theories, and its correctness has been verified with extremely high accuracy. From this undeniable fact, methods of renormalization that underlies quantification of the theory have become increasingly reliable. For the achievements in the study of renormalization, Tomonaga, Schwinger, and Feynman were awarded the 1965 Nobel Prize in Physics. The intellectual giant Dyson, who passed away at the age of 96 in 2020, was also a strong candidate.

Four dimensional quantum field theory that deals with continuous infinite-dimensional degrees of freedom normally accompanies ultraviolet divergences. Renormalizable means that types of ultraviolet divergences are finite and they can be appropriately removed by introducing counterterms. Appropriately means that the counterterm has the same form as original action, and removing divergences can be carried out by absorbing them in fields and coupling constants, that is, by renomalizing them. The reason why Einstein's theory of gravity is not renormalizable is that if you start from the Einstein-Hilbert action, then new divergences that cannot be offset in the form of this action occur one after another, and after all you have to prepare an infinite number of counterterms.

Unlike clear rules like symmetry, renormalizability was a property found by trial and error at first. In fact, renormalization at the one-loop level only looks like removing divergences by hand. The essence of renormalization becomes apparent at two or more loop levels, and its specific procedure is as follows.

First of all, prepare all possible local actions that are allowed by symmetries and field properties that the theory has. Original fields and coupling constants before processing are called "bare fields" and "bare coupling constants", and are collectively called "bare quantities". Then, formally express bare quantities as products of "renormalized quantities" and "renormalization factors" (sometimes given in sum). At this point, renormalized quantities are not substantial yet and are just named. Loop calculations are performed using renormalized quantities, then renormalization factors are determined in order, so that divergences yielded by the calculations can be removed. That is, the divergences are dealt with as if the original bare quantities were diverging. A task of determining renormalization factors is nothing but determining counterterms mentioned above. In this way, a finite quantity described only by renormalized quantities is obtained, thereby revealing physical properties that renormalized quantities have, such as the asymptotic freedom.

As can be seen from this procedure, since the original action is local, there is a restriction that all ultraviolet divergences must be local as a condition of renormalizability. This is a natural condition considering that the ultraviolet represents a narrow area. On the other hand, if you recall that effective actions including quantum corrections, which are described in terms of running coupling constants, are nonlocal, you can understand that this is not necessarily a trivial condition.

Symmetry is a main factor in defining quantum field theory, while renormalizability is subordinate. If an action has no dimensional parameters, renormalizability becomes obvious in a theory whose action is uniquely determined by symmetry that does not break even when quantized. Gauge theories such as QED and QCD are exactly like that.[1] If not so, you have to find the theory by groping. Therefore, even now, without accepting renormalizability as a guiding principle, quests for a manifestly finite theory continue, especially in theories concerned with gravity.

The renormalization method has become more believable after an idea of "renormalization group" has been established. When performing renormalization calculations, a new arbitrary scale that did not exist in original actions is introduced, written as μ on mass scale here. Renormalization group represents a relationship between worlds with different μ, and the "renormalization group equation" is a partial differential equation that expresses response to changes in μ. One of the most important functions that appears when defining the equation is the beta function, which represents response of the renormalized coupling constant to μ.

Well then, how does that the arbitrary mass scale arise? In order to explain this fact, it is necessary to describe regularization, which is a task to make the theory finite once in order to handle ultraviolet divergences. If you do not specify its method, you cannot say that the theory is truly defined. The simplest example is to introduce an ultraviolet cutoff for energy. Renormalization is to remove terms that diverge when you bring the cutoff to infinity. In addition, there are lattice regularization and dimensional regularization. Each has its advantages and disadvantages, but one of criteria for selection is whether or not symmetries, which are considered to be the most important for the theory, are preserved. The following describes the individual regularization methods.

If the theory is regularized by introducing an ultraviolet cutoff $\Lambda_{\rm UV}$, divergences in the form of $\log \Lambda_{\rm UV}^2$ will appear. Since an argument of logarithm must be dimensionless, a new scale μ will be introduced to offset the dimension of the cutoff, and the divergence will be processed in the form of $\log(\Lambda_{\rm UV}^2/\mu^2)$.[2] If renormalization is executed so that the cutoff dependency

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[1] Even if fermions have mass, it does not change the structure of ultraviolet divergences and thus does not breaks renormalizability.

[2] For simplicity, mass parameters included in the action are considered to be zero here.

is removed in this form, then to make up for the lack of dimensions the scale μ remains as a nonlocal quantum correction term $\log(q^2/\mu^2)$ in the effective action. Eventually it replaced with a physical scale that appears in the running coupling constant.

An advantage of introducing an ultraviolet cutoff is that it is simple, but there is a big problem that it breaks diffeomorphism invariance as well as gauge invariance, and such undesired divergences will occur in a power of $\Lambda_{\rm UV}$, not in the logarithm. If all divergences could be properly renormalized, the broken symmetry could be restored by bringing the ultraviolet cutoff to infinity. However, it is difficult to actually do it with the cutoff method, thus this method is generally applied to non-renormalizable field theories that are quantized while keeping the cutoff finite.

The same applies to the method of discretizing spacetime in a lattice pattern and making the theory regular. In this case, a lattice spacing corresponds to the reciprocal of the cutoff $\Lambda_{\rm UV}$, and an arbitrary scale will be introduced to offset the dimension. This is a method of approximating the neighborhood of each lattice point with an average value and finally taking a limit of zero lattice spacing to restore continuity. At this time, renormalizability represents that the continuum limit can be taken appropriately. Unlike the simple cutoff method, an advantage of this method is that a lattice version of gauge symmetry is mathematically well-defined. Although the translation and rotation symmetries are broken, it is also believed that they will recover in the continuum limit.

The formulation of gauge field theory on a lattice was proposed by Wilson in the 1970s, and its concrete numerical calculation based on the Monte Carlo method was started by Creutz in 1980. The usefulness of this regularization method is evident in the success of lattice gauge theory. It is nowadays possible to calculate hadron mass spectra with fairly high accuracy. This success also shows that gauge symmetry is a crucial symmetry that determines physics.

Finally, dimensional regularization proposed by 't Hooft and Veltman in 1972 is described. This is a method to make the theory finite by performing an analytic continuation that makes spacetime dimension a little smaller than 4 while holding the continuity of fields, then ultraviolet divergences arise in the form of poles such as $1/(D-4)^n$, where D is the spacetime dimension and n is a natural number. After removing the divergences according to the renormalization procedure, you can return the dimension to 4 and obtain physical quantities. The arbitrary mass scale μ is introduced to make up for the lack of mass dimension caused by changing the spacetime dimension. An advantage of this method is that it preserves gauge symmetry and diffeomorphism invariance strictly. On the other hand, when you apply it to a theory with symmetry such as chiral symmetry or supersymmetry that holds only in a specific integer dimension, you have to determine whether it is okay that the symmetry is broken or whether it is likely to be restored when the dimension returns to 4.

In addition, methods of Pauli-Villars regularization and zeta-function regularization are known. The former was a mainstream regularization applied to gauge field theory before dimensional regularization was devised. The latter is a method often used in two-dimensional quantum field theory.

Now, let us return to the story of the renormalization group. It should be noted that renormalizable theories are originally defined by bare quantities. It means that bare quantities depend only on scales and parameters existing in its original action. From this fact, a condition that they do not depend on the arbitrary mass scale μ introduced upon quantization is derived, so that $\mu d(\text{bare quantities})/d\mu = 0$. The so-called renormalization group equation is obtained by rewriting it in terms of renormalized quantities, which is a partial differential equation that describes how parameters in the theory respond to changes in scale. The beta function introduced at that time is the most important function, which is defined by $\beta_{\mathfrak{g}} = \mu d\mathfrak{g}/d\mu$ for a renormalized coupling constant $\mathfrak{g}$.

The partition function or the effective action given by its logarithm, which is originally defined as bare quantities, is an quantity that does not depend on the arbitrary mass scale μ. On the other hand, renormalizable means that it is finite, that is, represented by renormalized quantities. Among renormalized quantities, the one that does not depend on the scale μ is specially called a renormalization group invariant. The effective action is exactly a renormalization group invariant. The energy-momentum tensor is also a renormalization group invariant which is a bare quantity as well as a finite quantity.

