Maps used in SAD (Draft)

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Abstract

This note summarizes the transfer maps used in SAD code. Note that this note is far from finished, and any comments for improving it are highly appreciated.

1 General theories

1.1 Hamiltonian formalism[1, 2]

The Lagrangian for a charged particle of rest mass m_0 and charge q traveling in an electromagnetic field with speed v is

$$L = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} - q\phi + q\vec{v} \cdot \vec{A}, \tag{1}$$

where $\vec{A}$ is the electromagnetic vector potential and ϕ is the scalar potential from which the electric and magnetic fields $\vec{E}$ and $\vec{B}$ are derived as

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}, \tag{2}$$

$$\vec{B} = \nabla \times \vec{A}. \tag{3}$$

First, let us consider a straight coordinate system (x, y, z) moving in the z-direction. The canonical momentum from classical mechanics is defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \tag{4}$$

where the overdot denotes $\frac{d}{dt}$. The Lorentz equation of motion

$$\vec{F} = \frac{d}{dt}\vec{p} = q(\vec{E} + \vec{v} \times \vec{B}). \quad (5)$$

Using the Lagrangian Eq.(1), one gets

$$p_i = mv_i + qA_i. \quad (6)$$

The Hamiltonian H is derived from the Lagrangian Eq.(1) as

$$H = \sum_i \dot{q}_i p_i - L(q, \dot{q}, t). \quad (7)$$

Inserting Eq.(1) in Eq. (7) gives

$$H = v_x p_x + v_y p_y + v_z p_z + \frac{m_0 c^2}{\gamma} - q\vec{v} \cdot \vec{A} + q\phi = mv^2 + \frac{m_0 c^2}{\gamma} = \gamma m_0 c^2. \quad (8)$$

Simplification of the above equation gives

$$H = mv^2 + \frac{m_0 c^2}{\gamma} + q\phi = \gamma m_0 c^2 + q\phi. \quad (9)$$

One can see that the above Hamiltonian has the value of the total energy, including the rest energy. In terms of $\vec{p}$ and $\vec{A}$, the general form of the Hamiltonian for a relativistic particle in an electromagnetic field in Cartesian coordinatesis is

$$H = \sqrt{(\vec{p} - q\vec{A})^2 c^2 + m_0^2 c^4} + q\phi, \quad (10)$$

where $\vec{p}$ is the canonical momentum. In terms of the kinetic momentum and the vector potential

$$\vec{p} = \vec{P} + q\vec{A}. \quad (11)$$

The Hamilton equations of motion are obtained by

$$\dot{q}_i = \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad (12)$$

$$\dot{p}_i = \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad (13)$$

and the conjugate phase-space coordinates are (x, p_x), (y, p_y), and (z, p_z).

The Poisson bracket of any two dynamical variables $f(q_i, p_i, t)$ and $g(q_i, p_i, t)$ is defined by

$$[f, g] \equiv \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right). \tag{14}$$

The time derivative of f is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) = \frac{\partial f}{\partial t} + [f, H]. \tag{15}$$

In particular, there is

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}. \tag{16}$$

If H does not depend explicitly on time t, it is a constant of the motion. Any invariant of the motion not containing t explicitly, has a vanishing Poisson bracket with H.

1.2 Canonical transformations

There are four standard mixed-variable generating functions that can be used to construct canonical transformations:

- Generating function of the first kind: $F_1 = F_1(q, Q, t)$, $p_i = \frac{\partial F_1}{\partial q_i}$, $P_i = -\frac{\partial F_1}{\partial Q_i}$, $K = H + \frac{\partial F_1}{\partial t}$.
- Generating function of the second kind: $F_2 = F_2(q, P, t)$, $p_i = \frac{\partial F_2}{\partial q_i}$, $Q_i = \frac{\partial F_2}{\partial P_i}$, $K = H + \frac{\partial F_2}{\partial t}$.
- Generating function of the third kind: $F_3 = F_3(p, Q, t)$, $q_i = -\frac{\partial F_3}{\partial p_i}$, $P_i = -\frac{\partial F_3}{\partial Q_i}$, $K = H + \frac{\partial F_3}{\partial t}$.
- Generating function of the fourth kind: $F_4 = F_4(p, P, t)$, $q_i = -\frac{\partial F_4}{\partial p_i}$, $Q_i = \frac{\partial F_4}{\partial P_i}$, $K = H + \frac{\partial F_4}{\partial t}$.

1.3 Accelerator Hamiltonian

In this section, we follow S.Y. Lee's (2nd edition) and H. Wiedemann's (3rd edition) textbooks [Note: The coordinate system used by SAD is the same as Wiedemann's textbook, different from that of Lee's] to derive the Hamiltonian for an accelerator. The Hamiltonian Eq.(10) is expressed with time t as the independent coordinate. Usually it is favorable to use the position along the accelerator s as the independent coordinate.

Let $\vec{r}_0(s)$ be the reference orbit, with s the arc length measured along the closed orbit from a reference initial point. The tangent unit vector to the closed orbit is given by

$$\vec{e}_s(s) = \frac{d\vec{r}_0(s)}{ds}. \tag{17}$$

The unit vector perpendicular to the tangent vector is $\vec{e}_x$. With the third unit binormal vector $\vec{e}_y(s) = \vec{e}_s(s) \times \vec{e}_x(s)$, they form an orthogonal coordinate system moving along the trajectory of the reference particle at $\vec{r}_0(s)$. It is natural that $(\vec{e}_x, \vec{e}_y, \vec{e}_s)$ forms a right-handed orthonormal basis, and thre are $\vec{e}_s(s) = \vec{e}_x(s) \times \vec{e}_y(s)$, and $\vec{e}_x(s) = \vec{e}_y(s) \times \vec{e}_s(s)$. In accelerator physics, the plane defined by vectors $\vec{e}_x(s)$ and $\vec{e}_s(s)$ as the horizontal plane, and the plane orthogonal to it as the vertical plane, parallel to $\vec{e}_y(s)$. The changes in vectors are determined by curvatures

$$\frac{d\vec{e}_x(s)}{ds} = \kappa_x \vec{e}_s(s) \quad \text{and} \quad \frac{d\vec{e}_y(s)}{ds} = \kappa_y \vec{e}_s(s), \tag{18}$$

$$\frac{d\vec{e}_s}{ds} = -\kappa_x \vec{e}_x - \kappa_y \vec{e}_y \tag{19}$$

where $(\kappa_x, \kappa_y) = (\frac{1}{\rho_x}, \frac{1}{\rho_y})$ are the curvatures in the horizontal and vertical plane, respectively. In general, these parameters vary along the orbit position s. The particle trajectory is then described by

$$\vec{r}(x, y, s) = \vec{r}_0(s) + x(s)\vec{e}_x(s) + y(s)\vec{e}_y(s). \tag{20}$$

With the above definitions, there is

$$d\vec{r} = \vec{e}_x dx + \vec{e}_y dy + h\vec{e}_s ds, \tag{21}$$

where

$$h = 1 + \kappa_{0x} x + \kappa_{0y} y. \tag{22}$$

The velocity of the particle is given by

$$\frac{d\vec{r}}{dt} = \frac{ds}{dt}(x'\vec{e}_x + y'\vec{e}_y + h\vec{e}_s), \tag{23}$$

$$\frac{dr}{dt} = \frac{ds}{dt}\sqrt{x'^2 + y'^2 + h^2} \tag{24}$$

with

$$x' = \frac{dx}{ds}, \quad \text{and} \quad y' = \frac{dy}{ds}. \tag{25}$$

Consider a differential function $f(x, y, s)$, we define the differential operator ∇ as

$$df \equiv (\nabla f) \cdot d\vec{r}. \tag{26}$$

We suppose that ∇f is given by

$$\nabla f = a_x \vec{e}_x + a_y \vec{e}_y + a_s \vec{e}_s, \tag{27}$$

where a_x, a_y, and a_s are the coefficients. From the definition of $d\vec{r}$, we have

$$df = (a_x\vec{e}_x + a_y\vec{e}_y + a_s\vec{e}_s) \cdot (\vec{e}_x dx + \vec{e}_y dy + h\vec{e}_s ds) = a_x dx + a_y dy + h a_s ds. \tag{28}$$

On the other hand, df is given by the total derivative with respect to (x, y, s), e.g.

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial s}ds. \tag{29}$$

Then, there are

$$a_x = \frac{\partial f}{\partial x}, \quad a_y = \frac{\partial f}{\partial y}, \quad a_s = \frac{1}{h}\frac{\partial f}{\partial s}.$$

Consequently, the gradient of the function f is given by

$$\nabla f = \frac{\partial f}{\partial x}\vec{e}_x + \frac{\partial f}{\partial y}\vec{e}_y + \frac{1}{h}\frac{\partial f}{\partial s}. \tag{30}$$

The divergence for a vector $\vec{A} = A_x\vec{e}_x + A_y\vec{e}_y + A_s\vec{e}_s$ is given by

$$\begin{aligned} \nabla \cdot \vec{A} &= \left(\vec{e}_x\frac{\partial}{\partial x} + \vec{e}_y\frac{\partial}{\partial y} + \vec{e}_s\frac{1}{h}\frac{\partial}{\partial s}\right) \cdot (A_x\vec{e}_x + A_y\vec{e}_y + A_s\vec{e}_s) \\ &= \frac{1}{h}\left[\frac{\partial(hA_x)}{\partial x} + \frac{\partial(hA_y)}{\partial y} + \frac{\partial A_s}{\partial s}\right]. \end{aligned} \tag{31}$$

The curl of a vector $\vec{A}$ is

$$\nabla \times \vec{A} = \vec{e}_x \frac{1}{h}\left[\frac{\partial(hA_s)}{\partial y} - \frac{\partial A_y}{\partial s}\right] + \vec{e}_y \frac{1}{h}\left[\frac{\partial(A_x)}{\partial s} - \frac{\partial(hA_s)}{\partial x}\right] + \vec{e}_s\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right). \tag{32}$$

The Laplacian for a scalar function is given by

$$\nabla^2 f = \nabla \cdot (\nabla f) = \frac{1}{h}\left[\frac{\partial}{\partial x}\left(h\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial s}\left(\frac{1}{h}\frac{\partial f}{\partial s}\right)\right]. \tag{33}$$

