Begun to write since Aug. 8th, 2011. Revised on Aug. 10th, 2011.

※注：20110801 時点においては、未だ個人見解レベルの検討書です。

Particle Accelerator Development Note

Analytic solution of frequency dependence on distributed delayed line magnets
～ No.49：ラダー回路型電磁石の周波数特性の解析解 ～

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要約

分布定数型電磁石の周波数特性の解析解を示す。シミュレータが発達してきたので、近年は解析的な手法がみられなくなったが、公式が存在することも示しておきたくここにまとめる。LC 回路の段数を増やせばその分の次数で関数が複雑化するが、とりあえず実用的な１０段まではやっておく。

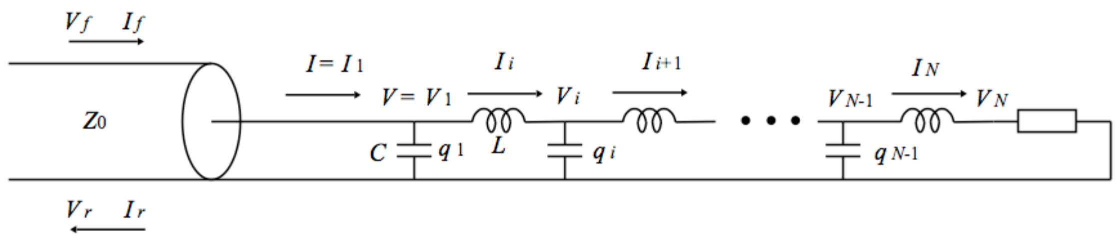
一般的なパルス伝送方程式

$$\begin{cases} V_f + V_r = V \\ I_f - I_r = I \end{cases} \quad ; \quad \begin{cases} V_f = Z_0 I_f \\ V_r = Z_0 I_r \end{cases}$$

を整理する。この議論で必要の無い反射の項を消去すると

$$2V_f = V + Z_0 I \ .$$

さて、梯子回路を適用するわけだが、概ね下記のように考える。



便宜上、全ての関数は Fourier 変換してあるものとする。伝送方程式は、

$$\begin{cases} V_i - V_{i+1} = j\omega L I_{i+1} = j\omega L (I_i - j\omega C V_i) \\ I_i - I_{i+1} = j\omega q_i = j\omega C V_i \end{cases} \Rightarrow \begin{cases} V_{i+1} = (1 - \omega^2 CL) V_i - j\omega L I_i \\ I_{i+1} = I_i - j\omega C V_i \end{cases}$$

Matrix 表現が可能であり、その transfer matrix を M とすると、

$$\begin{pmatrix} I_{i+1} \\ V_{i+1} \end{pmatrix} = M \begin{pmatrix} I_i \\ V_i \end{pmatrix} = \begin{pmatrix} 1 & -j\omega C \\ -j\omega L & 1 - \omega^2 CL \end{pmatrix} \begin{pmatrix} I_i \\ V_i \end{pmatrix} \equiv \begin{pmatrix} 1 & ja \\ j\dfrac{x}{a} & (1-x) \end{pmatrix} \begin{pmatrix} I_i \\ V_i \end{pmatrix} ; M^{-1} = \begin{pmatrix} 1 - \omega^2 CL & j\omega C \\ j\omega L & 1 \end{pmatrix}$$

$$\det M = 1$$

となる。

特性インピーダンス整合条件により C を定めるのが一般的なので、ここでもその条件を適用して簡略化する（但し、ミスマッチでも、式は複雑化するが、同様な議論で OK)。
同時に、次元を合わせるため、行列 G を導入する。

$$2V_f = V + Z_0 I$$

$$\Phi = \frac{V - V_N}{j\omega}$$

$$Z_M = \frac{V_f}{\Phi} = \frac{j\omega(V + Z_0 I)}{2(V - V_N)}$$

$$G \equiv \begin{pmatrix} 1-\omega^2(L/Z_0)^2 & j\omega L/Z \\ j\omega L/Z_0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1-x^2 & jx \\ jx & 1 \end{pmatrix}; \det G = 1$$

$$G^2 = \begin{pmatrix} 1-x^2 & jx \\ jx & 1 \end{pmatrix}\begin{pmatrix} 1-x^2 & jx \\ jx & 1 \end{pmatrix} = \begin{pmatrix} (1-x^2)^2 - x^2 & jx(2-x^2) \\ jx(2-x^2) & 1-x^2 \end{pmatrix} \equiv AG + BE$$

$$\left\{ \begin{array}{r} (1-x^2)^2 - x^2 = A(1-x^2) + B \\ x(2-x^2) = jxA \\ 1-x^2 = A+B \end{array} \right. \Rightarrow \left\{ \begin{array}{l} A = (2-x^2) \\ B = -1 \end{array} \right. \Rightarrow G^2 = (2-x^2)G - E$$