A renormalized parameter that is not a bare one but a renormalization group invariant is called a physical constant, namely, a true constant which does not change anywhere. Writing it as $M_{\rm phys}$, it satisfies

$$\mu \frac{d}{d\mu} M_{\rm phys} = 0. \tag{10-1}$$

The physical constant can be constructed by combining the arbitrary scale μ, the renormalized coupling constant and so on, and this operation is called "dimensional transmutation". For example, in QCD, if the one-loop beta function is given by $\beta_{\mathfrak{g}} = -\beta_0 \mathfrak{g}^3$, the dynamical energy scale is described in the form $\Lambda_{\rm QCD} = \mu\, e^{-1/2\beta_0 \mathfrak{g}^2}$ as a renormalization group invariant. Its higher order terms in perturbation are more complex, but can be systematically defined. Using this expression, it can be shown that nonlocal logarithmic quantum corrections are grouped in the form of the running coupling constant (7-1), and the effective action can be written in the renormalization group invariant form in which the reciprocal of the running coupling squared is extracted in front, as pointed out in Chapter 7.

Renormalizability and quantum gravity

Although renormalization is incompatible with Einstein's theory of gravity, it never inconsistent with diffeomorphism invariance. Renormalization in the asymptotically background-free quantum gravity introduced in this book is carried out by employing the dimensionless coupling constant t in a diffeomorphism invariant way, same to the renormalization procedure described above. Renormalization in quantum gravity discussed from a different perspective will be touched upon in the next section.

One of the conditions that determine quantum gravity actions is, of course, diffeomorphism invariance. However, why quantum gravity is so difficult is that the action cannot be determined only from the invariance. In fact, there are three candidates for the fourth-derivative actions of the gravitational field: the square of the Riemann curvature tensor, the square of the Ricci tensor, and the square of the scalar curvature, and at first glance a combination of them can be chosen arbitrarily. Here, I introduce a work that the action is determined to a certain combination from renormalizability conditions.

In order to determine the gravitational action, first consider QED or QCD in curved spacetime. That is, consider gauge field theories in which the gravitational field remains classical and all other fields are quantized. Then, among various counterterms required to remove ultraviolet divergences generated as quantum corrections, those composed only of the gravitational field are called gravitational counterterms. The reason for considering such a theory is that imposing both gauge invariance and diffeomorphism invariance uniquely determines the form of (bare) actions for gauge fields and fermions, including how to couple with the gravitational field. In other words, the only parts that have not been determined at this point are the gravitational counterterms. However, since interactions between the gravitational field and the other quantum fields are known, if you find forms of ultraviolet divergences by calculating Feynman diagrams in which outer lines are the gravitational fields and inner lines are gauge fields and fermions, you should be able to determine the gravitational counterterms uniquely as well.

Calculations of the gravitational counterterm are performed using dimensional regularization that preserves both gauge invariance and diffeomorphism invariance. Renormalizability can be expressed in terms of renormalization group equations, which impose strict restrictions on the counterterm form. In fact, in order to determine forms of conformal anomalies, Hathrell derived in the 1980s a special renormalization group equation that correlation functions containing various normal products, such as energy-momentum tensors, should satisfy. At that time, the condition that ultraviolet divergences are local, which was mentioned in the previous section, plays an important role. Recent studies have shown that solving this renormalization group equation with all orders in perturbation brings together the gravitational counterterms in only two forms. They are in the form of the conformally invariant gravitational actions described in Chapter 4, when returned to four dimensions. More precisely, parts that deviate from four dimensions are also determined completely, and these parts exactly contain information of the Wess-Zumino actions S (4-5) such as the Riegert action.[3] It can be said to be a natural result because the theory is quantized so as to preserve diffeomorphism invariance. And you can see that the result does not depend on gauge group and kinds of fermions in gauge field theory.

The fact that the gravitational counterterms are given by two conformally invariant forms reflects that interactions contributing to ultraviolet divergences of gauge field theory has a conformally invariant form in four dimensions. If interactions with dimensionless coupling constants that are not conformally invariant manifestly are added, then a counterterm of the scalar curvature squared will be added to these two. What emphasizing here is that they are classified into these definite three types.

Since quantum field theory in curved spacetime can be regarded as part of quantum gravity theory, the quantum gravity action also has to be the same form as the gravitational counterterm above if renormalizability is required. Therefore, you have to adopt it as a bare quantum gravity action.

Once the form of the action is determined, you can find how to introduce coupling constants. Here, it is assumed in advance that conformal invariance is important in the ultraviolet limit. That is, as explained in Chapter 4, the reciprocal of the square of the dimensionless gravitational coupling constant t is introduced before the Weyl action with a correct sign so as to be a perturbation expansion around a conformally flat spacetime.

Next, using this action, it is necessary to calculate and confirm whether there is no contradiction as a quantum theory of gravity and whether the beta function is correctly negative so that conformal invariance is realized in the ultraviolet limit. Actually, it can be shown that the beta function of the coupling constant t becomes negative regardless of matter field contents couped with gravity. This indicates that, like QCD, the dynamical energy scale Λ_{QG} of the quantum gravity can be defined, and conformal invariance will appear at higher energy than that.

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[3] When dimensional regularization is employed, the Schwinger-Dyson equation (5-2) holds as it is, including contributions from the measure, without rewriting as in (5-3).

Conformal invariance also appears in ordinary gauge field theory, but a big difference is that here it appears as the background-metric independence. It is a quantum diffeomorphism invariance that occurs for the first time by quantizing the gravitational field, and is called the BRST conformal invariance to emphasize that it is a gauge symmetry. It has already been stated everywhere that all ghost modes become unphysical because of the presence of this gauge symmetry.

On renormalization in a different perspective

As mentioned in Chapter 3, if you try to formulate quantum gravity by simply perturbing Einstein's theory of gravity, it will be expanded employing the Newton constant, which has dimension of length squared, as a coupling constant. Therefore, it is easy to find from dimensional analysis that higher-derivative gravitational counterterms composed of multiple products of curvatures are required. It is said that renormalization is impossible because if the order of perturbation is increased, eventually an infinite number of new counterterms will have to be introduced. Nevertheless, many physicists still try to formulate quantum gravity based on Einstein's theory of gravity. Below, I will comment on such attempts.

First of all, consider a system with only the gravitational field, and an attempt was made to eliminate gravitational counterterms by imposing the Ricci flat condition, that is the Einstein equation, on the gravitational field in external lines of Feynman diagrams. This approach could eliminate the gravitational counterterm at the one-loop level, but failed soon because some of higher-derivative gravitational counterterms required in higher loops did not disappear.

Then, supergravity theory with supersymmetry that restricts the theory more strongly has been considered. Supersymmetry is a symmetry that connects bosons and fermions discovered in the 1970s, and combining this symmetry and gravitational equations of motion, it was expected that the gravitational counterterms would be limited to a finite number. This approach has been actively studied because there was Hawking's prediction that raising the level of supersymmetry to the highest would determine everything, but did not work out. This is a method different from normal renormalization, and it can be said that it is a desperate resort devised for quantizing gravity. This approach is eventually absorbed in string theory, and supergravity then emerges as its effective field theory.

As a different perspective, there is an idea of "asymptotic safety" advocated by Weinberg in the 1970s in the approach of introducing an ultraviolet cutoff. It was proposed as a property that correct quantum theories of gravity should have when you bring the cutoff to infinity, which is that an infinite number of gravitational counterterms are allowed, but there have to exist a non-trivial ultraviolet fixed point in renormalization group flows for newly introduced coupling constants associated with the counterterms. It represents an expectation that at the ultraviolet limit, the theory converges to that point and then diffeomorphism invariance lost by the cutoff will be restored.

The research is still being actively conducted to show that such a fixed point exists. At this time, a method called functional renormalization group, which was conceived to investigate response of effective field theory to a change of the cutoff, is often used. This is a broader concept than the renormalization group that appears in renormalizable quantum field theories described in the first half, and is also applied to non-renormalizable field theories such as Einstein's theory of gravity.

If such a fixed point exists, it is reasonable to assume that some conformal invariance is realized there. The asymptotically background-free quantum gravity is a renormalizable field theory in which the core part is given by conformal field theory and perturbation expansion is defined as deviation from it. Therefore, it can be said that the asymptotic background freedom is a representation of the asymptotic safety.

