PADN- 211 (PADN211CyrCoorv1.doc)
Begun to write since March 15th, 2010. Reformed on. Revised on.
※注：20090717 時点においては、未だ個人見解レベルの検討書です。

Particle Accelerator Development Note

物理数学基礎
～ No.11: 円筒座標系 ～

報告者： 中村英滋（KEK 加速器研究施設）

要約
円筒座標系の座標変換に関する公式の確認。

３次元空間 $R\,(x,\ y\ ,z)$ を $R\,(r,\ \theta\ ,z)$ に変換する。自明の事だが、下記のように記述。

CylindricalCoordinates

$\mathbf{r} = (x, y, z) = (r\cos\theta, r\sin\theta, z)$

$x = x(r, \theta, z),\ y = y(r, \theta, z),\ z = z(r, \theta, z).$

$r = r(x, y, z),\ \theta = \theta(x, y, z),\ z = z(x, y, z).$

$x = x(r, \theta),\ y = y(r, \theta),\ z = z(z).$

$r = r(x, y),\ \theta = \theta(x, y),\ z = z(z).$

ここでそれぞれの基準となる基礎物理量は、下記のように変換される。

$\mathbf{e}_x = (1, 0, 0)$

$\mathbf{e}_y = (0, 1, 0)$

$\mathbf{e}_z = (0, 0, 1)$

$\mathbf{e}_r = (\cos\theta, \sin\theta, 0) = \cos\theta\ \mathbf{e}_x + \sin\theta\ \mathbf{e}_y$

$\mathbf{e}_\theta = (-\sin\theta, \cos\theta, 0) = -\sin\theta\ \mathbf{e}_x + \cos\theta\ \mathbf{e}_y$

$\boldsymbol{A} = (A_x, A_y, A_z) = A_x\,\mathbf{e}_x + A_y\,\mathbf{e}_y + A_z\,\mathbf{e}_z$

$= (A_r, A_\theta, A_z) = A_r\,\mathbf{e}_r + A_\theta\,\mathbf{e}_\theta + A_z\,\mathbf{e}_z$

$= A_r(\cos\theta\ \mathbf{e}_x + \sin\theta\ \mathbf{e}_y) + A_\theta(-\sin\theta\ \mathbf{e}_x + \cos\theta\ \mathbf{e}_y) + A_z\,\mathbf{e}_z$

$= (A_r\cos\theta - A_\theta\sin\theta)\,\mathbf{e}_x + (A_r\sin\theta + A_\theta\cos\theta)\,\mathbf{e}_y + A_z\,\mathbf{e}_z$

$$\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}A_r\\ A_\theta\end{pmatrix} = \begin{pmatrix}A_x\\ A_y\end{pmatrix}\ ;\quad \begin{pmatrix}A_r\\ A_\theta\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}A_x\\ A_y\end{pmatrix}$$

次に、微分量に関しては、座標変換の係数として関数θが入っているため下記のようになる。

$\mathbf{e}_r{}' = -\sin\theta\ \theta'\ \mathbf{e}_x + \cos\theta\ \theta'\ \mathbf{e}_y = \theta'\ \mathbf{e}_\theta$

$\mathbf{e}_\theta{}' = -\cos\theta\ \theta'\ \mathbf{e}_x - \sin\theta\ \theta'\ \mathbf{e}_y = -\theta'\ \mathbf{e}_r$

これを用いて、空間微分は下記のようになる。

$$= \quad r\,\frac{\partial}{\partial r} + \quad \theta\,\frac{\partial}{\partial \theta} + \quad z\,\frac{\partial}{\partial z}$$
$$= (\cos\theta,\ \sin\theta,\ 0)\frac{\partial}{\partial r} + \frac{1}{r}(-\sin\theta,\ \cos\theta,\ 0)\frac{\partial}{\partial \theta} + (0,0,1)\frac{\partial}{\partial z}$$
$$= \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z\frac{\partial}{\partial z}$$

ラプラシンは、上記の内積をとることで得られる。

$$= \left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z\frac{\partial}{\partial z}\right)\quad\left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z\frac{\partial}{\partial z}\right)$$
$$= \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} + \left(\mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}\right)\quad\left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}\right)$$
$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} + \left(\mathbf{e}_\theta\frac{1}{r}\quad\frac{\partial\,\mathbf{e}_r}{\partial \theta}\frac{\partial}{\partial r}\right)$$
$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} + \left(\mathbf{e}_\theta\frac{1}{r}\quad\mathbf{e}_\theta\frac{\partial}{\partial r}\right)$$
$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}$$
$$^2 = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

物理ベクトル量 B に対する作用は以下のようになる。

$\boldsymbol{B} \equiv B_r(r,\theta,z)\,\mathbf{e}_r + B_\theta(r,\theta,z)\,\mathbf{e}_\theta + B_z(r,\theta,z)\,\mathbf{e}_z$

$$\boldsymbol{B} = \left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z\frac{\partial}{\partial z}\right)\quad(B_r\,\mathbf{e}_r + B_\theta\,\mathbf{e}_\theta + B_z\,\mathbf{e}_z)$$
$$= \frac{\partial B_r}{\partial r} + \left(\mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}\right)\quad(B_r\,\mathbf{e}_r + B_\theta\,\mathbf{e}_\theta) + \frac{\partial B_z}{\partial z}$$
$$= \frac{\partial B_r}{\partial r} + \frac{B_r}{r} + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z}$$