$$2V_f = V + Z_0 I$$

$$\Phi = \frac{V - V_N}{j\omega}$$

$$Z_M = \frac{V_f}{\Phi} = \frac{j\omega(V + Z_0 I)}{2(V - V_N)}$$

$$G \equiv \begin{pmatrix} 1-\omega^2(L/Z_0)^2 & j\omega L/Z \\ j\omega L/Z_0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1-x^2 & jx \\ jx & 1 \end{pmatrix}; \det G = 1$$

$$\begin{pmatrix} Z_0 I \\ V \end{pmatrix} = G^{(N-1)} \begin{pmatrix} V_N \\ V_N \end{pmatrix} = \begin{pmatrix} 1-\omega^2(L/Z_0)^2 & j\omega L/Z \\ j\omega L/Z_0 & 1 \end{pmatrix}^{(N-1)} \begin{pmatrix} V_N \\ V_N \end{pmatrix} \equiv \begin{pmatrix} \alpha & j\beta \\ j\beta & \gamma \end{pmatrix}\begin{pmatrix} V_N \\ V_N \end{pmatrix} : \alpha\delta + \beta^2 = 1$$

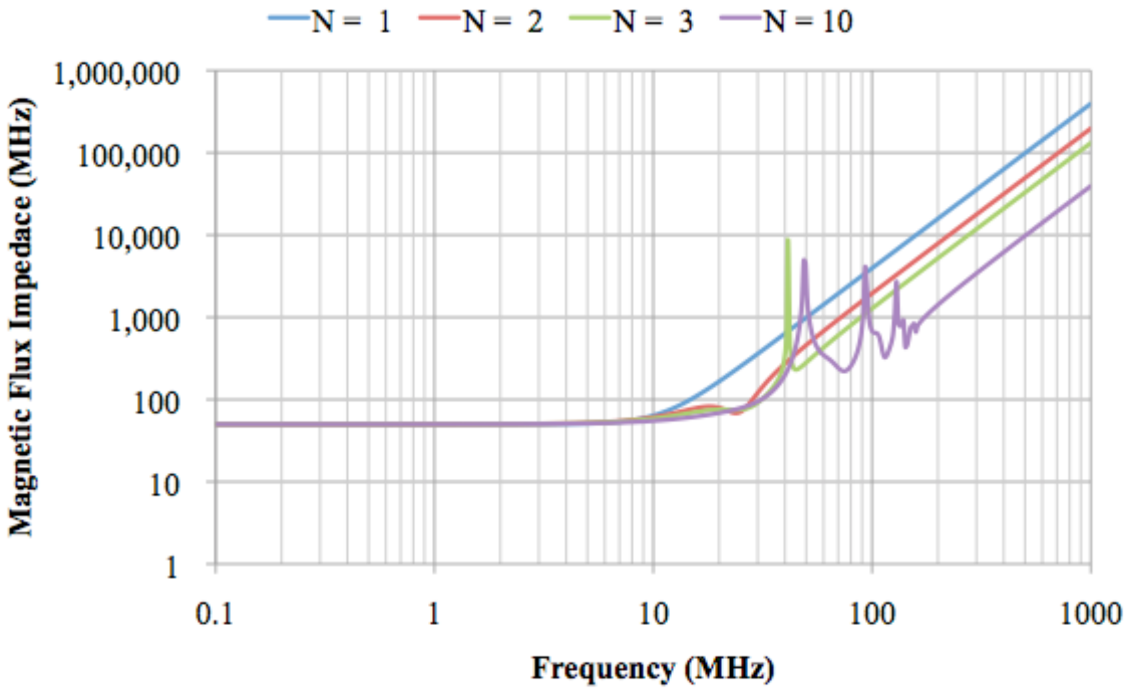
$$Z_M = \frac{j\omega(j\beta + \gamma + \alpha + j\beta)}{2(j\beta + \gamma - 1)} = \frac{j\omega(\alpha + \gamma + j2\beta)}{2(\gamma - 1 + j\beta)} = \frac{\omega}{2}\frac{-2\beta + j(\alpha+\gamma)}{(\gamma - 1 + j\beta)}$$