Finally, as already mentioned, the choice of regularization method depends on what symmetry determines the theory. There may not be a suitable method other than the cutoff. In that case, experience matters. However, it creates ambiguity, and as a result the theory is broad interpreted and becomes uncertain. In the next chapter, the cosmological constant problem will be described as one such problem.

Chapter 11

What is The Cosmological Constant Problem?

Some people say that the cosmological constant problem is a big problem, while others say it is just a parameter problem. As I have already pointed out, I am in the latter position. The root cause of the problem is that the existence of a world beyond the Planck scale is ignored. Here, I will present an answer to this problem from the standpoint of quantum gravity theory.

Discovery of the cosmological constant

First, the history of the cosmological constant will be described briefly. The cosmological constant was introduced by Einstein himself in 1917, the year after the general theory of relativity was published, in order to realize a static universe. That in itself is nothing surprising, because the existence of the cosmological constant cannot be denied from diffeomorphism invariance, and also many researchers in those days, including Einstein, thought that the universe would not change over time.

The static universe image was shattered by observation results published by astronomer Hubble in 1929. He measured velocities of many galaxies and found that all galaxies were moving away from the earth. It has been shown that the farther the galaxy is, the faster its recession velocity is, that is, the space is expanding everywhere and distance between all objects is increasing. A few years after this discovery, Einstein allegedly described the introduction of the cosmological constant as "the biggest mistake in life."

It was not until the late 1990s that the cosmological constant came into limelight again. Observations of brightness of Type Ia supernovae, which are used as standard candles, have shown that the expansion rate of the universe is increasing (see Fig. 11-1). A standard candle is a celestial body whose absolute luminosity, corresponding to wattage of a light bulb, is known, and difference in brightness can be regarded as difference in distance from the earth. This fact, which indicates that the universe is currently accelerated expanding, can be explained using the cosmological constant. For this discovery, Perlmutter, who led the Supernova Cosmology Project, and Riess and Schmidt, who led the Supernova Search Team, were awarded the 2011 Nobel Prize. The existence of the cosmological constant was further confirmed by observations of CMB fluctuations by WMAP. In those days, many people still believed that the cosmological constant did not exist. This discovery was a big shock, as supersymmetric field theories that makes the cosmological constant exactly zero were being actively studied.

The observed cosmological constant has a dimension of the fourth power of energy, which is about 120 orders of magnitude smaller than the value of the fourth power of the Planck mass. It is extremely small and negligible in the early universe, and does not contribute to the inflation scenario described in Chapter 9. The reason why the small cosmological constant contributes greatly to the current accelerated expansion is as follows. In the Einstein equation that determines the evolution of the universe after the Big Bang, energy density of matters is overwhelmingly large at the beginning, but its contribution becomes smaller and smaller as the universe expands. On the other hand, since the contribution of the cosmological constant is constant, the contributions of both are reversed near the present.

Is the cosmological constant problem really a problem?

Until the cosmological constant was discovered, whether it had a finite value was a major issue. However, it is a subject that has nothing to do with quantum theory and can be discussed within the framework of general relativity. The cosmological constant problem, which is the main subject of this chapter, is what will be described below.

The first to point out the cosmological constant problem is said to be Pauli.[1] That was in the 1920s when quantum mechanics was born. When

[1] See, for example, "Cosmic Conundrum" Scientific American, February 2021 and N.

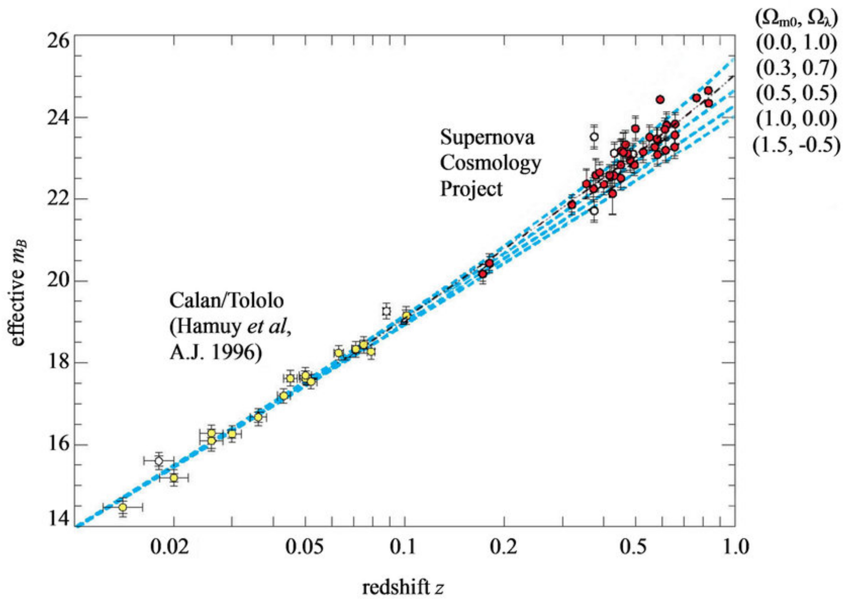

Figure 11-1: Observation results indicating the existence of the cosmological constant, where the horizontal and vertical axises represent redshift (recession velocity) and luminosity distance, respectively. The points are observed Type Ia supernovae. If expansion rate of the universe is constant, the slope will be constant, but since most of the observed values are above it, the universe is accelerated expanding. [Supernova Cosmology Project, S. Perlmutter, *et.al.*, Astrophys. J. **157** (1999) 565.]

he calculated a repulsive force due to zero-point energy, he got a very large value. He recognized it as a serious problem because it implies that the universe would expand rapidly and celestial bodies would soon be far apart and could not recognize each other. In those days, it seemed that nobody really took the story severely because it was too eccentric, but he might have been amused and challenged the task. A perfectionist known for Pauli's exclusion principle, he is also famous for leaving behind the harsh word "not even wrong", meaning that it does not make sense in the first place before

Straumann, *The History of the Cosmological Constant Problem*, arXiv:gr-qc/0208027.

deciding whether it is right or wrong.

The first concrete calculation of zero-point energy was done by Zel'dovich in the 1960s when quantum field theory became widely accepted. As a result, a very large cosmological constant that did not match the observation was obtained, and then the cosmological constant has come to be discussed as a problem of quantum field theory.

The description so far is just a summary of what is widely known (see Weinberg's review in 1989). The question here is not the magnitude of the value, but how the calculation was performed. The essence of the cosmological constant problem is that when an ultraviolet cutoff was introduced to the Planck scale and zero-point energy was calculated, a large cosmological constant proportional to the fourth power of the Planck mass was obtained.

In this calculation method, the cutoff is finite and has a physical meaning. In other words, it is presumed that there is no field in a world beyond the Planck scale. Needless to say, it comes from the prejudice that the gravitational field cannot be renormalized. Considering discrete spacetime is often called quantization of gravity, but denying field continuity like this breaks diffeomorphism invariance. Hence, the problem with diffeomorphism invariance is hidden behind the cosmological constant problem.

When the cutoff method is applied to gauge field theory, a mass term of gauge field proportional to the cutoff squared will appear and gauge invariance is broken. Mass terms will also appear in other matter fields. At that time, a choice is often made to manually remove the terms that obviously break gauge invariance and leave the terms that do not. The cosmological constant in question is that kind. In the first place, there is no guarantee that the cutoff dependence will be in the form of the cosmological term, which is not a simple mass term, because diffeomorphism invariance is broken.

Despite the fact that what predicts the existence of the cosmological term is diffeomorphism invariance, if the quantization method breaks this invariance, it is a wrong way for the end. Actually, if you use dimensional regularization, which can make the theory finite while preserving gauge invariance and diffeomorphism invariance, ultraviolet divergences in a power of the cutoff that break the invariance do not appear.

To solve the cosmological constant problem, you have to step into the trans-Planckian regions. In general, quantum theory of gravity that holds even in such regions shows that zero-point energy vanishes as a result of diffeomorphism invariance, as was shown in Chapter 5 using the Schwinger-Dyson equation. That is, it means that the cause of the problem itself disappears.

Well then, what is the cosmological constant? In the renormalizable quantum gravity it becomes a physical constant. To be precise, it is a renormalization group invariant, that is, a true constant whose value does not change in the evolution of the universe. It is a constant that is determined experimentally, as is the Newton constant.