$$\times\boldsymbol{B} = \left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z\frac{\partial}{\partial z}\right)\times(B_r\,\mathbf{e}_r + B_\theta\,\mathbf{e}_\theta + B_z\,\mathbf{e}_z)$$
$$= \mathbf{e}_r\frac{\partial}{\partial r}\times(B_r\,\mathbf{e}_r + B_\theta\,\mathbf{e}_\theta + B_z\,\mathbf{e}_z)$$
$$+ \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}\times(B_r\,\mathbf{e}_r + B_\theta\,\mathbf{e}_\theta + B_z\,\mathbf{e}_z)$$
$$+ \mathbf{e}_z\frac{\partial}{\partial z}\times(B_r\,\mathbf{e}_r + B_\theta\,\mathbf{e}_\theta + B_z\,\mathbf{e}_z)$$
$$= \mathbf{e}_z\frac{\partial B_\theta}{\partial r} - \mathbf{e}_\theta\frac{\partial B_z}{\partial r}$$
$$- \mathbf{e}_z\frac{1}{r}\frac{\partial B_r}{\partial \theta} + \mathbf{e}_z\frac{B_\theta}{r} + \mathbf{e}_r\frac{1}{r}\frac{\partial B_z}{\partial \theta}$$
$$+ \mathbf{e}_\theta\frac{\partial B_r}{\partial z} - \mathbf{e}_r\frac{\partial B_\theta}{\partial z}$$
$$= \mathbf{e}_r\left(\frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z}\right) + \mathbf{e}_\theta\left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right) + \mathbf{e}_z\left(\frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r} - \frac{1}{r}\frac{\partial B_r}{\partial \theta}\right)$$

(例)　$E = \mathbf{0}$; $B\theta = 0$ の系での荷電粒子の運動は、γが一定の条件の下、下記のように表される。

Kineticequtioninstatic magnetic field

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = q\,(\boldsymbol{E} + \upsilon \times \boldsymbol{B}) \;;\quad \boldsymbol{p} = m_0\,\gamma\,\upsilon = m_0\,\gamma\,\boldsymbol{r}'$$

$$\boldsymbol{r}'' = \frac{q}{m_0\,\gamma}\,(\boldsymbol{E} + \boldsymbol{r}' \times \boldsymbol{B})$$

$$\boldsymbol{r} = r\,\mathbf{e}_r + z\,\mathbf{e}_z \;;\quad \boldsymbol{r}' = r'\,\mathbf{e}_r + r\,\mathbf{e}_r' + z'\mathbf{e}_z = r'\,\mathbf{e}_r + r\,\theta'\,\mathbf{e}_\theta + z'\,\mathbf{e}_z$$

$$\boldsymbol{r}'' = r''\,\mathbf{e}_r + r'\,\mathbf{e}_r' + r'\,\theta'\,\mathbf{e}_\theta + r\,\theta''\,\mathbf{e}_\theta + r\,\theta'\,\mathbf{e}_\theta' + z''\mathbf{e}_z$$

$$= r''\,\mathbf{e}_r + 2\,r'\,\theta'\,\mathbf{e}_\theta + r\,\theta''\,\mathbf{e}_\theta - r\,\theta'^2\,\mathbf{e}_r + z''\mathbf{e}_z$$

$$(r'' - r\,\theta'^2)\,\mathbf{e}_r + (2\,r'\,\theta' + r\,\theta'')\,\mathbf{e}_\theta + z''\mathbf{e}_z$$

$$= \frac{q}{m_0\,\gamma}\,\{\boldsymbol{E} + (r'\,\mathbf{e}_r + r\,\theta'\,\mathbf{e}_\theta + z'\,\mathbf{e}_z) \times \boldsymbol{B}\}$$

For fields $\mathbf{E} = \mathbf{0}$ *and* $\boldsymbol{B} = B_r(r,z)\,\mathbf{e}_r + B_z(r,z)\,\mathbf{e}_z$

$$(r'' - r\,\theta'^2)\,\mathbf{e}_r + (2\,r'\,\theta' + r\,\theta'')\,\mathbf{e}_\theta + z''\mathbf{e}_z$$

$$= \frac{q}{m_0\,\gamma}\,\{(r'\,\mathbf{e}_r + r\,\theta'\,\mathbf{e}_\theta + z'\,\mathbf{e}_z) \times (B_r\,\mathbf{e}_r + B_z\,\mathbf{e}_z)\}$$

$$= \frac{q}{m_0\,\gamma}\,(-\,r'\,B_z\,\mathbf{e}_\theta - r\,\theta'\,B_r\,\mathbf{e}_z + r\,\theta'\,B_z\,\mathbf{e}_r + z'\,B_r\,\mathbf{e}_\theta)$$

$$\begin{cases} r'' - r\,\theta'^2 = \dfrac{q}{m_0\,\gamma}\,r\,\theta'\,B_z \\ (2\,r'\,\theta' + r\,\theta'') = \dfrac{q}{m_0\,\gamma}\,(-\,r'\,B_z + z'\,B_r) \\ z'' = -\dfrac{q}{m_0\,\gamma}\,r\,\theta'\,B_r \end{cases}$$

$$\upsilon_\theta \equiv r\,\theta'$$

$$\begin{cases} r'' - \dfrac{\upsilon_\theta^{\,2}}{r} = \dfrac{q}{m_0\,\gamma}\,\upsilon_\theta\,B_z \\ \dfrac{(r\,\upsilon_\theta)'}{r} = \dfrac{q}{m_0\,\gamma}\,(-\,r'\,B_z + z'\,B_r) \\ z'' = -\dfrac{q}{m_0\,\gamma}\,\upsilon_\theta\,B_r \end{cases}$$