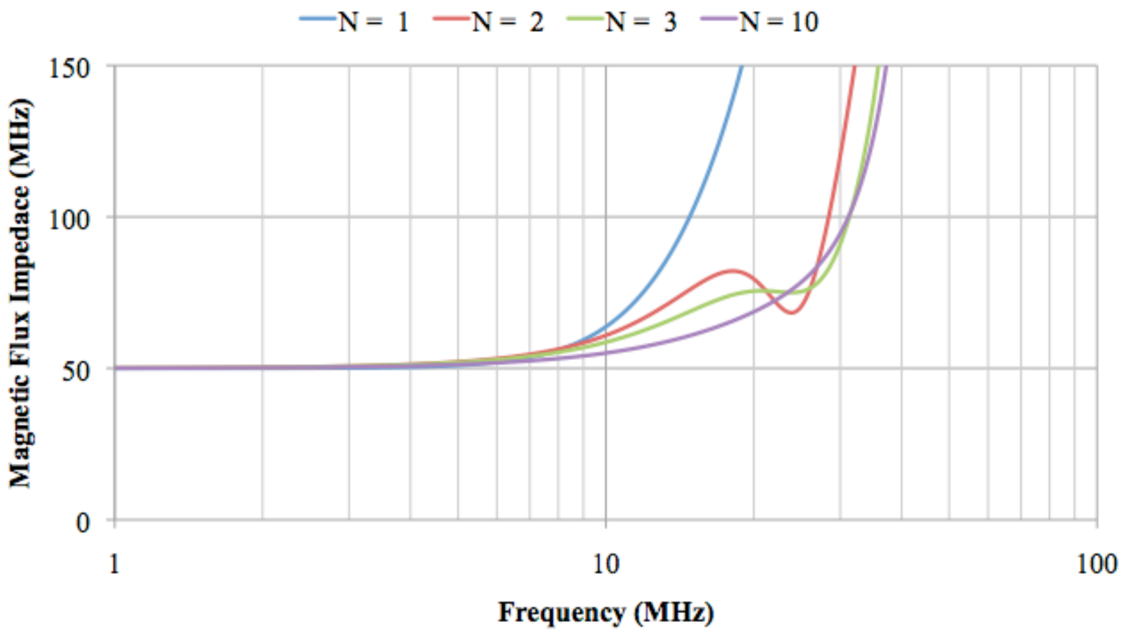
$$Z_M = \frac{\omega}{2}\frac{\{-2\beta + j(\alpha+\gamma)\}(\gamma - 1 - j\beta)}{(\gamma-1)^2 + \beta^2} = \frac{\omega}{2}\frac{\beta(2+\alpha-\gamma) + j\{2\beta^2 + (\alpha+\gamma)(\gamma-1)\}}{(\gamma-1)^2 + \beta^2}$$

$$|Z_M| = \frac{\omega}{2}\frac{\sqrt{(1-\alpha\gamma)(2+\alpha-\gamma)^2 + \{2(1-\alpha\gamma) + (\alpha+\gamma)(\gamma-1)\}^2}}{(\gamma-1)^2 + 1 - \alpha\gamma}$$

$$|Z_M| = \frac{\omega}{2}\frac{\sqrt{(1-\alpha\gamma)(2+\alpha-\gamma)^2 + \{2(1-\alpha\gamma) + (\alpha+\gamma)(\gamma-1)\}^2}}{(\gamma-1)^2 + 1 - \alpha\gamma}$$