The Laplacian for a vector $\vec{A}$ in the Frenet-Serret coordinate system is defined by

$$\nabla^2 \vec{A} \equiv \nabla(\nabla \cdot \vec{A}) - \nabla \times (\nabla \times \vec{A}), \tag{34}$$

which gives

$$(\nabla^2 \vec{A})_x = \frac{\partial}{\partial x}\left[\frac{1}{h}\frac{\partial(hA_x)}{\partial x}\right] + \frac{1}{h}\frac{\partial}{\partial y}\left(h\frac{\partial A_x}{\partial y}\right) + \frac{1}{h}\frac{\partial}{\partial s}\left(\frac{1}{h}\frac{\partial A_x}{\partial s}\right) - \frac{2}{h^2\rho_x}\frac{\partial A_s}{\partial s} - \frac{A_s}{h}\frac{\partial}{\partial s}\left(\frac{1}{h}\frac{\partial h}{\partial x}\right) - \frac{A_y}{h^2\rho_x\rho_y}, \tag{35}$$

$$(\nabla^2 \vec{A})_y = \frac{1}{h}\frac{\partial}{\partial x}\left(h\frac{\partial A_y}{\partial x}\right) + \frac{\partial}{\partial y}\left[\frac{1}{h}\frac{\partial(hA_y)}{\partial y}\right] + \frac{1}{h}\frac{\partial}{\partial s}\left(\frac{1}{h}\frac{\partial A_y}{\partial s}\right) - \frac{2}{h^2\rho_y}\frac{\partial A_s}{\partial s} - \frac{A_s}{h}\frac{\partial}{\partial s}\left(\frac{1}{h}\frac{\partial h}{\partial x}\right) - \frac{A_x}{h^2\rho_x\rho_y}, \tag{36}$$

$$(\nabla^2 \vec{A})_s = \frac{\partial}{\partial x}\left[\frac{1}{h}\frac{\partial(hA_s)}{\partial x}\right] + \frac{\partial}{\partial y}\left[\frac{1}{h}\frac{\partial(hA_s)}{\partial y}\right] + \frac{1}{h}\left(\frac{1}{h}\frac{\partial A_s}{\partial s}\right) + \frac{2}{h^2\rho_x}\frac{\partial A_x}{\partial s} + \frac{A_x}{h}\frac{\partial}{\partial s}\left(\frac{1}{h}\frac{\partial h}{\partial x}\right) + \frac{2}{h^2\rho_y}\frac{\partial A_y}{\partial s} + \frac{A_y}{h}\frac{\partial}{\partial s}\left(\frac{1}{h}\frac{\partial h}{\partial y}\right). \tag{37}$$

To derive the equation of motion in the Frenet-Serret coordinate system, we tentatively rewrite the Hamiltonian Eq.(10) in Cartesian coordinate system as

$$H(X, Y, Z, P_X, P_Y, P_Z; t) = \sqrt{(\vec{P} - q\vec{A})^2 c^2 + m_0^2 c^4} + q\phi, \tag{38}$$

with the canonical variables (X, P_X, Y, P_Y, Z, P_Z). To find the canonical transformation to (x, y, s, p_x, p_y, p_s), we perform a canonical transformation by using the generating function

$$F_3(x, y, s, P_X, P_X, P_X; t) = -\vec{P} \cdot \left[\vec{r}_0(s) + x\vec{e}_x(s) + y\vec{e}_y(s)\right], \tag{39}$$

where $\vec{P}$ is the canonical momentum in the Cartesian coordinate system. The conjugate momenta for the coordinates (x, y, s) are

$$p_x = -\frac{\partial F_3}{\partial x} = \vec{P} \cdot \vec{e}_x, \tag{40}$$

$$p_y = -\frac{\partial F_3}{\partial y} = \vec{P} \cdot \vec{e}_y, \tag{41}$$

$$p_s = -\frac{\partial F_3}{\partial s} = h\vec{P} \cdot \vec{e}_s. \tag{42}$$

The new Hamiltonian becomes

$$\begin{aligned} H_1(x, y, s, p_x, p_y, p_s; t) &= H + \frac{\partial F_3}{\partial t} \\ &= c\sqrt{(p_x - qA_x)^2 + (p_y - qA_y)^2 + \frac{1}{h^2}(p_s - qA_s)^2 + m_0^2c^2} + q\phi, \end{aligned} \tag{43}$$

where A_x, A_y and A_s are obtained by

$$A_x = \vec{A} \cdot \vec{e}_x, \quad A_y = \vec{A} \cdot \vec{e}_y, \quad A_s = h\vec{A} \cdot \vec{e}_s. \tag{44}$$

Note that the choice of above definitions are so-called "coordinate basis" (Ref. T. Agoh's PhD thesis). When one adopts the "natural basis", the choice is $A_x = \vec{A} \cdot \vec{e}_x, A_y = \vec{A} \cdot \vec{e}_y, A_s = \vec{A} \cdot \vec{e}_s$. In the new coordinate system the Hamilton equations are

$$\dot{x} = \frac{dx}{dt} = \frac{\partial H_1}{\partial p_x}, \qquad \dot{p}_x = \frac{dp_x}{dt} = -\frac{\partial H_1}{\partial x}; \tag{45}$$

$$\dot{y} = \frac{dy}{dt} = \frac{\partial H_1}{\partial p_y}, \qquad \dot{p}_y = \frac{dp_y}{dt} = -\frac{\partial H_1}{\partial sy}; \tag{46}$$

$$\dot{s} = \frac{ds}{dt} = \frac{\partial H_1}{\partial p_s}, \qquad \dot{p}_s = \frac{dp_s}{dt} = -\frac{\partial H_1}{\partial s}. \tag{47}$$

The next step is to replace the independent varial time t by orbit distance s. From the above equations, it is trivial to find

$$x' = \frac{dx}{ds} = \frac{\frac{dx}{dt}}{\frac{ds}{dt}} = \frac{\partial H_1}{\partial p_x}\left(\frac{\partial H_1}{\partial p_s}\right)^{-1} = \frac{\partial(-p_s)}{\partial p_x}, \tag{48}$$

$$p_x' = \frac{dp_x}{ds} = \frac{dp_x}{dt}\left(\frac{ds}{dt}\right)^{-1} = \frac{\partial p_s}{\partial x}, \tag{49}$$

$$y' = \frac{dy}{ds} = \frac{dy}{dt}\left(\frac{ds}{dt}\right)^{-1} = \frac{\partial H_1}{\partial p_y}\left(\frac{\partial H_1}{\partial p_s}\right)^{-1} = \frac{\partial(-p_s)}{\partial p_y}, \tag{50}$$

$$p_y' = \frac{dp_y}{ds} = \frac{dp_y}{dt}\left(\frac{ds}{dt}\right)^{-1} = \frac{\partial p_s}{\partial y}, \tag{51}$$

$$t' = \frac{dt}{ds} = \frac{\partial p_s}{\partial H_1}, \qquad H_1' = \frac{dH_1}{ds} = -\frac{\partial p_s}{\partial t}. \tag{52}$$

Note that the fact of $dH_1 = \frac{\partial H_1}{\partial p_x}dp_x + \frac{\partial H_1}{\partial p_s}dp_s = 0$ and similar relations are used for obtaining the above equations. These equations are the Hamilton equations of motion with s as the independent variable, and $-p_s$ as the new Hamiltonian. The corresponding conjugate phase-space coordinates are in pairs given by (x, p_x), (y, p_y), and $(t, -H_1)$. Now we have a new Hamiltonian defined by

$$\begin{aligned} H_2(x, y, t, p_x, p_y, -H_1; s) &= -p_s \\ &= -h\sqrt{\frac{(H_1 - q\phi)^2}{c^2} - (p_x - qA_x)^2 - (p_y - qA_y)^2 - m_0^2c^2} - qA_s. \end{aligned} \tag{53}$$

The total energy and kinetic momentum of the particle are $E = H_1 - q\phi$ and $p = \sqrt{E^2/c^2 - m_0^2c^2}$.

The canonical variable pair $(t, -H_1)$ is not very convenient to use.

1.4 Basic theory of magnetic field

This section gives a brief description of the basic theory of magnetic field in accelerators by referring Ref. [5]. Usually a two dimensional description is valid for most of the magnet, except at the ends. The multipole expansion of complex magnetic field $\vec{B}(z)$ in terms of the normal and skew compoents, in U.S. convention, is

$$\vec{B}(z) = B_y + iB_x = \sum_{n=0}^{\infty} [B_n + iA_n] \left(\frac{z}{R_{ref}} \right)^n , \tag{54}$$

with $z = x + iy = re^{i\theta}$. The parameter R_{ref} is an arbitrary reference radius, typically chosen to be 50-70% of the magnet aperture. The coefficients in the above equation are defined as

$$B_n = \frac{R_{ref}^n}{n!} \left(\frac{\partial^n B_y}{\partial x^n} \right) \Big|_{x=0;y=0} , \tag{55}$$

$$A_n = \frac{R_{ref}^n}{n!} \left(\frac{\partial^n B_x}{\partial x^n} \right) \Big|_{x=0;y=0} . \tag{56}$$

According to the U.S. convention, $n = 0$ indicates dipole field. The magnet field is also written as

$$B_y + iB_x = B_{ref} \sum_{n=0}^{\infty} [b_n + ia_n] \left(\frac{z}{R_{ref}} \right)^n , \tag{57}$$

with

$$b_n = \frac{B_n}{B_{ref}}, a_n = \frac{A_n}{B_{ref}} .$$

In SAD, the normal component is defined as

$$K_n = \frac{B^{(n)} L}{B\rho} . \tag{58}$$

Then the relation of K_n and b_n can be found as

$$b_n = \frac{B\rho}{B_{ref} L} \frac{R_{ref}^n}{n!} K_n . \tag{59}$$

1.5 Magnetic field in Frenet-Serret coordinate system

1.6 Accelerator coordinate system in SAD

2 Maps for particle tracking

In SAD source codes, the subroutines for particle tracking are tbend.f, tquad.f, tdrift.f, etc.. They are called by the tturn.f file. For initializing the input parameters for these subroutines, the relevant source files are: initb1.f, tpara.f, etc..

General formula for the dimensionless Hamiltonian:

$$\begin{aligned} H(x, p_x, y, p_y, z, p) = \frac{E}{P_0 v_0} - (1 + \frac{x}{\rho_x} + \frac{y}{\rho_y})\sqrt{p^2 - (p_x - \hat{A}_x)^2 - (p_y - \hat{A}_y)^2} \\ - (1 + \frac{x}{\rho_x} + \frac{y}{\rho_y})\hat{A}_s, \end{aligned} \tag{60}$$

where P_0 and v_0 are respectively the momentum and velocity of a reference synchronous particle. p is the magnitude of the normalized momentum of a particle. $p_x = P_x/P_0$ and $p_y = P_y/P_0$ are the normalized canonical conjugate momenta. $\hat{A}_x = \frac{qA_x}{P_0} = \frac{A_x}{B\rho}$, $\hat{A}_y = \frac{qA_y}{P_0} = \frac{A_y}{B\rho}$, and $\hat{A}_s = \frac{qA_s}{P_0} = \frac{A_s}{B\rho}$ are the components of the dimensionless vector potential. q is the charge of the particle. A_x, A_y, and A_s are components of the vector potential in the Frenet-Serret coordinate system. ρ_x and ρ_y are the radii of curvature in x and y directions, respectively. For electron, $q = -e(?)$.