The discovery of the cosmological constant is similar to the discovery of neutrino mass that led to Kajita's 2015 Nobel Prize. The neutrino mass has long been believed to be zero, but that was just a desire, only because the number of free parameters was reduced and the theory became simpler. There is no symmetry that requires the mass to vanish. These discoveries tell us that if a quantity does not denied by some symmetry, it should exist. In this case, what you need to find is merely whether its value is large or small. Conversely, this indicates that there is a world of difference between zero and small.

Finally, as an alternative to the cosmological constant, the term "dark energy" is often used in recent years. The cosmological constant is literally a physical constant, but this term is used more generally including a possibility that it is not a constant. It is impressive when used in combination with the word dark matter. Also, inflation theory discussed in the framework of Einstein's theory of gravity generally uses the cosmological constant, so there seems to be a theoretical background that wants to distinguish it from the current cosmological constant.

Chapter 12

Other Issues in Quantum Gravity

This chapter devotes to other issues that are believed to occur near the Planck scale. One is the Landau pole singularity, which is known as a theoretical difficulty that arises in QED, and the other is the issue on topology. Let us see how these problems are reinterpreted in the quantum gravity theory.

QED and Landau poles

The Landau pole singularity is a problem that is thought to occur generally in asymptotically non-free quantum field theory whose beta function is positive, and refers to having a singular point where the running coupling constant incorporating quantum corrections diverges on the high energy side. QED is known as a representative of theories with Landau poles. Such a theory is not defined in regions beyond energy at which the singularity appears, so it has to be replaced with a new theory there. Hence, a theory with the Landau pole singularity is not appropriate for a fundamental theory that describes a microscopic world.

Such a bad thing is usually thought to occur near the Planck scale. Therefore, you have to consider a scenario that the theory is integrated into another particle model that shows asymptotic freedom such as grand unified theories (GUT) before it breaks down, otherwise restrict parameters of the theory so that it works up to the Planck scale. The latter is a negative way of thinking that if the theory holds until the Planck scale, then it is not a matter, from the perspective that there is no point in thinking about a world beyond it.

However, the dynamical scale $\Lambda_{\rm QG}$ is set to 10^{17} GeV here, which is two orders of magnitude lower than the Planck energy. Therefore, the theoretical situation changes because quantum gravity corrections are added before reaching the Planck scale.

Now consider whether Weinberg's asymptotic safety requirements mentioned in Chapter 10: 10.3, can be applied not only to quantum gravity but also to QED. That of quantum gravity has already been revealed as the BRST conformal invariance, and here I expect that QED with quantum gravity corrections also has a non-trivial ultraviolet fixed-point where its beta function β_e vanishes. Although non-perturbative calculations to confirm this are so difficult, it is known that quantum gravity loop corrections in the expansion by t described in Chapters 4, 7, and 10 give a negative contribution to the QED beta function. Therefore, it can be expected that β_e will decrease and reach an ultraviolet fixed-point when energy exceeds Λ_{QG}.

During the inflation period, the conformally invariant dynamics in which the Weyl tensor disappears are well retained and the running coupling constant $\bar{t}(Q)$ does not increase until near the point where the spacetime phase transition occurs. Let us then assume that the QED beta function β_e is also mostly stuck at a fixed point, maintaining a zero value which indicates the presence of conformal invariance. Approaching the phase transition point, the running coupling constant increases rapidly, and eventually the conformally invariant dynamics of gravity disappear completely, shifting to a low-energy effective field theory based on Einstein's theory of gravity. At that time, if β_e also becomes non-zero and the Wess-Zumino interaction $\sqrt{-g}\,\beta_e \phi F_{\mu\nu}^2$ opens, you can consider a scenario that photons are generated from the scalar-like quantum gravity fluctuation through this interaction.

The mechanism of matter creation during the spacetime phase transition also applies to the QCD and GUT models if the photon field strength $F_{\mu\nu}$ is replaced with that of the Yang-Mills field. In these cases, the Landau pole singularity does not occur because the beta functions disappear in the high energy limit.

Topology and gravitational instantons

Not a few researchers think that quantization of gravity is to do a summation over topologies. In lower-dimensional quantum gravity, where there is no local dynamics of the gravitational field, topology may be essential, but in four dimensions, it is not a major topic and the local dynamics of the gravitational field is much more important. However, if you are looking for dynamics causing CP violation, for instance, in the early universe, spacetime with a special topology may contribute to it. In general, such spacetime configurations are called gravitational instantons.

There are a number of gravitational instanton solutions, which were investigated mainly in the 1970s and summarized in a report by Eguchi, Gilkey, and Hanson. Most of them are given as solutions of Einstein equations with satisfying a self-dual condition of the Riemann curvature tensor, and many of them have boundaries.

Here consider a particular solution closely related to the Weyl action, which is very similar to the Yang-Mills instanton solution discovered by Belavin, Polyakov, Schwartz, and Tyupkin in 1975. It is a compact four-manifold with no boundaries that is obtained as a solution that satisfies both the self-dual condition for the Weyl tensor and the equation of motion of the gravitational field derived from the Weyl action.[1] The known solution is only the two-dimensional complex projective space ($\mathbb{CP}^2$) examined by Eguchi and Freund.

The full Weyl action in $\mathbb{CP}^2$ space may be given by adding a gravitational version of the theta term, $i\theta\tau$, where τ is the Hirzebruch signature, which denotes the number of the gravitational instantons and is given here with $\tau = \pm 1$ depending on orientation. Up to this point, it is similar to normal instanton solutions, simply the field strength of the Yang-Mills field is replaced with the Weyl tensor, which is the field strength of the traceless tensor field.

The big difference from when dealing with normal instantons is that

[1] Discuss in Euclid spacetime as in the Yang-Mills instanton, which is obtained by Wick rotating the background spacetime.

there is no spin structure in the $\mathbb{CP}^2$ space, that is, fermions cannot be correctly defined in the space. This can be seen from the fact that Atiyah-Singer's spin-1/2 Dirac index, which must be an integer, becomes a fraction as $\mathcal{I}_{1/2} = -\tau/8$. Being an integer is a necessary condition, and can be overcome by the number of Weyl fermions to a multiple of 8, but it is not enough yet. When parallel-transporting a fermion over a non-contractible two-sphere ($\mathbb{CP}^1$) subspace in $\mathbb{CP}^2$, you find another contradiction that an extra phase arises from the spin connection.

If fermions do not exist in space, the real world cannot be described. So Hawking and Pope came up with so-called generalized spin structures to accommodate fermions. That is to introduce a background $U(1)$ gauge field on the $\mathbb{CP}^2$ space and modify the spacetime structure so that fermions are affected not only by the spin connection but also by the gauge field. If fermion charges are odd, the problem can be solved by offsetting the extra phase resulting from the spin connection when parallel-transported over the $\mathbb{CP}^1$ subspace with that from the background gauge field. At that time, it also adds a restriction that charges of boson fields coupled with the background gauge field must be even numbers.

What you can see from the discussion so far is that the Standard Model with $SU(3) \times SU(2) \times U(1)$ does not have spin structures in the $\mathbb{CP}^2$ space. If you just want to make the Atiyah-Singer index an integer, you can do it by considering right-handed neutrinos because the total number of right-handed and left-handed fermions becomes 16 per generation. However, since the right-handed neutrino has no gauge charge, the phase of the spin connection cannot be erased by introducing a background gauge field. For other fermions and boson fields, their charges are also not assigned well, so that there is no spin structure. Therefore, there is no choice but to modify the theory so that spin structures can exist by adding an extra $U(1)$ gauge field and appropriately assigning its charge to fields. Also, the $SU(5)$ GUT with 15 fermions for each generation does not have spin structures because the Atiyah-Singer index does not become an integer.

Of various GUT models that contain the Standard Model as part, the simplest gauge group that can have spin structures on $\mathbb{CP}^2$ is $SO(10)$. In this case, there are several possible ways to make it have spin structures, and they depend on which $U(1)$ subgroup of $SO(10)$ the background gauge field is introduced into. In addition, the Atiyah-Singer index must be chosen to be zero, in order for the partition function to exist with a non-zero value.[2] Therefore, for example, considering $SO(10) \to SU(4) \times SU(2) \times SU(2)$ as a symmetry breaking pattern and introducing the background gauge fields into $U(1)$ subgroups of both $SU(2)$s, it can have spin structures.