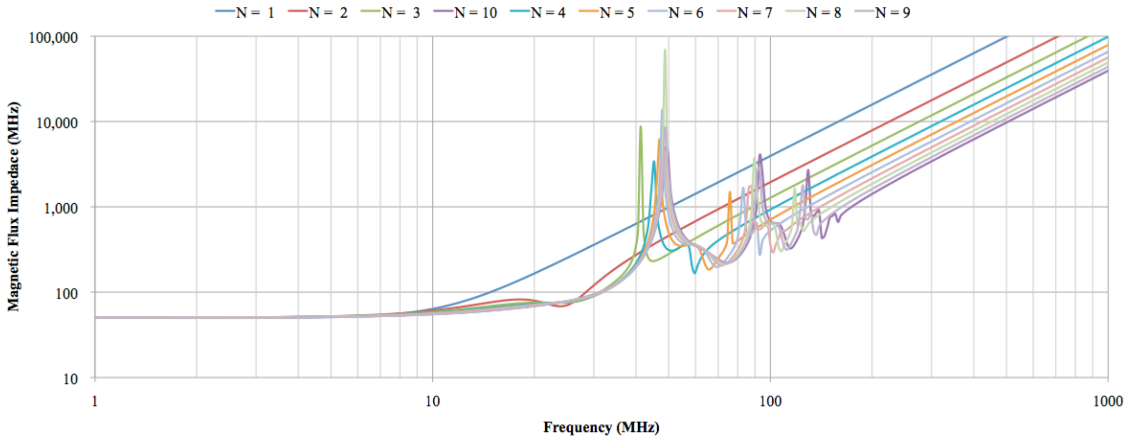
１、２、３、１０ステージの計算結果を下記に示す。

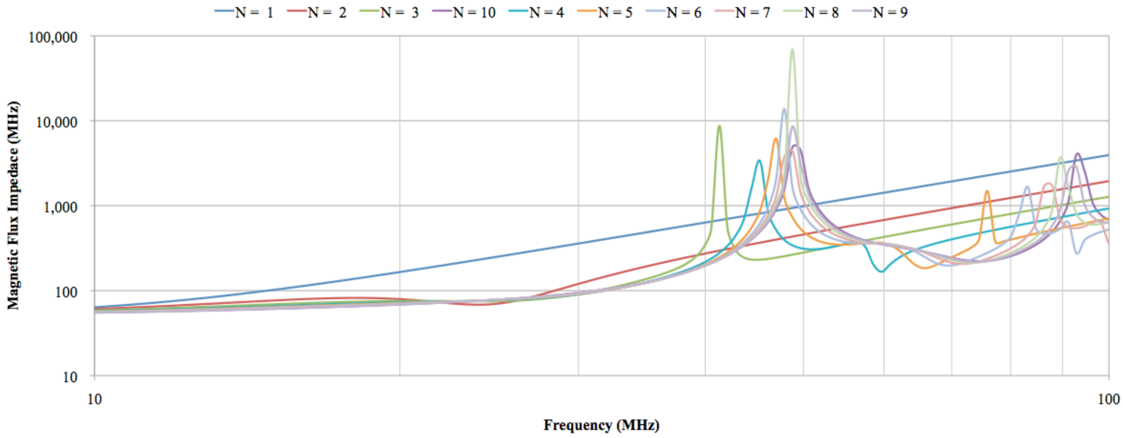


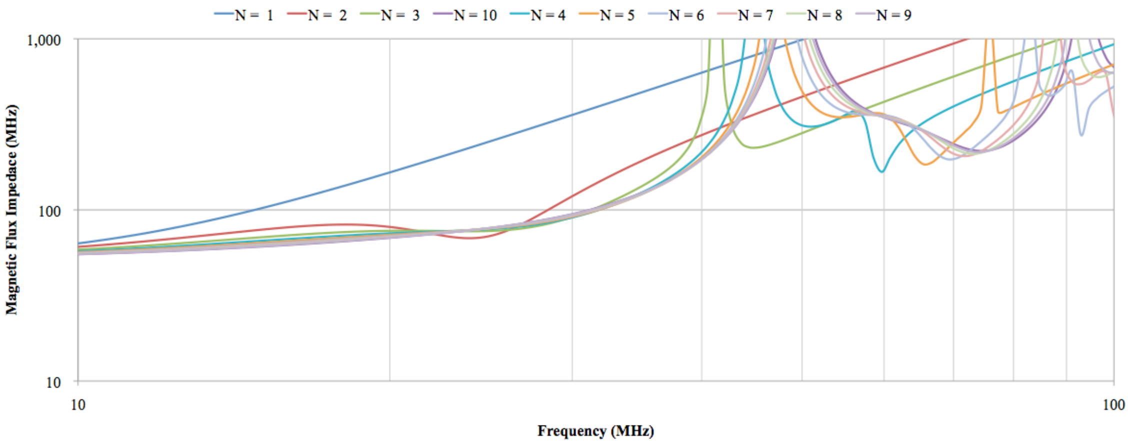


※ここからはメモです。

ちなみに、段数を細かくみると下記のようになる。
N = 3 以上であれば 30MHz あたりまでは変化がなく、実用上は問題ない。







※印象：解析解とは思えないほどの複雑な特性。

ちなみに、行列 M の特徴は下記のようになる。

$$M = \begin{pmatrix} 1 & ja \\ j\dfrac{x}{a} & (1-x) \end{pmatrix}$$

$$M^2 = \begin{pmatrix} 1 & ja \\ j\dfrac{x}{a} & (1-x) \end{pmatrix}\begin{pmatrix} 1 & ja \\ j\dfrac{x}{a} & (1-x) \end{pmatrix} = \begin{pmatrix} 1-x & ja(2-x) \\ j\dfrac{x}{a}(2-x) & -x+(1-x)^2 \end{pmatrix} \equiv AM + BE$$

$$\left\{\begin{array}{l} 1-x = A+B \\ ja(2-x) = jaA \\ j\dfrac{x}{a}(2-x) = j\dfrac{x}{a}A \\ -x+(1-x)^2 = (1-x)A+B \end{array}\right. \Rightarrow \left\{\begin{array}{l} A = 2-x \\ B = -1 \end{array}\right.$$

$$M^2 = (2-x)M - E \equiv yM - E$$

$$M^2 = yM - E$$
$$M^4 = y^2M^2 - 2yM + E = y^2(yM-E) - 2yM + E = y(y^2-2)M + (1-y^2)E$$
$$M^5 = y(y^2-2)M^2 + (1-y^2)M = (y^2-2)y(yM-E) - (y^2-1)M = (y^4-3y^2+1)M - (y^2-2)yE$$
$$M^{10} = (y^4-3y^2+1)^2M^2 - 2(y^2-2)y(y^4-3y^2+1)M + (y^2-2)^2y^2E$$
$$= (y^4-3y^2+1)^2(yM-E) - 2(y^2-2)y(y^4-3y^2+1)M + (y^2-2)^2y^2E$$
$$= (y^4-3y^2+1)y\{y^4-3y^2+1-2y^2+4\}M + \{(y^3-2y)^2 - (y^4-3y^2+1)^2\}E$$
$$= (y^4-3y^2+1)y(y^4-5y^2+5)M - (y^4+y^3-3y^2-2y+1)(y^4-y^3-3y^2+2y+1)E$$