For a quadrupole, there is $\vec{A} \equiv (A_x, A_y, A_s) = (0, 0, \frac{1}{2}B_1(y^2 - x^2))$ with $B_1 = \partial B_y/\partial x$. Then $K_1 = \frac{B_1}{B_0\rho} = \frac{eB_1}{P_0}$.

For a solenoid, there is $\vec{B} = (0, 0, B_s)$, and $\vec{A} \equiv (A_x, A_y, A_s) = (-\frac{1}{2}B_s y, \frac{1}{2}B_s x, 0)$.

2.1 Map for DRIFT

Variables used in a DRIFT element:

- L: The element length
- RADIUS: Radius of the vacuum chamber. Effective when SPAC is ON.

2.1.1 Pure drift

Hamiltonian for a pure drift in SAD:

$$H(x,p_x,y,p_y,z,\delta)=\frac{E}{P_0v_0}-\sqrt{p^2-p_x^2-p_y^2} \tag{61}$$

with $p=P/P_0=1+\delta$, $p_x=P_x/P_0$, $p_y=P_y/P_0$. $E=\sqrt{P^2c^2+(m_0c^2)^2}=$ is the particle's total energy. Note that Eq.(61) is consistent with that used in Ohmi's SCTR code, but different from that defined in the SAD manual. Equation (61) can be rewritten as

$$H(x,p_x,y,p_y,z,\delta)=\frac{1}{v_0}\sqrt{p^2c^2+(\frac{m_0c^2}{P_0})^2}-\sqrt{p^2-p_x^2-p_y^2}. \tag{62}$$

Note that $P_0=\gamma_0m_0v_0$, $E_0^2=P_0^2c^2+m_0^2c^4$. In SAD, v_0 is defined as

$$v_0=\frac{P_0c^2}{E_0}=\frac{P_0c}{\sqrt{P_0^2+(m_0c)^2}}. \tag{63}$$

In the SAD sources, there is definition $mass=m_0c$.

Comment 1: I don't understand why the term of $\frac{E}{P_0v_0}$ in Eq.(61) has the minus sign in the SAD manual.

Comment 2: The global variable "MOMENTUM" in SAD is defined as $P_0=\gamma_0m_0v_0c=\beta_0\gamma_0m_0c^2$.

From Eq.(62), one can derive the map as following

$$e^{-:H:L}x=x-[HL,x]=x-L(\frac{\partial H}{\partial x}\frac{\partial x}{\partial p_x}-\frac{\partial H}{\partial p_x}\frac{\partial x}{\partial x})=x+L\frac{\partial H}{\partial p_x}, \tag{64}$$

$$e^{-:H:L}y=y-[HL,y]=y-L(\frac{\partial H}{\partial y}\frac{\partial y}{\partial p_y}-\frac{\partial H}{\partial p_y}\frac{\partial y}{\partial y})=y+L\frac{\partial H}{\partial p_y}, \tag{65}$$

$$e^{-:H:L}z=z-[HL,z]=z-L(\frac{\partial H}{\partial z}\frac{\partial z}{\partial \delta}-\frac{\partial H}{\partial \delta}\frac{\partial z}{\partial z})=z+L\frac{\partial H}{\partial \delta}. \tag{66}$$

The relevant maps from the above equations are

$$x_2=x_1+\frac{p_{x1}}{\sqrt{p^2-p_{x1}^2-p_{y1}^2}}L \tag{67}$$

$$p_{x2} = p_{x1} \tag{68}$$

$$y_2 = y_1 + \frac{p_{y1}}{\sqrt{p^2 - p_{x1}^2 - p_{y1}^2}} L \tag{69}$$

$$p_{y2} = p_{y1} \tag{70}$$

$$z_2 = z_1 - \left(\frac{p}{\sqrt{p^2 - p_{x1}^2 - p_{y1}^2}} - \frac{v}{v_0}\right)L = z_1 + \left(1 - \frac{p}{\sqrt{p^2 - p_{x1}^2 - p_{y1}^2}}\right)L - \frac{v_0 - v}{v_0} L$$

(71)

The above equations are consistent with the maps used in file tdrift.f. Note that, in SAD, $\Delta v = dv = \frac{v_0 - v}{v_0}$ is defined in the file tconvm.f for tracking, and is defined in tsetdv.f for emittance calculation. In the file tconvm.f, there is

$$dv = -\frac{g(1 + p_r)}{h_1(h_1 + p_r h_0)} + dvfs \tag{72}$$

with $dvfs$ = FSHIFT, $g = \delta$, $p_r = 1 + \delta$, $h_1 = \sqrt{1 + p_0^2(1 + \delta)^2}$, $h_0 = \sqrt{1 + p_0^2}$. With $p_0 = P_0/(m_0 c) = \frac{\gamma_0 v_0}{c}$, $p_r = 1 + \delta = \frac{\gamma v}{\gamma_0 v_0}$, and $\gamma = \sqrt{1 - v^2/c^2}$, one can find $h_1 = \gamma$, $h_0 = \gamma_0$, and $\Delta v = \frac{v_0 - v}{v_0}$.

2.1.2 Drift with $B_z \neq 0$

Hamiltonian for a drift with $B_z \neq 0$ in SAD:

$$H(x, p_x, y, p_y, z, \delta) = \frac{E}{P_0 v_0} - \sqrt{p^2 - (p_x + \frac{1}{2} b_z y)^2 - (p_y - \frac{1}{2} b_z x)^2} \tag{73}$$

with $b_z = \frac{eB_z}{P_0}$. Note that the sign of b_z is different from that in Ohmi's SCTR code.

Question: Should e take value of the charge, e.g. positive for positron, and negative for electron?

The relevant maps used in SAD:

$$x_2 = x_1 + \frac{(1 + \delta)\sin\phi}{b_z} p_{xi} + \frac{(1 + \delta)(1 - \cos\phi)}{b_z} p_{yi}, \tag{74}$$

$$y_2 = y_1 - \frac{(1+\delta)(1-\cos\phi)}{b_z} p_{xi} + \frac{(1+\delta)\sin\phi}{b_z} p_{yi}, \tag{75}$$

$$p_{x2} = p_{xi}\cos\phi + p_{yi}\sin\phi - \frac{b_z}{2(1+\delta)} y_2, \tag{76}$$

$$p_{y2} = -p_{xi}\sin\phi + p_{yi}\cos\phi + \frac{b_z}{2(1+\delta)} x_2, \tag{77}$$

$$z_2 = z_1 + \left[\frac{\sqrt{1-p_{xi}^2-p_{yi}^2}-1}{\sqrt{1-p_{xi}^2-p_{yi}^2}} - \Delta v \right] L \tag{78}$$

with

$$\phi = \frac{b_z L}{(1+\delta)\sqrt{1-p_{xi}^2-p_{yi}^2}}, \tag{79}$$

$$p_{xi} = p_{x1} + \frac{b_z y_1}{2(1+\delta)}, \tag{80}$$

$$p_{yi} = p_{y1} - \frac{b_z x_1}{2(1+\delta)}, \tag{81}$$

$$\Delta v = \frac{v_0 - v}{v_0}. \tag{82}$$

2.1.3 Drift with $B_z \neq 0$, $K_0 \neq 0$, and $SK_0 \neq 0$

This case can be handled by SAD, but the relevant maps are skipped here.

2.2 Map for BEND

Variables used in a BEND element:

- AE1: The absolute face angle at the entrance. The effective face angle is E1 * ANGLE + AE1, and a positive angle at the entrance corresponds to a surface with dx/ds > 0.

- AE2: The absolute face-angle at the exit to the bending angle. The effective face angle is E2 * ANGLE + AE2, and a positive angle at the exit corresponds to a surface with dx/ds < 0.

- ANGLE: The bending angle. If positive, it bends the orbit in x-s plane toward negative-x-direction. ANGLE determines the geometry of the beam line, while K0 represents a dipole kick on top of the bending angle given by ANGLE, i.e., the total deflection of the beam is given of ANGLE + K0.

- DISFRIN: If nonzero, the nonlinear Maxwellian fringe (hard-edge fringe model) is suppressed.

- DISRAD: If nonzero, the synchrotron radiation in the particle-tracking is inhibited.

- DROTATE: Additional rotation in x-y plane to simulate a rotation error. DROTATE does not afgfect the geometry of the ring.

- DX: Horizontal displacement of magnet. This applied before the rotation by ROTATE.

- DY: Vertical displacement of magnet. This applied before the rotation by ROTATE.

- E1: The ratio of the face-angle at the entrance to the bending angle. The effective face angle is E1 * ANGLE + AE1, and a positive angle at the entrance corresponds to a surface with dx/ds $>$ 0. For example, a symmetrically-placed rectangular magnet has E1 = 0.5 and E2 = 0.5.

- E2: The ratio of the face-angle at the exit to the bending angle. The effective face angle is E2 * ANGLE + AE2, and a positive angle at the exit corresponds to a surface with dx/ds $<$ 0. For example, a symmetrically-placed rectangular magnet has E1 = 0.5 and E2 = 0.5.

- F1: Length of the slope of the field at the edge. For the soft-edge fringe effects, only the effects up to y^4 in Hamiltonian are taken into account. More rigorous definition is

$$\mathrm{F1} = 6\int_{s=-\infty}^{+\infty} ds \left(\frac{B_y(s)}{B_0} - \frac{B_y^2(s)}{B_0^2}\right), \tag{83}$$

  where integration is done over one fringe. This soft fringe also changes the path length in the body of the bend. To maintain the geometric position of the design orbit, i.e., one has to increase the bend field a little bit to keep the orbit unchanged. Unlike a quadrupole, the effect of linear fringe is always applied at both the entrance and the exit, otherwise one cannot obtain a circular design orbit.