Thus, there are some interesting limitations, such as that $SO(10)$ is selected, which has not yet been excluded from experiments, and the number of fermions is required to be a multiple of 16. However, this does not force $SO(10)$ by any means. If spin structures do not exist, this topology is merely excluded from the path integral. Conversely, if $SO(10)$ is chosen as physics, then $\mathbb{CP}^2$ space may also have to be considered. The application of gravitational instantons to physics is a research field that has not progressed much, and is a future task.

[2] This is a necessary condition for absence of fermion zero-modes. If there is a zero-mode, the partition function disappears due to its Grassmann integral. If there is a mass term, it can be avoided to be vanishing, but here is considering in high energy regions where the mass can be regarded as substantially zero.

Chapter 13

Lattice Quantum Gravity

The asymptotically background-free quantum gravity is a renormalizable quantum field theory that is formulated in a hybrid form combining nonperturbative conformal field theory and perturbation theory. As an approach to formulate this theory completely nonperturbatively, a method based on dynamical triangulations has been proposed. As in the case of lattice gauge field theory, this is a method of considering Euclidean spacetime and dividing it into random lattices for examination.

The method of formulating gravity theory by discretizing spacetime was started by Regge in the 1960s. Initially, various spacetimes were expressed by changing length of each side while fixing how to connect lattices. However, it is recognized that this method cannot sufficiently express all possible spacetime configurations required for path integral over the gravitational field. There is also a question of what integral measure to use when performing it.

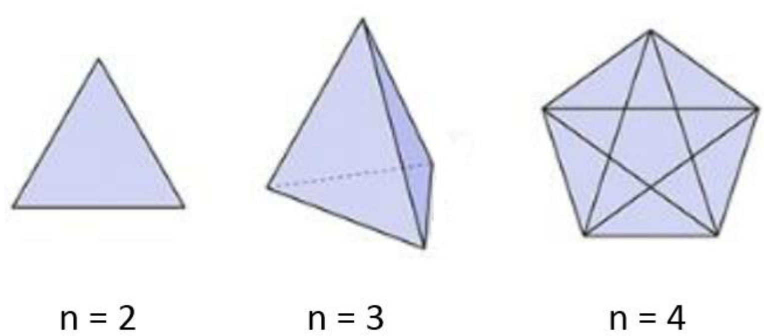

Figure 13-1: Element of n-dimensional simplicial quantum gravity

As a method different from the Regge calculus, a "simplicial quantum gravity" has been proposed, in which spacetime is divided into a random lattice, instead of changing length of each side. An element that composes a lattice is given by a simplex as shown in Fig. 13-1, and n-dimensional spacetime is made by gluing n-simplices. For example, when triangles, which are representative of 2-simplices, are glued, a two-dimensional surface as shown in Fig. 13-2 can be constructed. Here, a spacetime without boundaries is considered and its topology is fixed.

What is important here is the variety of possible patterns for gluing simplices, not the shape of each simplex, which is considered equivalent with the same volume. Spacetime exists inside the simplex on average, and it is thought that a smooth spacetime can be obtained by taking a continuum limit. Different gluing patterns represent locally different spacetime configurations, and the path integral for gravity is expressed by taking the sum of all possible configurations. For topology, first consider the sphere, and if necessary, consider other topologies and take the sum over them.

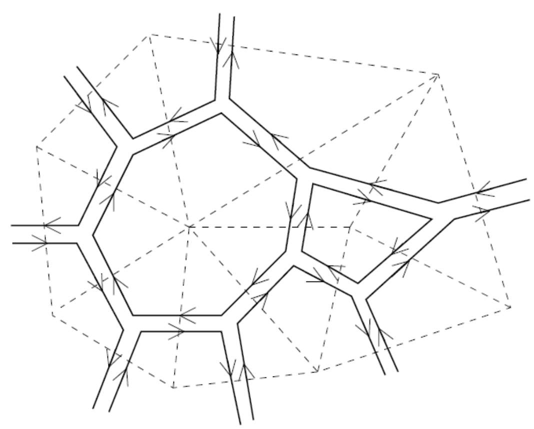

Figure 13-2: A two-dimensional random surface (dotted line) and Feynman diagram dual to it (double line).

This method was first performed successfully in two dimensions by Weingarten, David, Kazakov, and others. Let the number of triangles be N_2 and denote their possible gluing patterns as $\mathcal{T}_2$, then the partition function is defined by

$$Z_{\mathrm{SG}_2}(\lambda) = \sum_{N_2=0}^{\infty} \sum_{\mathcal{T}_2} e^{-\lambda N_2} = \sum_{N_2=0}^{\infty} \Omega(N_2) e^{-\lambda N_2}, \tag{13-1}$$

where $\Omega(N_2)$ is the partition number of $\mathcal{T}_2$. This is known as a statistical model in which a second-order phase transition occurs. Denoting the critical point by $\lambda = \lambda^c$, the partition number behaves in the large N_2 as $\Omega(N_2) \sim N_2^{\gamma_{\mathrm{st}}-3} e^{\lambda^c N_2}$, where the exponent γ_{st} is called the string susceptibility. Letting area of each triangle be 1, N_2 becomes area of the surface, and thus λ acts as a cosmological constant that controls it. Introducing a scale a corresponding to lattice spacing, two-dimensional quantum gravity is then derived taking a continuum limit that $N_2 \to \infty$ at the same time as $a \to 0$ while retaining $(\lambda - \lambda^c)/a^2 \to \lambda_{\cos}$ at the critical point, where $\lambda_{\cos}$ is a physical cosmological constant.

Two-dimensional random surfaces have a one-to-one correspondence with Feynman diagrams when considering dual diagrams as shown in Fig. 13-2, thus the partition function (13-1) can be expressed using a matrix model with three-point interaction (like a 0-dimensional Yang-Mills theory) that generates the Feynman diagrams. Using this method, the summation of random surfaces can be performed analytically as an integral of the matrix variable, and you can show that the second-order phase transition actually occurs. Furthermore, it was shown by three groups, Brezin and Kazakov, Douglas and Schenker, Gross and Migdal that taking the continuum limit just on the phase transition point gives a result completely in agreement with the two-dimensional quantum gravity mentioned at the end of Chapter 4.

In the 1990s, following the success in two dimensions, extension to higher dimensions was actively examined. Numerical calculations of four-dimensional quantum gravity by the dynamical triangulation method were started by groups such as Ambjørn and Jurkiewicz, Agishtein and Migdal, and were executed as follows. Letting N_4 be the number of 4-simplices representing total volume and N_2 be the number of 2-simplices contained in it, the

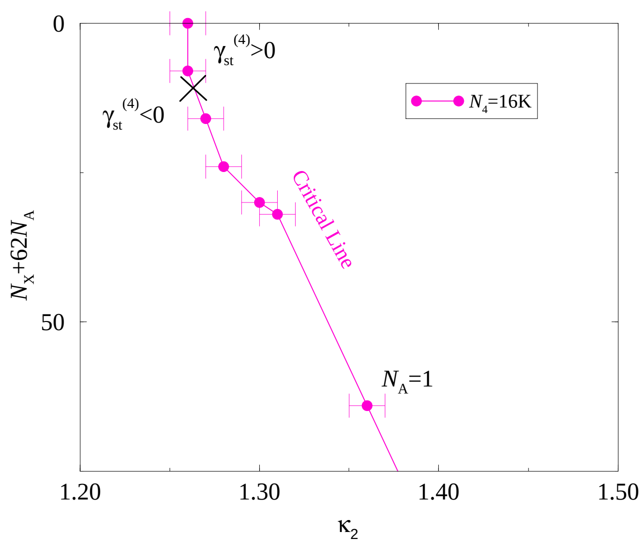

Figure 13-3: Phase transition point κ_4^c and its κ_2-dependence in numerical calculation when $N_X + 62N_A$ matter fields are introduced, where N_X and N_A represent the number of scalar and $U(1)$ gauge fields, respectively. Below the $\times$ mark, the susceptibility becomes negative and a second-order phase transition point appears. [K. Hamada, S. Horata and T. Yukawa, *Focus on Quantum Gravity Research* (Nova Science Publisher, NY, 2006), Chap. 1.]

partition function is then defined by

$$Z_{\mathrm{SG}_4}(\kappa, \lambda) = \sum_{N_4=0}^{\infty} \sum_{\mathcal{T}_4} e^{\kappa N_2 - \lambda N_4} = \sum_{N_4=0}^{\infty} \Omega(\kappa, N_4) e^{-\lambda N_4}, \qquad (13\text{-}2)$$

where $\mathcal{T}_4$ denotes possible gluing patterns of N_4 4-simplices. κ corresponds to the square of the Planck mass, and λ corresponds to the cosmological constant. If a second-order phase transition occurs at $\lambda = \lambda^c(\kappa)$, the partition number behaves like $\Omega(\kappa, N_4) \sim N_4^{\gamma_{\mathrm{st}}^{(4)}-3} e^{\lambda^c(\kappa) N_4}$ at that point, where the exponent $\gamma_{\mathrm{st}}^{(4)}$ is a four-dimensional version of the string susceptibility.