$$\therefore M^{10} = (y^4-3y^2+1)y(y^4-5y^2+5)M - (y^4+y^3-3y^2-2y+1)(y-1)(y^3-3y-1)E$$

$$M^2 = yM - E$$
$$M^3 = yM^2 - M = y(yM-E) - M = (y^2-1)M - yE$$
$$M^4 = (y^2-1)M^2 - yM = (y^2-1)(yM-E) - yM = (y^2-2)yM - (y^2-1)E$$
$$M^5 = (y^2-2)yM^2 - (y^2-1)M = (y^2-2)y(yM-E) - (y^2-1)M = (y^4-3y^2+1)M - (y^2-2)yE$$
$$M^2 = yM - E$$
$$M^{10} = (y^4-3y^2+1)^2M^2 - 2(y^4-3y^2+1)(y^2-2)yM + (y^2-2)^2y^2E$$
$$= (y^4-3y^2+1)^2(yM-E) - 2(y^4-3y^2+1)(y^2-2)yM + (y^2-2)^2y^2E$$
$$= y(y^4-3y^2+1)(y^4-5y^2+3)M + \{(y^2-2)^2y^2 - (y^4-3y^2+1)^2\}E$$

ここからは行列の一般式を求める試み。

$$A_m \equiv \begin{pmatrix} a_n & jb_n \\ jb_n & c_n \end{pmatrix} \equiv G^n = \begin{pmatrix} 1-x^2 & jx \\ jx & 1 \end{pmatrix} G^{n-1} = \begin{pmatrix} 1-x^2 & jx \\ jx & 1 \end{pmatrix} \begin{pmatrix} a_{n-1} & jb_{n-1} \\ jb_{n-1} & c_{n-1} \end{pmatrix}$$

$$a_{n+1} = (1-x^2)a_n - xb_n$$

$$\{ \quad b_{n+1} = b_n + xa_n$$

$$b_{n+1} = (1-x^2)b_n + xc_n$$

$$c_{n+1} = c_n - xb_n$$

$$a_{n+2} - a_{n+1} = (1-x^2)(a_{n+1} - a_n) - x(b_{n+1} - b_n) = (1-x^2)(a_{n+1} - a_n) - x^2 a_n$$

$$a_{n+2} + (x^2-2)a_{n+1} + a_n = 0$$

$$a_{n+2} + (x^2-2)a_{n+1} + a_n = 0; a_1 = 1-x^2, a_2 = (1-x^2)^2 - x^2$$

$$u \equiv \frac{2-x^2+\sqrt{(x^2-2)^2-4}}{2}; v \equiv \frac{2-x^2-\sqrt{(x^2-2)^2-4}}{2}$$

$$u+v = 2-x^2; uv = 1$$

$$\alpha_n \equiv a_{n+1} - ua_n$$

$$\alpha_1 \equiv a_2 - ua_1 = (1-x^2)^2 - x^2 - \frac{2-x^2+\sqrt{(x^2-2)^2-4}}{2}(1-x^2) = \frac{x^2(x^2-3)+(1-x^2)\sqrt{(x^2-2)^2-4}}{2}$$

$$\alpha_{n+1} - v\alpha_n = (a_{n+2} - va_{n+1}) - u(a_{n+1} - va_n) = a_{n+2} - (u+v)a_{n+1} + uva_n = a_{n+2} + (x^2-2)a_{n+1} + a_n = 0$$

$$\alpha_{n+1} - v\alpha_n = 0$$

$$\alpha_n = v^{n-1}\alpha_1 = a_{n+1} - ua_n$$

$$a_n \equiv \beta_n u^n$$

$$\beta_{n+1}u^{n+1} - \beta_n u^{n+1} = v^{n-1}\alpha_1 \Rightarrow \beta_{n+1} - \beta_n = v^{n-1}\alpha_1 / u^{n+1} = v^{2n}\alpha_1$$

$$\beta_n - \beta_1 + \alpha_1 = \alpha_1 \sum_{i=0}^{n-1} (v^2)^i = \alpha_1 \frac{v^{2n}-1}{v-1}$$

$$\beta_n = \frac{a_1}{u} - \alpha_1 + \alpha_1 \frac{v^{2n}-1}{v-1} = va_1 + \alpha_1 \frac{v^{2n}-v}{v-1}$$