- FB1: F1 at the entrance. Actually F1 + FB1 is used at the entrance.
- FB2: F1 at the exit. Actually F1 + FB2 is used at the exit.
- FRINGE: When FRINGE is non-zero, the effect of the linear fringe F1 is taken into account both at the entrance and the exit.
- K0: The normal 2-pole magnetic field component (times the length L). It is defined as
$$\mathrm{K0} = \frac{B^{(0)}\mathrm{L}}{B\rho}, \tag{84}$$
where L is the length of the component. Positive sign means horizontal focusing.
- K1: The normal 4-pole magnetic field component (times the length L). It is defined as
$$\mathrm{K1} = \frac{B^{(1)}\mathrm{L}}{B\rho}, \tag{85}$$
where L is the length of the component. Positive sign means horizontal focusing.
- L: Length along the arc of the orbit.
- ROTATE: Rotation in x-y plane. After displacing the magnet by DX and DY, rotate the magnet around the local s-axis by an amount given by ROTATE, then place the component. At the exit rotate back the magnet around the local s-axis at the exit, then take out displacement.

The transformation of a bend depends on the value of K1. If K1 $\neq 0$, the transformation map is defined in the source file tbendi.f. If K1 is zero, it is a series of transformations as following defined in the source file tbend.f.

- Step 1: Kick due to rotation error:
$$p_{x2} = p_{x1} + \Delta\phi_x, \tag{86}$$
$$p_{y2} = p_{y1} + \Delta\phi_y, \tag{87}$$

where $\Delta\phi_x$ and $\Delta\phi_y$ are rotation errors of the magnet in horizontal and vertical directions, respectively. Those parameters are defined in source file tpara.f as following

$$\Delta\phi_x = \mathrm{ANGLE} \cdot \sin^2\left(\frac{\mathrm{DROTATE}}{2}\right), \tag{88}$$

$$\Delta\phi_y = \frac{\mathrm{ANGLE}}{2} \cdot \sin(\mathrm{DROTATE}). \tag{89}$$

- Step 2: Drift to the entrance face:

$$x_2 = \frac{x_1}{\cos(\psi_1) - \sin(\psi_1)(p_{x1}/p_{z1})}, \tag{90}$$

$$p_{x2} = p_{x1}\cos(\psi_1) + p_{z1}\sin(\psi_1), \tag{91}$$

$$y_2 = y_1 + \frac{p_{y1}}{p_{z1}} x_2 \sin(\psi_1), \tag{92}$$

$$z_2 = z_1 - \frac{1+\delta}{p_{z1}} x_2 \sin(\psi_1), \tag{93}$$

with

$$p_{z1} = \sqrt{(1+\delta_1)^2 - p_{x1}^2 - p_{y1}^2}, \tag{94}$$

where $\psi_1 = \mathrm{ANGLE} * \mathrm{E1} + \mathrm{AE1}$, and δ_1 is the momentum deviation at position 1.

- Step 3: Soft-edge fringe at entrance face (Based on tbend.f.):

$$x_2 = x_1 + \frac{\delta_1}{1+\delta_1}\Delta x_{fr1}, \tag{95}$$

$$p_{y2} = p_{y1} + \frac{\Delta y_{fr1} - \Delta y_{fra1} y_1^2}{1+\delta_1} y_1, \tag{96}$$

$$z_2 = z_1 + \frac{1}{(1+\delta_1)^2}\Delta x_{fr1} p_{x1} + \frac{\Delta y_{fr1} - \Delta y_{fra1} y_1^2/2}{2(1+\delta_1)^2} y_1^2 - \Delta z_{fr1}, \tag{97}$$

where

$$\Delta x_{fr1} = \frac{\mathrm{FB1}^2}{24\rho_b}, \tag{98}$$

$$\Delta y_{fr1} = \frac{\mathrm{FB1}}{6\rho_b^2}, \tag{99}$$

$$\Delta y_{fra1} = \frac{2}{3\text{FB1}\rho_b^2}, \tag{100}$$

$$\Delta z_{fr1} = \Delta x_{fr1} \cdot \sin(\psi_1). \tag{101}$$

$$\rho_b = \frac{\text{L'}}{\text{ANGLE} + \text{K0}}, \tag{102}$$

$$\text{L'} = \text{L} - \frac{(\text{ANGLE F1})^2}{24\text{L}} \cdot \frac{\sin((\text{ANGLE}(1 - \text{E1} - \text{E2}) - \text{AE1} - \text{AE2})/2)}{\sin(\text{ANGLE}/2)}. \tag{103}$$

Note that L' is defined in the source file tpara.f (comment: There is discrepancy between teh source code and the above equation!). When ANGLE=0, there is L'=L. The relevant map is $e^{:f:}$ with generating function

$$f = \frac{\text{FB1}^2}{24\rho_b}\frac{\delta}{1+\delta}p_x - \frac{\text{FB1}}{12\rho_b^2}\frac{y^2}{1+\delta} + \frac{1}{6\rho_b^2\text{FB1}}\frac{y^4}{1+\delta}. \tag{104}$$

- Step 4: Maxwellian hard-edge fringe at entrance:

$$x_2 = x_1 + y_1^2\frac{(1 - y_1^2/\rho_b^2/12)(1+\delta)^2}{2\rho_b\left[(1+\delta)^2 - p_{x1}^2\right]^{3/2}}, \tag{105}$$

$$p_{y2} = p_{y1} - p_{x1}\frac{(1 - y_1^2/\rho_b^2/6)y_1}{\rho_b\sqrt{(1+\delta)^2 - p_{x1}^2}}, \tag{106}$$

$$z_2 = z_1 - y_1^2\frac{(1 - y_1^2/\rho_b^2/12)(1+\delta)p_{x1}}{2\rho_b\left[(1+\delta)^2 - p_{x1}^2\right]^{3/2}}. \tag{107}$$

The map is $e^{:f:}$ with gerating function

$$f = \frac{(1 - \frac{y^2}{12\rho_b^2})y^2 p_x}{2\rho_b\sqrt{p^2 - p_x^2}}. \tag{108}$$

- Step 5: Body of bend:

$$p_{x2} = -\frac{\rho_0}{\rho_b}\left[\sin(\psi_2) + \sin(\Omega + \psi_1)\right] + p_{z1}\sin(\Omega) + p_{x1}\cos(\Omega) - \frac{x_1}{\rho_b}\sin(\Omega), \tag{109}$$

$$x_2 = x_1\cos(\Omega) + \rho_b\left[p_{z2} - p_{z1}\cos(\Omega) + p_{x1}\sin(\Omega)\right] + \rho_0\left[\cos(\Omega + \psi_1) - \cos(\psi_2)\right], \tag{110}$$

$$y_2 = y_1 + \frac{p_{y1}}{\sqrt{p_1^2 - p_{y1}^2}} s, \tag{111}$$

$$z_2 = z_1 - s\frac{p_1}{\sqrt{p_1^2 - p_{y1}^2}} + \frac{v_1}{v_0}\text{L'}, \tag{112}$$

where

$$\rho_0 = \frac{\text{L'}}{\text{ANGLE}}, \tag{113}$$

$$\Omega = \text{ANGLE} - \psi_1 - \psi_2, \tag{114}$$

$$s = \rho_b \cdot \text{ANGLE}\left[\arcsin(\frac{p_{x1}}{\sqrt{p_1^2 - p_{y1}^2}}) - \arcsin(\frac{p_{x2}}{\sqrt{p_2^2 - p_{y2}^2}}) + \Omega\right]. \tag{115}$$

v0 is the velocity of the on-momentum particle. v1 is the velocity of the particle being tracked at position 1.

- Step 6: Maxwellian hard-edge fringe at exit:

$$x_2 = x_1 - y_1^2 \frac{(1 - y_1^2/\rho_b^2/12)(1+\delta)^2}{2\rho_b\left[(1+\delta)^2 - p_{x1}^2\right]^{3/2}}, \tag{116}$$

$$p_{y2} = p_{y1} + p_{x1}\frac{y_1(1 - y_1^2/\rho_b^2/6)}{\rho_b\sqrt{(1+\delta)^2 - p_{x1}^2}}, \tag{117}$$

$$z_2 = z_1 + \frac{p_{x1}y_1^2(1 - y_1^2/\rho_b^2/12)(1+\delta)}{2\rho_b\left[(1+\delta)^2 - p_{x1}^2\right]^{3/2}}. \tag{118}$$

- Step 7: Soft-edge fringe at exit face:

$$x_2 = x_1 - \Delta x_{fr2}\frac{\delta}{1+\delta}, \tag{119}$$

$$p_{y2} = p_{y1} + \frac{1}{1+\delta}(\Delta y_{fr2} - \Delta y_{fra2} y_1^2) y_1, \tag{120}$$

$$z_2 = z_1 + \frac{1}{(1+\delta)^2}\left[-\Delta x_{fr2}p_{x1} + \frac{1}{2}(\Delta y_{fr2} - \Delta y_{fra2}y_1^2/2)y_1^2\right] - \Delta z_{fr2}, \tag{121}$$

where

$$\Delta x_{fr2} = \frac{\mathrm{FB2}^2}{24\rho_b}, \tag{122}$$

$$\Delta y_{fr2} = \frac{\mathrm{FB2}}{6\rho_b^2}, \tag{123}$$

$$\Delta y_{fra2} = \frac{2}{3\mathrm{FB2}\rho_b^2}, \tag{124}$$

$$\Delta z_{fr2} = \Delta x_{fr2} \cdot \sin(\psi_2). \tag{125}$$

The relevant map is $e^{:f:}$ with generating function

$$f = -\frac{\mathrm{FB2}^2}{24\rho_b}\frac{\delta}{1+\delta}p_x - \frac{\mathrm{FB2}}{12\rho_b^2}\frac{y^2}{1+\delta} + \frac{1}{6\rho_b^2\mathrm{FB2}}\frac{y^4}{1+\delta}. \tag{126}$$

- Step 8: Drift from the exit face:

$$p_{x2} = p_{x1}\cos(\psi_2) + p_{z1}\sin(\psi_2), \tag{127}$$

$$x_2 = x_1\left[\cos(\psi_2) + \frac{p_{x2}}{p_{z2}}\sin(\psi_2)\right], \tag{128}$$

$$y_2 = y_1 + \frac{p_{y2}}{p_{z2}}x_1\sin(\psi_2), \tag{129}$$

$$z_2 = z_1 - \frac{1+\delta}{p_{z2}}x_1\sin(\psi_2), \tag{130}$$

where

$$\psi_2 = \mathrm{ANGLE} \cdot \mathrm{E2} + \mathrm{AE2}, \tag{131}$$

$$p_{z2} = \cos(\psi_2)\sqrt{(1+\delta)^2 - p_{x1}^2 - p_{y1}^2} + p_{x1}\sin(\psi_2). \tag{132}$$

- Step 9: Kick due to rotation error:

$$p_{x2} = p_{x1} + \Delta\phi_x, \tag{133}$$

$$p_{y2} = p_{y1} + \Delta\phi_y. \tag{134}$$

If K1 is nonzero, the effects from E1 and E2 are approximated by thin quadrupoles. Then the body is subdivided into 1 + floor(sqrt(abs(K1 L')/(12 10^-5 EPS))) pieces (EPS = 1 is used when EPS = 0), and the bend-body transformation above is done for each piece and the kick from K1 is applied alternatively.