In 4-simplicial quantum gravity, the existence of the second-order phase transition required to take the continuum limit is not as obvious as in the two-dimensional model, but its existence is numerically suggested by Horata,

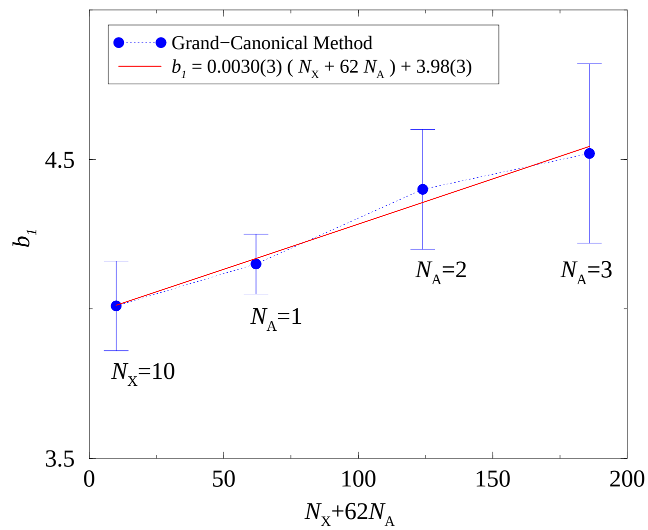

Figure 13-4: Numerical results of the susceptibility $\gamma_{\rm st}^{(4)}$. By matching the calculated susceptibility with the formula of $\gamma_{\rm st}^{(4)} = 2 - (b_1 + \sqrt{b_1^2 - 4b_1})/2$ derived from the continuum theory, a functional form of b_1 shown in the figure is obtained, where b_1 is the coefficient of the Riegert action. As predicted in (B-3), it can be seen that it is represented by a linear function of $N_X + 62N_A$. [K. Hamada, S. Horata and T. Yukawa, *Focus on Quantum Gravity Research* (Nova Science Publisher, NY, 2006), Chap. 1.]

Egawa, and Yukawa (see Fig. 13-3). That is the result of numerical simulation performed by gluing more than 10,000 4-simplices to form a manifold with four-sphere topology and further incorporating matter fields into it. Fig. 13-4 shows that the four-dimensional string susceptibility obtained by evaluating just at the phase transition point is consistent with that derived from the asymptotically background-free quantum gravity in four dimensions.

It should be noted in comparing the two methods that the weight $e^{\kappa N_2 - \lambda N_4}$ of the 4-simplicial quantum gravity has only the two parts corresponding to the Einstein-Hilbert action and the cosmological term in the continuum theory. In other words, information on the fourth-derivative gravitational terms will be included in the gluing patterns denoted by $\mathcal{T}_4$. This is linked to the fact that the fourth-derivative gravitational action $I^{(4)}$ and the Wess-Zumino action S are purely quantum-mechanical quantities independent of $\hbar$, and can be regarded as part of the path integral measure, as mentioned at the end of the first section of Chapter 4.

Lastly, three-dimensional quantum gravity will be briefly described below. Unlike even dimensions, there is no gravitational scalar quantity that becomes dimensionless when volume integration is done in three dimensions. Therefore, it is necessary to introduce a dimensionful parameter to the gravitational action. From this, it can be seen that the methods employed in two and four dimensions cannot be applied to three dimensions. As one of the methods using continuum fields, Witten's approach describing the gravity theory without the cosmological constant and matter fields as a Chern-Simons theory is widely known. It represents that three-dimensional gravity is a topological field theory with no local degrees of freedom. Corresponding to this, there are some models expressing topological invariants as 3-simplicial gravity such as the model by Ponzano and Regge constructed by using the Racah-Wigner $6j$ symbol to represent 6 sides of a tetrahedron and the model by Turaev and Viro that is a more mathematical development of it. Research on the relationship between them and the Chern-Simons theory is still ongoing.

In addition, Boulatov's model proposed as a 3-simplicial quantum gravity is well known, which is defined by doing the sum over all possible gluing patterns of 3-simplices, with fixing topology and adding the weights due to the cosmological term and the Einstein-Hilbert action as in (13-1) and (13-2). He generalized the matrix model that succeeded in 2-simplices to a model with tensor legs, and aimed to execute the sum in 3-simplices. However, since it is a fairly complicated model, it cannot be said that the analysis is progressing. Also, it seems that a large-scale numerical calculation for 3-simplicial gravity has not yet been performed, probably because proper physical quantities that can be compared and examined are not known. Three-dimensional quantum gravity is mainly studied as a mathematical object.

Chapter 14

Concluding Remarks

There is an old scientific thought called "Ockham's razor". That is a principle advocated by William, a philosopher in the village of Ockham, England in the 14th century, stating that you should not make more assumptions than necessary to explain an event. A razor is used to mean cutting off unnecessary things.

The Ptolemaic theory, for instance, can be made as precise as you like by adding circular motions one after another if all the planets are on the same disk. This is, however, typical of parameter physics which introduces variables one after another to match observations, thus is against the thought of Occam's razor. In fact, when Pluto and comets with oblique orbits are added, it can no longer deal with them, and if you consider outside the solar system, it breaks down. Occam's razor does not judge authenticity, but it makes sense to think that it is better to be able to explain by simpler principles.

The guiding principle that has been respected throughout this book is diffeomorphism invariance. However, it seems that many people in the world put the Einstein equation first. Since the Einstein equation is beautiful and so clear that it can be used as a T-shirt design, there is a certain idolatry aspect, and there is a tendency that denying this equation is denying Einstein himself. That is, of course, wrong, and Einstein's greatest achievement is undoubtedly the discovery that diffeomorphism invariance is one of the fundamental principle of physics. That is never in conflict with quantum theory. The Einstein equation is one of equations derived from the principle, but it is incomplete.

The "asymptotically background-free quantum gravity", proposed to describe the trans-Planckian world, is a quantum field theory which has been formulated so that diffeomorphism invariance holds everywhere. This is a renormalizable theory that can handle infinite degrees of freedom systematically and is described using continuum fields without introducing ultraviolet cutoffs. This nature is, of course, guaranteed by diffeomorphism invariance. The background-metric independence that characterizes quantum gravity is also contained in properties that diffeomorphism invariance acquires at the quantum level. I have shown that these properties provide new solutions different from before to the problems with gravity. All are derived from diffeomorphism invariance.

The novelty in this approach is the use of conformal invariance. The background-metric independence hidden in diffeomorphism invariance is revealed in the form of conformal invariance by non-perturbatively treating the most important conformal factor in the gravitational field that governs distance, where the other tensor modes are handled perturbatively as having less role in the ultraviolet limit. The quantization of gravity in this case is not a simple task that is automatically established if diffeomorphism invariant actions are prepared. It is necessary to correctly incorporate contributions from the path integral measure, which is generally called conformal anomaly. When that is executed, the background-metric independence is expressed as the BRST conformal invariance. That is, all theories with different backgrounds connected to each other by conformal transformations become gauge equivalent. The world beyond the Planck scale is then described as a special conformal field theory with this property.

The condition that the theory should be renormalizable provides a rationale that the dimension of spacetime is four. It seems that many people unknowingly see the universe from outside the four-dimensional spacetime. An idea that the universe was born at a certain moment is that you are looking at that moment from the outside. However, since you are in a four-dimensional world, you cannot know if there is an outside world in the universe. Extra dimensions are like unknown continents. It adds an infinite number of extra degrees of freedom than what needs to be explained.