$$a_n = (va_1 + \alpha_1 \frac{v^{2n}-v}{v-1})u^n = a_1 u^{n-1} + (a_2 - ua_1)\frac{v^n - u^{n-1}}{v-1} = a_2 \frac{v^n - u^{n-1}}{v-1} - a_1(\frac{v^{n-1} - u^n}{v-1} - u^{n-1})$$

$$2V_f = V + Z_0 I$$

$$\Phi = \frac{V - V_N}{j\omega}$$

$$Z_M = \frac{V_f}{\Phi} = \frac{j\omega(V + Z_0 I)}{2(V - V_N)}$$

$$G \equiv \begin{pmatrix} 1-\omega^2(L/Z_0)^2 & j\omega L/Z \\ j\omega L/Z_0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1-x^2 & jx \\ jx & 1 \end{pmatrix}; \det G = 1$$

$$\begin{pmatrix} Z_0 I \\ V \end{pmatrix} = G^{(N-1)} \begin{pmatrix} V_N \\ V_N \end{pmatrix} = \begin{pmatrix} 1-\omega^2(L/Z_0)^2 & j\omega L/Z \\ j\omega L/Z_0 & 1 \end{pmatrix}^{(N-1)} \begin{pmatrix} V_N \\ V_N \end{pmatrix} \equiv \begin{pmatrix} \alpha & j\beta \\ j\beta & \gamma \end{pmatrix} \begin{pmatrix} V_N \\ V_N \end{pmatrix} : \alpha\delta + \beta^2 = 1$$

$$Z_M = \frac{j\omega(j\beta + \gamma + \alpha + j\beta)}{2(j\beta + \gamma - 1)} = \frac{j\omega(\alpha + \gamma + j2\beta)}{2(\gamma - 1 + j\beta)} = \frac{\omega}{2} \frac{-2\beta + j(\alpha + \gamma)}{(\gamma - 1 + j\beta)}$$

(1) N = 1 case:

$$2V_f = V + Z_0 I$$

$$\Phi = \frac{V - V_2}{j\omega}$$

$$Z_M = \frac{V_f}{\Phi} = \frac{j\omega(V + Z_0 I)}{2(V - V_2)}$$

$$\begin{pmatrix} I_2 \\ V_2 \end{pmatrix} = \begin{pmatrix} I_2 \\ Z_0 I_2 \end{pmatrix} = M\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 1 & -j\omega C \\ -j\omega L & 1 - \omega^2 CL \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}$$

$$\det M = 1$$

$$\begin{pmatrix} I \\ V \end{pmatrix} = M^{-1}\begin{pmatrix} I_2 \\ Z_0 I_2 \end{pmatrix} = \begin{pmatrix} 1 - \omega^2 CL & j\omega C \\ j\omega L & 1 \end{pmatrix}\begin{pmatrix} I_2 \\ Z_0 I_2 \end{pmatrix}$$

$$2V_f = V + Z_0 I$$

$$\Phi = \frac{V - V_2}{j\omega}$$

$$Z_M = \frac{V_f}{\Phi} = \frac{j\omega(V + Z_0 I)}{2(V - V_2)}$$

$$\begin{pmatrix} I \\ V \end{pmatrix} = M^{-1}\begin{pmatrix} I_2 \\ Z_0 I_2 \end{pmatrix} = \begin{pmatrix} 1 - \omega^2 CL & j\omega C \\ j\omega L & 1 \end{pmatrix}\begin{pmatrix} I_2 \\ Z_0 I_2 \end{pmatrix}$$

$$Z_M = \frac{j\omega(V + Z_0 I)}{2(V - V_2)} = \frac{j\omega\{j\omega L + Z_0 + Z_0(1 - \omega^2 CL + j\omega C Z_0)\}}{2(j\omega L + Z_0 - Z_0)} = \frac{Z_0\{j\omega\frac{L}{Z_0} + 1 + (1 - \omega^2 CL + j\omega C Z_0)\}}{2L}$$

$$Z_M = \frac{Z_0}{2L}\{2 - \omega^2 CL + j\omega(CZ_0 + \frac{L}{Z_0})\}$$

$$|Z_M| = \frac{Z_0}{2L}\sqrt{(2 - \omega^2 CL)^2 + \omega^2(CZ_0 + \frac{L}{Z_0})^2} = \frac{Z_0}{2L}\sqrt{4 + (CZ_0 - \frac{L}{Z_0})^2\omega^2 + (CL)^2\omega^4}$$