In FFS optics and Emittance calculations, or when the synchrotron radiation is turned on, the same algorithm as K1 $\neq$ 0 is applied.

2.3 Map for SOL

Unlike other elements, SOL elements inserted at boundaries or of a solenoid or at where the field changes. Between SOL elements DRIFT, BEND(straight bend only), QUAD, and MULT elements can be inserted. The longitudinal field of the solenoid overlaps on those elements.

In a SOL region, the coordinate is shifted on the axis of the solenoid, no matter how the design orbit bends there. The x-direction of the coordinate in a solenoid is so chosen as to CHI3 = 0. At the exit of a solenoid, the coordinate is shifted back to the design orbit, but the value of CHI3 is so determined as to set CHI3 zero at the nearest MARK element which has GEO = 1 after the exit. The offset and orientation of the design orbit can be given by keywords DX, DY, DPX, DPY at a SOL element with GEO = 1.

SOL can be used to shift the coordinate to the actual orbit even without BZ. It is useful to define the coordinate with magnets with DX and DY.

Variables used in a SOL element:

- BOUND: BOUND = 1 must be given at both sides of the boundaries of a solenoid, otherwise SOL only specifies the change of BZ.

- BZ: The longitudinal field of a solenoid. If a SOL is used with BOUND = 0 (default), only BZ is used to change the field, and no coordinate transformation is applied.

- DISFRIN: Disables the nonlinear fringe of solenoid if nonzero. The default is 0. The transformation for the nonlinear fringe is expressed by exp(:H:) with

$$\mathrm{H} = -\frac{B_z}{8B\rho p^2} p_\phi p_r, \tag{135}$$

where

$$p_\phi = x p_y - y p_x, \tag{136}$$

$$p_r = x p_x + y p_y, \tag{137}$$

whose canonical partners are

$$\phi = \arctan\left(\frac{y}{x}\right), \tag{138}$$

$$r = \frac{1}{2}\ln(x^2 + y^2), \tag{139}$$

respectively.

- DX: An x-offset of the design ORBIT relative to the solenoid center at SOL with GEO = 1.

- DY: A y-offset of the design ORBIT relative to the solenoid center at SOL with GEO = 1.

- F1: The length of fringe of the solenoide field. It affects only the EMITTANCE calculation. If F1 = 0 (default), no radiation arises at the fringe.

- GEO: One of boundaries (with GEO = 1) of a solenoid must have GEO = 1 to specify the alignment of the design orbit. At a SOL element with GEO = 1, the design orbit is determined by DX, DY, DPX, DPY parameters.

2.4 Map for MULT

This section discusses the maps for MULT elements in SAD. Actually, the detailed maps for MULT are quite complicated. For simplification, we only formulate the maps used in SuperKEKB lattices.

Keywords for MULT element:
L DX DY DZ CHI1 CHI2 ROTATE(=CHI3) K0..K21 SK0..SK21
DISFRIN F1 F2 FRINGE DISRAD EPS VOLT HARM PHI DPHI FREQ
RADIUS ANGLE E1 E2 AE1 AE2 DROTATE

The definitions of these keywords are described in the SAD manual.

In the SAD sources, the file tmulti.f contains the subroutine for tracking through a MULT elements. The input variables of subroutine *tmulti* include:

- np=NP: The number of macro-particles.

- (x, px, y, py, z, g): Coordinate lists of the macro-particles. g=dp/p0=(p-p0)/p0.

- (dv,pz): dv=(v-v0)/c, pz=sqrt(p^2-px^2-py^2).

- l: Integer number, indicating the type of element or position of the element along the beam line(?)

- al=L: Length of element

- ak: Strength of magnetic fields. ak_n=K_n+$i$$SK_n$, with K_n and SK_n the normal and skew 2(n+1)-pole magnetic field component (times the length L), respectively. $K_n = B^{(n)}L/(B\rho)$, SK_n=$B^{(n)}L/(B\rho)$ with ROTATE = (90/(n+1)) DEG. $0 \leq n \leq 21$.

- bz=BZ: The longitudinal field of a solenoid. Note that BZ is not a keyword of a MULT element. But its value is transferred from the previous SOL element if nonzero is defined.

- phia=ANGLE: The bending angle.

- (psi1, psi2)=(AE1, AE2): AE1 is the absolute face angle at the entrance. AE2 is the absolute face-angle at the exit to the bending angle

- (dx, dy, dz, chi1, chi2, theta, dtheta)=(DX, DY, DZ, CHI1, CHI2, ROTATE, DROTATE): Misalignment of the element. dx, dy, and dz are horizontal, vetical, and longitudinal displacement of the magnet, respectively. The displacements are applied before the rotation by ROTATE. chi1, chi2, and chi3=theta(=ROTATE) are the rotation angles around x-, y-, and z-axis, respectively.

- eps0=EPS: Variable for determining the number of slices used in tracking through the element.

- enarad=DISRAD: Flag for synchrotron radiation, default value=0. If nonzero, the synchrotron radiation in the particle-tracking is inhibited.

- fringe=DISFRIN: flag for nonlinear maxwellian fringe fields, default value=0. If nonzero, the nonlinear maxwellian fringe is suppressed.

- (f1, f2): Parameters to characterize the slope of the quadrupole field at the edges, calculated from F1 and F2. Detailed formula for f1 and f2 are given in the following.

- mfring=FRINGE: The effects of the linear fringe (characterized by F1 and F2), optional values=0, 1, 2, or 3.

- (fb1, fb2): F1 and F2 for the dipole magnet.

- (vc, harm, phirf, freq, dphirf)=(VOLT HARM PHI FREQ DPHI): Parameters for RF cavity. Usually not used for MULT elements.

- radius=RADIUS: Radius of the vacuum chamber. Effective when SPAC is ON.

- kptbl: It is a table to assign particles of the original order. The arrays x(i), px(i) always contain living particles up to np. When i-th particle is lost during a tracking, x(i),.. are replaced by x(np) and np is decreased by 1. Thus the order of x() is no longer equal to the original order. To recover the original order, kptbl keeps it as well as the reverse information.

Truths for the MULT elements defined in SuperKEKB lattices:

- ANGLE=0 for all MULT components. Consequently, the subroutine tmulta (defined in file tmulta.f) is not used in tracking through MULT elements.

- AE1=0, AE2=0, DROTATE=0, DX=0, DZ=0, F2=0, VOLT=0, HARM=0, PHI=0, DPHI=0, CHI1=0, and CHI2=0 for all MULT components.

- DY≠0, ROTATE≠0, K0≠0, and K1≠0 for QC* elements.

- K1=0, and ROTATE=0 for all EC* elements.

- DISRAD=0 for all MULT components by default. If one wants to tracking through the MULT elments without synchrotron radiation, he needs to use NORAD flag to turn SR off.

- BZ≠0 if the previous element is defined as SOL. In SAD, the BZ from SOL element is transferred to the following MULT.

Considering the above truths, the equations of motion for particle in MULT elements can be described as the following steps with L≠0:

- Step 1: Map for solenoid fringe at the entrance.

  The equations of motion are:

$$x_2 = x_1 - dx, \tag{140}$$

$$y_2 = y_1 - dy, \tag{141}$$

$$p_{x2} = p_{x1} + \frac{1}{2}B_z dy, \tag{142}$$

$$p_{y2} = p_{y1} - \frac{1}{2} B_z dx, \tag{143}$$

with B_z =bz, dx =dx=DX, and dy =dy=DY, $p_x = P_x/P_0$, $p_y = P_y/P_0$. The subtraction of closed orbit indicates that we are looking at the particles from the center of the solenoid magnet.

- Step 2: Rotation of particle coordinates at the entrance. This step is only for convenience of computations, and is irrelevant to the physics of particle motions.

  First we define the rotation angle as

$$\theta_1 = \begin{cases} \frac{1}{2} \arctan\left(\frac{SK_1}{K_1}\right) & \text{if} \quad K_1 > 0, \\ \frac{1}{2}\left[\arctan\left(\frac{SK_1}{K_1}\right) + \pi\right] & \text{if} \quad K_1 < 0, SK_1 > 0, \\ \frac{1}{2}\left[\arctan\left(\frac{SK_1}{K_1}\right) - \pi\right] & \text{if} \quad K_1 < 0, SK_1 < 0, \\ \frac{\pi}{4} \quad \text{if} \quad K_1 = 0, SK_1 > 0, \\ -\frac{\pi}{4} \quad \text{if} \quad K_1 = 0, SK_1 < 0 \end{cases} \tag{144}$$

  with $SK_1 \neq 0$ and the arctan function defined in the range of $(-\pi/2, \pi/2)$. If $SK_1 = 0$, then we define $\theta_1 = 0$. Here SK_1=SK1 and K_1=K1.

  Next we define

$$\theta_2 = \theta + \theta_1 \tag{145}$$

  with θ =ROTATE=CHI3. If $\theta_2 \neq 0$, then do the following transformations:

$$x_2 = x_1 \cos(\theta_2) - y_1 \sin(\theta_2), \tag{146}$$

$$y_2 = x_1 \sin(\theta_2) + y_1 \cos(\theta_2), \tag{147}$$

$$p_{x2} = p_{x1} \cos(\theta_2) - p_{y1} \sin(\theta_2), \tag{148}$$

$$p_{y2} = p_{x1} \sin(\theta_2) + p_{y1} \cos(\theta_2), \tag{149}$$

  with $p_x = P_x/P_0$, $p_y = P_y/P_0$. With rotation of particle coordinates, the effective field components felt by the particles also change. The effective field components are stored in akr(0:nmult) with nmult=21. Define

$$cr_1 = \cos(\theta_1) - i \sin(\theta_1), \tag{150}$$

there are

$$\text{akr}(0) = cr_1\text{ak}(0), \tag{151}$$

$$\text{akr}(1) = 0 \quad \text{if} \quad nmmax = 0, \tag{152}$$

$$\text{akr}(n) = cr_1^{n+1}\text{ak}(n) \quad \text{with} \quad n = 1, nmmax, \tag{153}$$

where *nmmax* is the maximum order of the MULT element's field components. Note that akr(1)=|ak(1)| with the above definition. It indicates that the skew-quadrupole component is eliminated. And without skew-quadrupole component, the analytic map used in tsolqu.f will be significantly simplified.