The asymptotic background freedom indicates the existence of a novel dynamical scale of quantum gravity. It is a physical energy scale that separates between quantum spacetime phase with the background-metric independence and classical spacetime phase, and its value has been determined to be 10^{17} GeV, which is two orders of magnitude smaller than the Planck mass $m_{\rm pl}$, from the scenario of inflation ignited by the quantum gravity effect. That is, it implies that the quantum effect of gravity begins to work before reaching the Planck scale. Therefore, problems in classical theory that were thought to be manifest at the Planck scale can be avoided.

The correlation length of quantum gravity is given by the reciprocal of this energy scale, which is about 100 times the Planck length. If a distance you are trying to measure is longer than the correlation length, you can measure it classically, but once inside, you will be in a quantum world where the spacetime phase changes and distance fluctuates greatly. The background-metric independent world will emerge as its extreme state. This means that physical distance is substantially quantized by the correlation length, though spacetime is described continuously.

It can be shown that as a consequence of diffeomorphism invariance, the Hamiltonian of the whole system vanishes, which means that zero-point energy disappears. Therefore, you cannot think of it as the origin of primordial fluctuations. The evolution of the universe is represented as a process that states of spacetime are changing while preserving the Hamiltonian to be zero. The evolution scenario becomes clearer when you consider that the universe began from a state where only quantum fluctuations of gravity exist. That indicates that quantum gravity is the source of everything.

The gravitational field is a dimensionless field and is also the only field that interacts with all matter directly. And the asymptotic background freedom says that scalar-like fluctuations by the conformal factor in the gravitational field become predominant in the early universe. Although matter fields still remain conformally invariant in the trans-Planckian region and are not coupled to the conformal-factor field there, when the spacetime phase transition begins to occur, interactions between the conformal-factor field and matter fields open through the Wess-Zumino action of conformal anomaly, and matters are generated. The primordial spectrum obtained in this way after the phase transition inherits the scalar-like and scale-invariant properties derived from the conformal invariance of quantum gravity. The essence of these properties originates from the fact that the gravitational field is dimensionless.

Nobody knows how wide the universe is. The observed region is only a part of it. In the inflation scenario, going back to the beginning of the universe, all currently visible regions fall within the correlation length of quantum gravity. That is, it suggests that the universe started with one of bubbles excited by quantum gravity. When inflation begins, it is considered that new bubbles fill gaps in space created by the expansion densely so that its energy density becomes almost constant. The universe that expanded rapidly like that undergoes the spacetime phase transition and shifts to the present universe, then bubbles are converted into matters all at once.

The trans-Planckian world with the background-metric independence is an unimaginable world as mentioned so far. That is why a solid guiding principle is necessary, and that is diffeomorphism invariance. There are attempts looking for new guiding principles, but the first thing you should do is to have a deep understanding of diffeomorphism invariance. What I had done was exactly to make sure that normal field methods work for quantum theory of gravity without doing any oddities, while adhering to diffeomorphism invariance. This has opened up new horizons in quantum gravity research.

Appendix A

Fundamental Constants

Reduced Planck constant	$\hbar$	$=$	1.055×10^{-27} cm^2 g s^{-1}
Speed of light	c	$=$	2.998×10^{10} cm s^{-1}
Newton's constant	G	$=$	6.672×10^{-8} cm^3 g^{-1} s^{-2}
Planck mass	$m_{\rm pl}$	$=$	2.177×10^{-5} g
		$=$	1.221×10^{19} GeV/c^2
Reduced Planck mass	$M_{\rm P}$	$=$	2.436×10^{18} GeV/c^2
Planck length	$l_{\rm pl}$	$=$	1.616×10^{-33} cm
Planck time	$t_{\rm pl}$	$=$	5.390×10^{-44} s
Boltzmann constant	$k_{\rm B}$	$=$	1.381×10^{-16} erg K^{-1}
Megaparsec	1Mpc	$=$	3.086×10^{24} cm
Hubble constant	H_0	$=$	$100h$ km s^{-1} Mpc^{-1}
Hubble distance	c/H_0	$=$	$2998h^{-1}$ Mpc
			(current observation: $h \simeq 0.7$)

Useful constants for converting to natural units ($c = \hbar = k_{\rm B} = 1$)

1 cm	$=$	5.068×10^{13} $\hbar$/GeV
1 s	$=$	1.519×10^{24} $\hbar$/GeV/c
1 g	$=$	5.608×10^{23} GeV/c^2
1 erg	$=$	6.242×10^{2} GeV
1 K	$=$	8.618×10^{-14} GeV/$k_{\rm B}$

Appendix B

Concise Summary of The Theory

The Minkowski metric used in this book is $\eta_{\mu\nu} = (-1, 1, 1, 1)$. The Riemann curvature tensor is defined by $R^\lambda{}_{\mu\sigma\nu} = \partial_\sigma \Gamma^\lambda{}_{\mu\nu} - \partial_\nu \Gamma^\lambda{}_{\mu\sigma} + \Gamma^\lambda_{\rho\sigma}\Gamma^\rho_{\mu\nu} - \Gamma^\lambda_{\rho\nu}\Gamma^\rho_{\mu\sigma}$ using the Christoffel symbol $\Gamma^\lambda_{\mu\nu} = g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})/2$. The Ricci tensor is $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$, and the scalar curvature is $R = R^\mu{}_\mu$. The covariant derivative is represented by $\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda$.

The action of the asymptopically background-free quantum gravity, $I = I^{(4)} + I_{\rm EG}/\hbar$, is given by

$$I = \int d^4x \sqrt{-g} \left[-\frac{1}{t^2} C^2_{\mu\nu\lambda\sigma} - bG_4 + \frac{1}{\hbar} \left(\frac{1}{16\pi G} R - \Lambda + \mathcal{L}_{\rm M} \right) \right]. \quad \text{(B-1)}$$

The first two terms gives the conformally invariant fourth-derivative action $I^{(4)}$, the first is the Weyl action, and the second $G_4 = R^2_{\mu\nu\lambda\sigma} - 4R^2_{\mu\nu} + R^2$ is the Euler density (Gauss-Bonnet combination). The action of the renormalizable theory will be described in terms of bare quantities, and usually symbols different from renormalized quantities are used, but they are not distinguished here for simplicity. Also, as mentioned in the text, $\hbar$ appears only before the lower derivative $I_{\rm EG}$. This comes from the fact that the gravitational field is a completely dimensionless field. All fourth-derivative gravitational actions describe purely quantum-mechanical dynamics and create entropy of spacetime. In the following, $\hbar = 1$.

Decompose the metric tensor field as $g_{\mu\nu} = e^{2\phi}\bar{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu} = (\hat{g}e^h)_{\mu\nu} = \hat{g}_{\mu\lambda}(\delta^\lambda{}_\nu + h^\lambda{}_\nu + \cdots)$, using the conformal-factor field ϕ, the traceless tensor field $h^\mu{}_\nu$, and the background metric $\hat{g}_{\mu\nu}$. Since the Weyl tensor is the field-strength of $h^\mu{}_\nu$, introducing the coupling constant t as in (B-1) is intended to be expanded in $h^\mu{}_\nu$. On the other hand, the conformal factor $e^{2\phi}$ is treated non-perturbatively as it is. Here, note that $I^{(4)}$ does not contain the ϕ-field, while the other lower-derivative action $I_{\rm EG}$ has an exponential factor of ϕ, and also fourth-derivative gravitational actions shown below are expressed by polynomials of ϕ.

The key to quantization is to rewrite the theory into a quantum field theory defined in the background spacetime with the practical metric $\hat{g}_{\mu\nu}$. The partition function is expressed in the path integral method by

$$Z = \int [dg]_g \, e^{iI(g)} = \int [d\phi dh]_{\hat{g}} \, e^{iI_{\rm QG}(\phi, \bar{g})}, \tag{B-2}$$

and then the action is changed to $I_{\rm QG}(\phi, \bar{g}) = S(\phi, \bar{g}) + I(g)$, where e^{iS} is a Jacobian that is necessary to preserve diffeomorphism invariance when rewriting the invariant measure of the full metric to a practical measure defined on the background metric. The S is called the Wess-Zumino action satisfying the Wess-Zumino consistency condition (see footnote 2 in Chapter 4), whose conformal variation yields conformal anomaly.