L, C が系の特性インピーダンスに整合しているときに周波数特性の２次の項が消去可能である。

$$\sqrt{\frac{L}{C}} = Z_0$$

$$|Z_M| = \frac{Z_0}{L}\sqrt{1 + (\frac{L}{2Z_0})^2\omega^4}$$

(2) N = 2 case:

$$2V_f = V + Z_0 I$$

$$\Phi = \frac{V - V_N}{j\omega}$$

$$Z_M = \frac{V_f}{\Phi} = \frac{j\omega(V + Z_0 I)}{2(V - V_N)} = \frac{j\omega(V + Z_0 I)}{2(V - Z_0 I_N)}$$

$$\begin{pmatrix} I_N \\ V_N \end{pmatrix} = \begin{pmatrix} I_N \\ Z_0 I_N \end{pmatrix} = M^2 \begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 1 & -j\omega C \\ -j\omega L & 1-\omega^2 CL \end{pmatrix} \begin{pmatrix} 1 & -j\omega C \\ -j\omega L & 1-\omega^2 CL \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}$$

$$\det M = 1; M^2 = (2 - \omega^2 CL)M - E; M^{-1} = (2 - \omega^2 CL)E - M$$

$$\begin{pmatrix} I \\ V \end{pmatrix} = M^{-2} \begin{pmatrix} I_N \\ Z_0 I_N \end{pmatrix} = M^{-1}\{(2 - \omega^2 CL)E - M\} \begin{pmatrix} I_N \\ Z_0 I_N \end{pmatrix} = \{(2 - \omega^2 CL)M^{-1} - E\} \begin{pmatrix} I_N \\ Z_0 I_N \end{pmatrix}$$

$$= [-(2 - \omega^2 CL)M + \{(2 - \omega^2 CL)^2 - 1\}E] \begin{pmatrix} I_N \\ Z_0 I_N \end{pmatrix}$$

$$= \begin{pmatrix} 1-\omega^2 CL & j\omega C \\ j\omega L & 1 \end{pmatrix} \begin{pmatrix} 1-\omega^2 CL & j\omega C \\ j\omega L & 1 \end{pmatrix} \begin{pmatrix} I_2 \\ Z_0 I_2 \end{pmatrix}$$

$$2V_f = V + Z_0 I$$

$$\Phi = \frac{V - V_2}{j\omega}$$

$$Z_M = \frac{V_f}{\Phi} = \frac{j\omega(V + Z_0 I)}{2(V - V_2)}$$

L,

$$\begin{pmatrix} I \\ V \end{pmatrix} = M^{-1} \begin{pmatrix} I_2 \\ Z_0 I_2 \end{pmatrix} = \begin{pmatrix} 1-\omega^2 CL & j\omega C \\ j\omega L & 1 \end{pmatrix} \begin{pmatrix} I_2 \\ Z_0 I_2 \end{pmatrix}$$

$$Z_M = \frac{j\omega(V + Z_0 I)}{2(V - V_2)} = \frac{j\omega\{j\omega L + Z_0 + Z_0(1 - \omega^2 CL + j\omega C Z_0)\}}{2(j\omega L + Z_0 - Z_0)} = \frac{Z_0\{j\omega \frac{L}{Z_0} + 1 + (1 - \omega^2 CL + j\omega C Z_0)\}}{2L}$$

$$Z_M = \frac{Z_0}{2L}\{2 - \omega^2 CL + j\omega(C Z_0 + \frac{L}{Z_0})\}$$

$$|Z_M| = \frac{Z_0}{2L}\sqrt{(2 - \omega^2 CL)^2 + \omega^2 (C Z_0 + \frac{L}{Z_0})^2} = \frac{Z_0}{2L}\sqrt{4 + (C Z_0 - \frac{L}{Z_0})^2 \omega^2 + (CL)^2 \omega^4}$$

C が系の特性インピーダンスに整合しているときに周波数特性の２次の項が消去可能である。

$$\sqrt{\frac{L}{C}} = Z_0$$

$$|Z_M| = \frac{Z_0}{L}\sqrt{1 + (\frac{L}{2Z_0})^2 \omega^4}$$

$$\begin{pmatrix} V_{N+1} \\ I_{N+1} \end{pmatrix} = M^N \begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = (\begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix}) \begin{pmatrix} V_1 \\ I_1 \end{pmatrix}$$

$$V_6 = \alpha V_1 + \beta I_1 = \alpha V + \beta(I - j\omega CV) = (\alpha - j\omega C)V + \beta I$$

$$\begin{pmatrix} V_5 \\ I_5 \end{pmatrix} = M^{-1} M^5 \begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = (\begin{matrix} 1 & j\frac{x}{a} \\ ja & 1-x \end{matrix})(\begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix}) \begin{pmatrix} V \\ I - j\omega CV \end{pmatrix}$$

$$I_5 = \{ja\alpha + \gamma(1-x)\}V + \{ja\beta + \delta(1-x)\}(I - j\omega CV)$$

$$V_6 = Z_0 I_5$$

$$(\alpha - j\omega C)V + \beta I = Z_0[\{ja\alpha + \gamma(1-x)\}V + \{ja\beta + \delta(1-x)\}(I - j\omega CV)]$$