- Step 3: Determine the number of slices *ndiv* when tracking through a MULT element. The source codes are:

  ndiv=1

  do n=2, nmmax

  ndiv=max(ndiv, int(sqrt(ampmax**(n-1)/6.d0/fact(n-1)/eps*abs(akr(n))*al))+1)

  enddo

  ndiv=min(ndivmax,ndiv)

  Here ampmax=0.05 m is the maximum aperture radius. fact(n-1)=(n-1)! is the factorial of n-1. In practice, the total number of slices is equal to ndiv+1. The first and last slices have length of L/ndiv/2, the other ndiv-1 slices have length of L/ndiv.

- Step 4: Map for nonlinear Maxwellian fringe field at the entrance.

It is assumed that $L > 0$, otherwise there is no need to execute this step. Suppose that K_n and SK_n are the normal and skew 2(n+1)-pole magnetic field components. B_z is the longitudinal magnetic field. Then we define

$$\theta = \begin{cases} \frac{1}{n+1}\arctan\left(\frac{SK_n}{K_n}\right), & \text{if} \quad K_n \neq 0 \quad \text{and} \quad SK_n \neq 0, \\ 0, & \text{if} \quad K_n \neq 0 \quad \text{and} \quad SK_n = 0, \\ \frac{\pi}{2(n+1)}, & \text{if} \quad K_n = 0 \quad \text{and} \quad SK_n \neq 0, \end{cases} \tag{154}$$

and

$$K_{an} = \begin{cases} \sqrt{K_n^2 + SK_n^2}, & \text{if} \quad K_n \neq 0 \quad \text{and} \quad SK_n \neq 0, \\ K_n, & \text{if} \quad K_n \neq 0 \quad \text{and} \quad SK_n = 0, \\ SK_n, & \text{if} \quad K_n = 0 \quad \text{and} \quad SK_n \neq 0. \end{cases} \tag{155}$$

If $\theta \neq 0$, we first rotate the particle coordinates by

$$x_2 = x_1 \cos\theta - y_1 \sin\theta, \tag{156}$$

$$y_2 = x_1 \sin\theta + y_1 \cos\theta, \tag{157}$$

$$p_{x2} = p_{x1} \cos\theta - p_{y1} \sin\theta, \tag{158}$$

$$p_{y2} = p_{x1} \sin\theta + p_{y1} \cos\theta. \tag{159}$$

With the above rotation, the particle only feels a kick from the normal component, and the skew-component is terminated. [Q: Why not remove the normal term? According to E. Forest, the maps for skew component is more simple.]

SAD uses the second-order implicit generating function to construct the map for the nonlinear Maxwellian fringe field of quadrupole ($n = 1$). The implicit scheme gives the general formulae for the transformation of coordinates:

$$\bar{q}_i = q_i + \frac{\partial H}{\partial \bar{p}_i} + \frac{1}{2}\sum_j \left[\frac{\partial^2 H}{\partial q_j \partial \bar{p}_i}\frac{\partial H}{\partial \bar{p}_j} + \frac{\partial H}{\partial q_j}\frac{\partial^2 H}{\partial \bar{p}_j \partial \bar{p}_i}\right], \tag{160}$$

$$p_i = \bar{p}_i + \frac{\partial H}{\partial q_i} + \frac{1}{2}\sum_j \left[\frac{\partial^2 H}{\partial q_i \partial q_j}\frac{\partial H}{\partial \bar{p}_j} + \frac{\partial H}{\partial q_j}\frac{\partial^2 H}{\partial q_i \partial \bar{p}_j}\right], \tag{161}$$

with $q = (x, y, z)$, $p = (p_x, p_y, \delta)$, and Hamiltonian $H = H(q, \bar{p})$. For higher-order components ($n \geq 2$), SAD uses the first-order implicit scheme

$$\begin{cases} \bar{q}_i = q_i + \frac{\partial H}{\partial \bar{p}_i}, \\ p_i = \bar{p}_i + \frac{\partial H}{\partial q_i}. \end{cases} \tag{162}$$

The case of dipole ($n = 0$) is special and is treated separately.

With orders of n from 0 to n_{max}, the maps are summarized as following. Note that the maps are executed in particle tracking by the order of following descriptions.

Quadrupole component with $n = 1$ *and* $B_z = 0$: The hamiltonian for the fringe field of a normal quadrupole is[3, 4]

$$H = \frac{K_{a1}}{12(1+\delta)L}[(x^3 + 3xy^2)p_x - (y^3 + 3x^2 y)p_y]. \tag{163}$$

Applying the above Hamiltonian to Eqs. (160)and (161), the map is obtained as

$$x_2 = x_1 + \Delta x_1, \tag{164}$$

$$y_2 = y_1 + \Delta y_1, \tag{165}$$

$$p_{x2} = \frac{1}{f}(f_y p_{x1} + h p_{y1}), \tag{166}$$

$$p_{y2} = \frac{1}{f}(f_x p_{y1} - h p_{x1}), \tag{167}$$

$$z_2 = z_1 - \frac{p_{x2}}{1+\delta}(\Delta x_1 + t x_1) - \frac{p_{y2}}{1+\delta}(\Delta y_1 + t y_1), \tag{168}$$

with

$$\Delta x_1 = x_1 \left[\frac{K_{a1} x_1^2}{12L(1+\delta)} + \frac{K_{a1} y_1^2}{4L(1+\delta)} + t \right], \tag{169}$$

$$\Delta y_1 = -y_1 \left[\frac{K_{a1} x_1^2}{4L(1+\delta)} + \frac{K_{a1} y_1^2}{12L(1+\delta)} - t \right], \tag{170}$$

$$f_x = 1 + \frac{K_{a1} x_1^2}{4L(1+\delta)} + \frac{K_{a1} y_1^2}{4L(1+\delta)} + \frac{K_{a1}^2 (x_1^2 - y_1^2)(5x_1^2 - y_1^2)}{96L^2(1+\delta)^2}, \tag{171}$$

$$f_y = 1 - \frac{K_{a1} x_1^2}{4L(1+\delta)} - \frac{K_{a1} y_1^2}{4L(1+\delta)} + \frac{K_{a1}^2 (x_1^2 - y_1^2)(x_1^2 - 5y_1^2)}{96L^2(1+\delta)^2}, \tag{172}$$

$$h = \frac{K_{a1} x_1 y_1}{2L(1+\delta)} \left[1 - \frac{K_{a1}(x_1^2 - y_1^2)}{12L(1+\delta)} \right], \tag{173}$$

$$f = f_x f_y + h^2, \tag{174}$$

$$t = \frac{K_{a1}^2 (x_1^2 - y_1^2)^2}{96L^2(1+\delta)^2}. \tag{175}$$

Quadrupole component with $n = 1$ *and* $B_z \neq 0$: The map is

$$x_2 = x_1 + \Delta x_1, \tag{176}$$

$$y_2 = y_1 + \Delta y_1, \tag{177}$$

$$p_{xa} = p_{x1} + \frac{B_z}{2} y_2, \tag{178}$$

$$p_{ya} = p_{y1} - \frac{B_z}{2} x_2, \tag{179}$$

$$p_{x2} = \frac{1}{f}(f_y p_{xa} + h p_{ya}) - \frac{B_z}{2} y_1, \tag{180}$$

$$p_{y2} = \frac{1}{f}(f_x p_{ya} - h p_{xa}) + \frac{B_z}{2} x_1, \tag{181}$$

$$z_2 = z_1 - \frac{p_{x2}}{1+\delta}(\Delta x_1 + t x_1) - \frac{p_{y2}}{1+\delta}(\Delta y_1 + t y_1), \tag{182}$$

with Δx_1, Δy_1, f_x, f_y, h, f, and t defined as the same as those of $B_z = 0$. [Q: How to derive the above equations?)

Dipole component with $n = 0$: The Hamiltonian for the fringe field of a normal dipole is

$$H = \frac{k_0}{2(1+\delta)} y^2 p_x. \tag{183}$$

The corresponding map calculated from Eq.(162) is

$$x_2 = x_1 + \frac{K_{a0}}{L(1+\delta)} y_1^2, \tag{184}$$

$$p_{y2} = p_{y1} - \frac{K_{a0}}{L(1+\delta)} y_1 p_{x1}, \tag{185}$$

$$z_2 = z_1 - \frac{K_{a0} y_1^2}{2L(1+\delta)^2} p_{x2}. \tag{186}$$

A different form of the hamiltonian for the fringe field of a normal dipole is[?]

$$H = \frac{k_0}{8(1+\delta)}(x^2 p_x - 2xy p_y + 3y^2 p_x). \tag{187}$$

Note that the above Hamiltonian given by Cai and Nosochkov is different from that by Forest and that in SAD. The corresponding map calculated from Eq.(162) is

$$x_2 = x_1 + \frac{K_{a0}}{8L(1+\delta)}(x_1^2 + 3y_1^2), \tag{188}$$

$$y_2 = y_1 - \frac{K_{a0}}{4L(1+\delta)} x_1 y_1, \tag{189}$$

$$p_{x2} = \frac{1}{d}\left[p_{x1} + \frac{K_{a0}}{4L(1+\delta)}(y_1 p_{y1} - x_1 p_{x1})\right], \tag{190}$$

$$p_{y2} = \frac{1}{d}\left[p_{y1} + \frac{K_{a0}}{4L(1+\delta)}(x_1 p_{y1} - 3y_1 p_{x1})\right], \tag{191}$$

$$z_2 = z_1 - \frac{K_{a0}(x_1^2 + 3y_1^2)}{8L(1+\delta)^2} p_{x2} + \frac{K_{a0} x_1 y_1}{4L(1+\delta)^2} p_{y2}, \tag{192}$$

with

$$d = 1 + \frac{K_{a0}^2(3y_1^2 - x_1^2)}{16L^2(1+\delta)^2}. \tag{193}$$

Sextupole component with $n = 2$: The map is

$$z_2 = z_1 - \frac{1}{1+\delta}\mathrm{Re}[c_a c_{p1}], \tag{194}$$

$$x_2 = \mathrm{Re}[c_{x1}], \tag{195}$$

$$y_2 = \mathrm{Im}[c_{x1}], \tag{196}$$

$$p_{x2} = \mathrm{Re}[c_{p1}], \tag{197}$$

$$p_{y2} = -\mathrm{Im}[c_{p1}], \tag{198}$$

with

$$c_x = x_1 + y_1 i, \tag{199}$$

$$c_p = p_{x1} - p_{y1} i, \tag{200}$$

$$c_{z1} = c_x^2, \quad c_z = c_x^3, \tag{201}$$

$$a = \frac{K_{a2}\mathrm{Im}[c_z]}{12L(1+\delta)}, \tag{202}$$

$$c_a = -\frac{K_{a2}c_x\left(\frac{1}{4}c_z - \overline{c_z}\right)}{24L(1+\delta)}, \tag{203}$$

$$c_{x1} = c_x + c_a, \tag{204}$$

$$d = 1 + a^2 - \frac{K_{a2}^2\left(\mathrm{Im}[c_z]^2 + \mathrm{Re}[c_z]^2\right)}{64L^2(1+\delta)^2}, \tag{205}$$

$$c_{p1} = \frac{1}{d}\left[(1+ia)c_p - \frac{K_{a2}c_{z1}\overline{c_p c_x}}{8L(1+\delta)}\right], \tag{206}$$

where the over-line indicates a complex conjugate, and the symbols Re[] and Im[] denote the real and imaginary part of a variable, respectely.