The Wess-Zumino action S is responsible for fourth-derivative dynamics of the conformal-factor field ϕ. The action that remains even at the zeroth order of the coupling constant t is particularly important, which is called the Riegert action, given by

$$S_{\rm R}(\phi, \bar{g}) = -\frac{b_1}{(4\pi)^2} \int d^4x \sqrt{-\bar{g}} \left(2\phi \bar{\Delta}_4 \phi + \bar{E}_4 \phi \right),$$

where $E_4 = G_4 - 2\nabla^2 R/3$ is the Euler density modified up to the total divergence and $\sqrt{-g}\Delta_4$ is a conformally invariant fourth-derivative operator for scalars, defined by $\Delta_4 = \nabla^4 + 2R^{\mu\nu}\nabla_\mu \nabla_\nu - 2R\nabla^2/3 + \nabla^\mu R \nabla_\mu/3$. The quantities with the bar are the ones defined by $\bar{g}_{\mu\nu}$, and these satisfy the relation $\sqrt{-g}E_4 = \sqrt{-\bar{g}}(4\bar{\Delta}_4\phi + \bar{E}_4)$. The lowest of the coefficient $b_1 = b_c + o(t^2)$ is given by

$$b_c = \frac{1}{360}\left(N_{\rm X} + 11N_{\rm F} + 62N_{\rm A}\right) + \frac{769}{180}, \tag{B-3}$$

where $N_{\rm X}$, $N_{\rm F}$, and $N_{\rm A}$ are the number of scalar fields, Dirac fermions, and gauge fields belonging to the matter term $\mathcal{L}_{\rm M}$, respectively. For example, b_c is 7.0 for the Standard Model, 9.1 for the $SU(5)$ GUT model, and 12.0 for the $SO(10)$ GUT model. In addition, interaction terms such as $\phi^{n+1}\sqrt{-\bar{g}}(2\bar{\Delta}_4\phi + \bar{E}_4)$, $\phi^n\sqrt{-\bar{g}}\,\bar{C}^2_{\mu\nu\lambda\sigma}$, and $\phi^n\sqrt{-\bar{g}}\bar{F}^2_{\mu\nu}$ for a gauge field

$F_{\mu\nu}$ with $n \geq 1$ arise as the Wess-Zumino actions at higher orders of t (note that $\sqrt{-g}\, C^2_{\mu\nu\lambda\sigma} = \sqrt{-\bar{g}}\, \bar{C}^2_{\mu\nu\lambda\sigma}$ and $\sqrt{-g} F^2_{\mu\nu} = \sqrt{-\bar{g}} \bar{F}^2_{\mu\nu}$).[1] The last one will contribute to the generation of matters at the spacetime phase transition.

The Riegert action, together with the kinetic term of the Weyl action, is a key part in realizing the BRST conformal invariance, which is an algebraic representation of the background-metric independence at the ultraviolet limit of $t \to 0$. It shows that physical results do not change even if the background metric is Weyl-rescaled like $\hat{g}_{\mu\nu} \to e^{2\sigma} \hat{g}_{\mu\nu}$. And therefore, it is justified to quantize gravity using a simple Minkowski metric as a background.

The beta function $\beta_t = \mu dt/d\mu = -\beta_0\, t^3 + o(t^5)$ is negative and the 1-loop coefficient is given by $\beta_0 = [(N_{\rm X} + 6N_{\rm F} + 12N_{\rm A})/240 + 197/60]/(4\pi)^2$, where μ is an arbitrary mass scale introduced upon quantization. The corresponding Weyl sector effective action is, in momentum space, given by

$$\begin{aligned} \Gamma_{\rm W} &= -\left[\frac{1}{t^2} - 2\beta_0 \phi + \beta_0 \log\left(\frac{q^2}{\mu^2} \right) \right] \sqrt{-g}\, C^2_{\mu\nu\lambda\sigma} \\ &= -\frac{1}{\bar{t}^2(Q)} \sqrt{-g}\, C^2_{\mu\nu\lambda\sigma}. \end{aligned} \tag{B-4}$$

The first in the first line is the tree action, the second is the Wess-Zumino action, and the third is a nonlocal loop correction, which put together in the running coupling constant as $\bar{t}^2(Q) = [\beta_0 \log(Q^2/\Lambda^2_{\rm QG})]^{-1}$, where $Q^2 = q^2/e^{2\phi}$ (7-3). The dynamical energy scale is then expressed as $\Lambda_{\rm QG} = \mu\, e^{-1/2\beta_0 t^2}$. This is one of the renormalization group invariants, satisfying $d\Lambda_{\rm QG}/d\mu = 0$. The same is true including higher order corrections, though expressions of the running coupling constant and the dynamical scale become more complicated.

The physical cosmological constant satisfying $d\Lambda_{\rm cos}/d\mu = 0$, which is

[1] Strictly speaking, in order to correctly perform higher-order path integral, including these terms as S, it is necessary to formulate (B-2) using dimensional regularization, which is known as the only method that can perform higher loop calculations while preserving diffeomorphism invariance (see Chapter 10).

defined by the effective action $\Gamma_\Lambda = -\Lambda_{\rm cos}\sqrt{-g}$, is given as follows:

$$\begin{aligned}\Lambda_{\rm cos} = \Lambda &+ (7-2\log 4\pi)\frac{\Lambda}{b_c} - \left(\frac{\Lambda}{b_c} - \frac{9\pi^2 M^4}{2b_c^2}\right)\log\left(\frac{64\pi^2}{\mu^4}\frac{\Lambda}{b_c}\right) \\ &-\frac{9\pi^2}{2}\left(\frac{25}{3} - 4\log 4\pi\right)\frac{M^4}{b_c^2} \\ &-6\pi\frac{M^2}{b_c}\sqrt{\frac{\Lambda}{b_c} - \frac{9\pi^2 M^4}{4b_c^2}}\arccos\left(\frac{3\pi M^2}{2\sqrt{b_c\Lambda}}\right) \\ &+\frac{5}{128}\alpha_t^2 M^4\left(\log\frac{\pi^2\alpha_t^2 M^4}{\mu^4} - \frac{21}{5}\right),\end{aligned}$$

where $\alpha_t = t^2/4\pi$. $M = 1/\sqrt{8\pi G}$ and Λ are renormalized quantities of the Planck mass and the cosmological constant in the action, respectively. To derive this expression, an approximation of large b_c, namely large number of matter fields, is employed, so that the ratios Λ/b_c and M^4/b_c^2 are comparable, and $\alpha_t/4\pi$ and $1/b_c$ are also so.

Finally, the equations of motion describing the static spherical excitation discussed in Chapter 8 are written below. As a coupled linear differential equation of gravitational potentials Φ and Ψ, it is given by

$$\frac{b_c}{8\pi^2}B(r)\left(-\frac{4}{3}\partial\!\!\!/^4\Phi - \frac{2}{3}\partial\!\!\!/^4\Psi\right) + M_{\rm P}^2\left(-4\partial\!\!\!/^2\Phi - 2\partial\!\!\!/^2\Psi\right) = 0$$

and

$$\begin{aligned}&\frac{b_c}{8\pi^2}B(r)\left(-\frac{8}{9}\partial\!\!\!/^4\Phi - \frac{4}{9}\partial\!\!\!/^4\Psi\right) + \frac{1}{\bar{t}^2(r)}\left(-\frac{8}{3}\partial\!\!\!/^4\Phi + \frac{8}{3}\partial\!\!\!/^4\Psi\right) \\ &+M_{\rm P}^2\left(-2\partial\!\!\!/^2\Phi - 2\partial\!\!\!/^2\Psi\right) = 0,\end{aligned}$$

where $\partial\!\!\!/^2 = \partial_r^2 + (2/r)\,\partial_r$ is the spatial Laplace operator. $B(r) = [1 + a_1\bar{t}^2(r)]^{-1}$ is a factor modeled by adding higher-order Wess-Zumino correction terms to the Riegert action. The term with $1/\bar{t}^2(r)$ is derived from the Weyl action and the term with $M_{\rm P}$ is from the Einstein tensor. The cosmological term and the matter part are set to be zero.

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