$$2V_f = V + Z_0 I$$

$$\Phi = \frac{V - V_6}{j\omega}$$

$$Z_M = \frac{V_f}{\Phi} = \frac{V + Z_0 I}{2(V - V_6)} = \frac{V + Z_0 I}{2\{V - (\alpha - j\omega C)V + \beta I\}}$$

$$(\alpha - j\omega C)V + \beta I = Z_0[\{ja\alpha + \gamma(1-x)\}V + \{ja\beta + \delta(1-x)\}(I - j\omega CV)]$$

$$M^2 = yM - E$$

$$M^{10} = (y^4 - 3y^2 + 1)^2 M^2 - 2(y^4 - 3y^2 + 1)(y^2 - 2)yM + (y^2 - 2)^2 y^2 E$$

$$= (y^4 - 3y^2 + 1)^2 (yM - E) - 2(y^4 - 3y^2 + 1)(y^2 - 2)yM + (y^2 - 2)^2 y^2 E$$

$$= y(y^4 - 3y^2 + 1)(y^4 - 5y^2 + 3)M + \{(y^2 - 2)^2 y^2 - (y^4 - 3y^2 + 1)^2\}E$$

$$M^2 = (2 - x)M - E$$

$$M^{10} = \{[(2-x)^2\{(2-x)^2 - 2\} - \{(2-x)^2 - 1\}]^2 - 2\{(2-x)^2 - 2\}[(2-x)^2\{(2-x)^2 - 2\} - \{(2-x)^2 - 1\}]\}(2-x)M$$

$$\{(2-x)^2\{(2-x)^2 - 2\}^2 - [(2-x)^2\{(2-x)^2 - 2\} - \{(2-x)^2 - 1\}]^2\}E$$

$$\begin{pmatrix} V_{N+1} \\ I_{N+1} \end{pmatrix} = M^N \begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = (\begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix}) \begin{pmatrix} V_1 \\ I_1 \end{pmatrix}$$

$$V_{11} = \alpha V_1 + \beta I_1 = \alpha V + \beta(I - j\omega CV) = (\alpha - j\omega C)V + \beta I$$

$$\begin{pmatrix} V_{10} \\ I_{10} \end{pmatrix} = M^{-1} M^{10} \begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = (\begin{matrix} 1 & j\frac{x}{a} \\ ja & 1 - x \end{matrix})(\begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix}) \begin{pmatrix} V \\ I - j\omega CV \end{pmatrix}$$

$$I_{10} = \{ja\alpha + \gamma(1 - x)\}V + \{ja\beta + \delta(1 - x)\}(I - j\omega CV)$$

$$V_{11} = Z_0 I_{10}$$

$$(\alpha - j\omega C)V + \beta I = Z_0[\{ja\alpha + \gamma(1 - x)\}V + \{ja\beta + \delta(1 - x)\}(I - j\omega CV)]$$

$$2V_f = V + Z_0 I$$

$$\Phi = \frac{V - V_{11}}{j\omega}$$

$$Z_M = \frac{V_f}{\Phi} = \frac{V + Z_0 I}{2(V - V_{11})} = \frac{V + Z_0 I}{2\{V - (\alpha - j\omega C)V + \beta I\}}$$

$$(\alpha - j\omega C)V + \beta I = Z_0[\{ja\alpha + \gamma(1 - x)\}V + \{ja\beta + \delta(1 - x)\}(I - j\omega CV)]$$

$$Z_M = \frac{V_f}{\Phi} = \frac{V + Z_0 I}{2(V - V_{11})} = \frac{V + Z_0 I}{2\{(1 - \alpha + j\omega C)V + \beta I\}}$$

$$(\alpha - j\omega C)V + \beta I = Z_0[\{ja\alpha + \gamma(1 - x)\}V + \{ja\beta + \delta(1 - x)\}(I - j\omega CV)]$$

$$\frac{V}{I} = \frac{Z_0\{ja\beta + \delta(1 - x)\} - \beta}{(\alpha - j\omega C) - Z_0\{ja\alpha + \gamma(1 - x)\} - Z_0 j\omega\{ja\beta + \delta(1 - x)\}}$$

$$Z_M = \frac{[Z_0\{ja\beta + \delta(1 - x)\} - \beta] + Z_0[(\alpha - j\omega C) - Z_0\{ja\alpha + \gamma(1 - x)\} - Z_0 j\omega\{ja\beta + \delta(1 - x)\}]}{2\{(1 - \alpha + j\omega C)[Z_0\{ja\beta + \delta(1 - x)\} - \beta] + \beta[(\alpha - j\omega C) - Z_0\{ja\alpha + \gamma(1 - x)\} - Z_0 j\omega\{ja\beta + \delta(1 - x)\}]\}}$$