Higher-order component with $n > 2$: The general formula for the Hamiltonian for the hard-edge fringe field of a normal multipole component is[4]

$$H = \frac{K_{an}}{4(1+\delta)L(n+2)n!}\mathrm{Re}\left[(x+iy)^{n+1}\left(xp_x + yp_y + \frac{i(n+3)}{n+1}(xp_y - yp_x)\right)\right]. \tag{207}$$

With some manipulations, the above equation can be re-written as

$$
\begin{aligned}
H =& \frac{K_{an}}{4(1+\delta)L(n+1)!} \\
& \times \left[\frac{1}{2}(xp_x + yp_y - ixp_y + iyp_x)\left(-\frac{(x+iy)^{n+1}}{n+2} + (x-iy)^{n+1}\right)\right. \\
& \left. +\frac{1}{2}(xp_x + yp_y + ixp_y - iyp_x)\left((x+iy)^{n+1} - \frac{(x-iy)^{n+1}}{n+2}\right)\right]. \\
=& \frac{K_{an}}{4(1+\delta)L(n+1)!}\left(G + \bar{G}\right),
\end{aligned} \tag{208}
$$

with

$$
G = \frac{1}{2}(xp_x + yp_y - ixp_y + iyp_x)\left(-\frac{(x+iy)^{n+1}}{n+2} + (x-iy)^{n+1}\right). \tag{209}
$$

One can see that, with $n = 1$, Eq.(207) and (208) reproduce Eq.(163). Using the coordinate transformation of

$$
\begin{cases}
u = x + iy, \\
v = x - iy, \\
p_u = \frac{1}{2}(p_x - ip_y), \\
p_v = \frac{1}{2}(p_x + ip_y),
\end{cases} \tag{210}
$$

one can re-write Eq.(208) as

$$
\begin{aligned}
H =& \frac{K_{an}}{4(1+\delta)L(n+1)!}\left[up_u\left(-\frac{u^{n+1}}{n+2} + v^{n+1}\right)\right. \\
& \left. +vp_v\left(u^{n+1} - \frac{v^{n+1}}{n+2}\right)\right].
\end{aligned} \tag{211}
$$

Applying the first-order scheme of coordinate transformation Eq.(162) to coordinates $(u, p_u, v, p_v, z, \delta)$, one can obtain

$$
u_2 = u_1\left[1 + \frac{K_{an}}{4(1+\delta)L(n+1)!}\left(-\frac{u_1^{n+1}}{n+2} + v_1^{n+1}\right)\right], \tag{212}
$$

$$
v_2 = v_1\left[1 + \frac{K_{an}}{4(1+\delta)L(n+1)!}\left(u_1^{n+1} - \frac{v_1^{n+1}}{n+2}\right)\right], \tag{213}
$$

$$p_{u2} = \frac{1}{d}\left[-\frac{K_{an}p_{v1}u_1^n v_1}{4(1+\delta)Ln!} + p_{u1}\left(1 + \frac{K_{an}(u_1^{n+1} - v_1^{n+1})}{4(1+\delta)L(n+1)!}\right)\right], \tag{214}$$

$$p_{v2} = \frac{1}{d}\left[-\frac{K_{an}p_{u1}u_1 v_1^n}{4(1+\delta)Ln!} + p_{v1}\left(1 + \frac{K_{an}(-u_1^{n+1} + v_1^{n+1})}{4(1+\delta)L(n+1)!}\right)\right], \tag{215}$$

$$z_2 = z_1 - \frac{K_{an}}{4(1+\delta)^2 L(n+1)!}\left[u_1 p_{u2}\left(-\frac{u_1^{n+1}}{n+2} + v_1^{n+1}\right) + v_1 p_{v2}\left(u_1^{n+1} - \frac{v_1^{n+1}}{n+2}\right)\right], \tag{216}$$

with

$$d = 1 - \frac{K_{an}^2}{16(1+\delta)^2 L^2[(n+1)!]^2}\left[(n+1)^2 u_1^{n+1} v_1^{n+1} + \left(u_1^{n+1} - v_1^{n+1}\right)^2\right]. \tag{217}$$

Traslating the above transformation back in terms of $(x, p_x, y, p_y, z, \delta)$, the map is

$$z_2 = z_1 - \frac{1}{1+\delta}\mathrm{Re}[c_a c_{p1}], \tag{218}$$

$$x_2 = \mathrm{Re}[c_{x1}], \tag{219}$$

$$y_2 = \mathrm{Im}[c_{x1}], \tag{220}$$

$$p_{x2} = \mathrm{Re}[c_{p1}], \tag{221}$$

$$p_{y2} = -\mathrm{Im}[c_{p1}], \tag{222}$$

with

$$c_x = x_1 + y_1 i, \tag{223}$$

$$c_p = p_{x1} - p_{y1} i, \tag{224}$$

$$c_{z1} = c_x^n, \quad c_z = c_x^{n+1}, \tag{225}$$

$$a = \frac{K_{an}\mathrm{Im}[c_z]}{2L(1+\delta)(n+1)!}, \tag{226}$$

$$c_a = -\frac{K_{an}c_x\left(\frac{1}{n+2}c_z - \overline{c_z}\right)}{4L(1+\delta)(n+1)!}, \tag{227}$$

$$c_{x1} = c_x + c_a, \tag{228}$$

$$d = 1 + a^2 - \frac{K_{an}^2\left(\mathrm{Im}[c_z]^2 + \mathrm{Re}[c_z]^2\right)}{16L^2(1+\delta)^2(n!)^2}, \tag{229}$$

$$c_{p1} = \frac{1}{d}\left[(1+ia)c_p - \frac{K_{an}c_{z1}\overline{c_p c_x}}{4L(1+\delta)n!}\right]. \tag{230}$$

Note that the above general map for a multipole is aslo valid for the case of $n = 0$ (dipole), $n = 1$ (quadrupole) and $n = 2$ (sextupole) in the first-order approximation based on Eq.(162).

Finally, we rotate back the particle coordinates by

$$x_2 = x_1 \cos\theta + y_1 \sin\theta, \tag{231}$$

$$y_2 = -x_1 \sin\theta + y_1 \cos\theta, \tag{232}$$

$$p_{x2} = p_{x1} \cos\theta + p_{y1} \sin\theta, \tag{233}$$

$$p_{y2} = -p_{x1} \sin\theta + p_{y1} \cos\theta. \tag{234}$$

- Step 5.0: Linear soft-edge focusing at the entrance

The conditions for activating the linear edge focusing map at the entrance are $L > 0$, FRINGE= 1 or 3, FB1$\neq$ 0,$K0 \neq 0$ (or $SK0 \neq 0$). The subroutine to be called is *tblfri* defined in source file *tmulti.f* . First, we define parameters

$$\Delta x_{fx} = \frac{K_{0N} F_{B1}^2}{24L}, \tag{235}$$

$$\Delta y_{fx} = \frac{K_{0N}^2 F_{B1}}{6L^2}, \tag{236}$$

$$\Delta y_{fax} = \frac{2K_{0N}^2}{3F_{B1}L^2}, \tag{237}$$

$$\Delta y_{fy} = \frac{SK_{0N} F_{B1}^2}{24L}, \tag{238}$$

$$\Delta x_{fy} = \frac{SK_{0N}^2 F_{B1}}{6L^2}, \tag{239}$$

$$\Delta x_{fay} = \frac{2SK_{0N}^2}{3F_{B1}L^2}, \tag{240}$$

where F_{B1}=FB1, K_{0N} = Re[akr(0)], and SK_{0N} = Im[akr(0)]. The equations of motion are (in canonical coordinates):

$$x_2 = x_1 + \frac{\delta_1}{1+\delta_1} \Delta x_{fx}, \tag{241}$$

$$y_2 = y_1 - \frac{\delta_1}{1+\delta_1}\Delta y_{fy}, \tag{242}$$

$$p_{x2} = p_{x1} + \frac{x_2}{1+\delta_1}\left(\Delta x_{fy} - \Delta x_{fay} x_2^2\right), \tag{243}$$

$$p_{y2} = p_{y1} + \frac{y_2}{1+\delta_1}\left(\Delta y_{fx} - \Delta y_{fax} y_2^2\right), \tag{244}$$

$$\begin{aligned} z_2 =& z_1 + \frac{1}{(1+\delta_1)^2}(\Delta x_{fx} p_{x2} - \Delta y_{fy} p_{y2}) + \frac{x_2^2}{(1+\delta_1)^2}\left(\frac{1}{2}\Delta x_{fy} - \frac{1}{4}\Delta x_{fay} x_2^2\right) \\ &+ \frac{y_2^2}{(1+\delta_1)^2}\left(\frac{1}{2}\Delta y_{fx} - \frac{1}{4}\Delta y_{fax} y_2^2\right). \end{aligned} \tag{245}$$

The relevant map $e^{:f:}$ with generating function

$$\begin{aligned} f =& \frac{\delta}{1+\delta}(\Delta x_{fx} p_x - \Delta y_{fy} p_y) - \frac{x^2}{1+\delta}\left(\frac{1}{2}\Delta x_{fy} - \frac{1}{4}\Delta x_{fay} x^2\right) \\ &- \frac{y^2}{1+\delta}\left(\frac{1}{2}\Delta y_{fx} - \frac{1}{4}\Delta y_{fax} y^2\right). \end{aligned} \tag{246}$$

- Step 5: Quad-like fringe map at the entrance with $L \neq 0$.

  Source codes:

  ```
  do i=1,np
  p=(1.d0+g(i))
  a=f1/p
  ea=exp(a)
  b=f2/p
  pxf=px(i)/ea
  pyf=py(i)*ea
  z(i)=z(i)-(a*x(i)+b*(1.d0+.5d0*a)*pxf)*px(i)
  +(a*y(i)+b*(1.d0-.5d0*a)*pyf)*py(i)
  x(i)=ea*x(i)+b*px(i)
  y(i)=y(i)/ea-b*py(i)
  ```

px(i)=pxf

py(i)=pyf

enddo

and the relevant equations:

$$x_2 = x_1 e^{\frac{f_1}{p/p_0}} + \frac{f_2}{p/p_0} p_{x1}, \tag{247}$$

$$y_2 = y_1 e^{-\frac{f_1}{p/p_0}} - \frac{f_2}{p/p_0} p_{y1}, \tag{248}$$

$$p_{x2} = p_{x1} e^{-\frac{f_1}{p/p_0}}, \tag{249}$$

$$p_{y2} = p_{y1} e^{\frac{f_1}{p/p_0}}, \tag{250}$$

$$\begin{aligned} z_2 = z_1 &- \left(\frac{f_1}{p/p_0} x_1 + \frac{f_2}{p/p_0} \left(1 + \frac{f_1}{2p/p_0} \right) p_{x1} e^{-\frac{f_1}{p/p_0}} \right) p_{x1} \\ &+ \left(\frac{f_1}{p/p_0} y_1 + \frac{f_2}{p/p_0} \left(1 - \frac{f_1}{2p/p_0} \right) p_{y1} e^{\frac{f_1}{p/p_0}} \right) p_{y1}, \end{aligned} \tag{251}$$

where f_1 and f_2 are defined by

$$f_1 = -\frac{1}{24} akk * \mathrm{F1} * |\mathrm{F1}|, \tag{252}$$

and

$$f_2 = akk * \mathrm{F2}, \tag{253}$$

with

$$akk = \begin{cases} \frac{\mathrm{K1}}{L} \quad \text{if} \quad \mathrm{SK1} = 0, \\ \frac{\sqrt{\mathrm{K1}^2 + \mathrm{SK1}^2}}{L} \quad \text{if} \quad \mathrm{SK1} \neq 0, \end{cases} \tag{254}$$

whre it has been assumed that $L \neq 0$. If $L = 0$, SAD defines $f_1 = 0$ and $f_2 = 0$, indicating that no quad-like linear fringe effect. From the SAD manual, $F1$ and $F2$ are defined as

$$\mathrm{F1} = \mathrm{Sign}(a)\sqrt{|a|} \tag{255}$$

with

$$a = 24 \left(\frac{1}{2} I_0^2 - I_1 \right), \tag{256}$$

and

$$\mathrm{F2} = I_2 - \frac{1}{3} I_0^3. \tag{257}$$

Here I_n is calculated from the s-dependent field distribution

$$I_n = \frac{1}{\mathrm{K1}} \int_{\infty}^{-\infty} (K_1(s) - \mathrm{K1})\,(s - s_0)^n ds \tag{258}$$

for normal quadrupole magnet as an example.

- Step 6: Map for the body part.

The body part is splitted into ndiv+1 slices, with the first and last slice having the length of L/ndiv/2. For each slice, the analytic map is used for field components up to quadrupole, including longitudinal magnetic field B_z. The maps for higher-order field components are interleaved between the ndiv+1 slices.

Step 6.1: Analytical map for $0 \leq s \leq \frac{\mathrm{L}}{2\mathrm{ndiv}}$ by calling subroutine tsolqu

Suppose that $L_s = \frac{\mathrm{L}}{2\mathrm{ndiv}}$, the map is furtherly divided into $2n_s + 1$ steps with $n_s = 1 + \mathrm{Int}[\sqrt{(K_1 L_s)^2 + (B_z L_s)^2}/\epsilon]$. Here ϵ is a real number with default value of 0.1. After dividing, the "Kick-Body-Kick" map is constructed as following with assumption of $K_1 > 0$.

If $K_1 < 0$, with replacements of $K_1 \to -K_1$, $B_z \to -B_z$, $K_0 \to -K_0$, $SK_0 \to -SK_0$, the subroutine will be re-called.

If $K_1 = 0$, the subroutine tdrift will be called. That is, the element is taken as a drift with dipole field allowed. See Section 2.1.3.

The first step is the nonlinear kinematic effect, in canonical coordinates the equations of motion are

$$x_2 = x_1 + \left(\frac{1}{\sqrt{p^2 - p_{x1}^2 - p_{y1}^2}} - \frac{1}{p} \right) p_{x1} L_{ss}, \tag{259}$$

$$y_2 = y_1 + \left(\frac{1}{\sqrt{p^2 - p_{x1}^2 - p_{y1}^2}} - \frac{1}{p} \right) p_{y1} L_{ss}, \tag{260}$$

$$z_2 = z_1 + \left(1 - \frac{p}{\sqrt{p^2 - p_{x1}^2 - p_{y1}^2}} + \frac{p_{x1}^2 + p_{y1}^2}{2p^2} \right) L_{ss}, \tag{261}$$

with $L_{ss} = \frac{L_s}{2n_s}$ and $p = 1 + \delta_1$.

The $2i$-th step with $i = 1, ..., n_s$ is the linear map of a quadrupole with dipole field:

$$x_{2i+1} = x_{2i} + \left(x_{2i} + \frac{K_0}{K_1} \right) [\cos\psi - 1] + \frac{p_{x2i}}{p\sqrt{\frac{K_1}{L_{ss}p}}} \sin\psi, \tag{262}$$

$$p_{x2i+1} = p_{x2i} + p_{x2i}(\cos\psi - 1) - p \left(x_{2i} + \frac{K_0}{K_1} \right) \sqrt{\frac{K_1}{L_{ss}p}} \sin\psi, \tag{263}$$

$$y_{2i+1} = y_{2i} + \left(y_{2i} + \frac{SK_0}{K_1} \right) [\cosh\psi - 1] + \frac{p_{y2i}}{p\sqrt{\frac{K_1}{L_{ss}p}}} \sinh\psi, \tag{264}$$

$$p_{y2i+1} = p_{y2i} + p_{y2i}(\cosh\psi - 1) + p \left(y_{2i} + \frac{SK_0}{K_1} \right) \sqrt{\frac{K_1}{L_{ss}p}} \sinh\psi, \tag{265}$$

$$
\begin{aligned}
z_{2i+1} =& z_{2i} - \Delta v L_{ss} - \frac{1}{4p^2} p_{x2i}^2 L_{ss} - \frac{1}{4p^2} p_{y2i}^2 L_{ss} \\
& - \frac{1}{4p} p_{x2i} (x_{2i} + \frac{K_0}{K_1})(\cos\psi - 1) \\
& - \frac{1}{4} \left[\left(x_{2i} + \frac{K_0}{K_1} \right) (\cos\psi - 1) + \frac{\sin\psi}{p\sqrt{\frac{K_1}{L_{ss}p}}} p_{x2i} \right] \\
& \times \left[\frac{1}{p} p_{x2i} \cos\psi - \sqrt{\frac{K_1}{L_{ss}p}} \left(x_{2i} + \frac{K_0}{K_1} \right) \sin\psi \right] \qquad (266) \\
& - \frac{1}{4} \left(x_{2i} + \frac{K_0}{K_1} \right)^2 \sqrt{\frac{K_1}{L_{ss}p}} (\psi - \sin\psi) \\
& - \frac{1}{4p} p_{y2i} (y_{2i} + \frac{SK_0}{K_1})(\cosh\psi - 1) \\
& - \frac{1}{4} \left[\left(y_{2i} + \frac{SK_0}{K_1} \right) (\cosh\psi - 1) + \frac{\sinh\psi}{p\sqrt{\frac{K_1}{L_{ss}p}}} p_{y2i} \right] \\
& \times \left[\frac{1}{p} p_{y2i} \cosh\psi + \sqrt{\frac{K_1}{L_{ss}p}} \left(y_{2i} + \frac{K_0}{K_1} \right) \sinh\psi \right] \qquad (267) \\
& + \frac{1}{4} \left(y_{2i} + \frac{SK_0}{K_1} \right)^2 \sqrt{\frac{K_1}{L_{ss}p}} (\psi - \sinh\psi) \qquad (268)
\end{aligned}
$$

with $\psi = \sqrt{\frac{K_1 L_{ss}}{1+\delta_1}}$, $p = 1 + \delta_1$, $\Delta v = \frac{v_0 - v}{v_0}$, and $L_{ss} = \frac{L_s}{n_s}$. Note that the trick of treating the dipoe field: the dipole field felt by a particle is equivalent to shifting its transverse orbit to an offset of $(S)K_0/K_1$ in a quadrupole (Question: Is this exact?). When $K_0 = 0$ and $SK_0 = 0$, the above maps in transverse directions reduce to the standard map for a thick-lens quadrupole. The transformation in z is quite complicated.

For the $2i + 1$-th step with $i = 1, ..., n_s - 1$, the map for the nonlinear kinematic term is performed:

$$x_{2i+2} = x_{2i+1} + \left(\frac{1}{\sqrt{p^2 - p_{x2i+1}^2 - p_{y2i+1}^2}} - \frac{1}{p} \right) p_{x2i+1} L_{ss}, \tag{269}$$

$$y_{2i+2} = y_{2i+1} + \left(\frac{1}{\sqrt{p^2 - p_{x2i+1}^2 - p_{y2i+1}^2}} - \frac{1}{p} \right) p_{y2i+1} L_{ss}, \tag{270}$$

$$z_{2i+2} = z_{2i+1} + \left(1 - \frac{p}{\sqrt{p^2 - p_{x2i+1}^2 - p_{y2i+1}^2}} + \frac{p_{x2i+1}^2 + p_{y2i+1}^2}{2p^2} \right) L_{ss}, \tag{271}$$

with $L_{ss} = \frac{L_s}{n_s}$ and $p = 1 + \delta_1$.

The map for the last $(2n_s + 1)$ step is the same as the first step:

$$x_{2n_s+2} = x_{2n_s+1} + \left(\frac{1}{\sqrt{p^2 - p_{x2n_s+1}^2 - p_{y2n_s+1}^2}} - \frac{1}{p} \right) p_{x2n_s+1} L_{ss}, \tag{272}$$

$$y_{2n_s+2} = y_{2n_s+1} + \left(\frac{1}{\sqrt{p^2 - p_{x2n_s+1}^2 - p_{y2n_s+1}^2}} - \frac{1}{p} \right) p_{y2n_s+1} L_{ss}, \tag{273}$$

$$z_{2n_s+2} = z_{2n_s+1} + \left(1 - \frac{p}{\sqrt{p^2 - p_{x2n_s+1}^2 - p_{y2n_s+1}^2}} + \frac{p_{x2n_s+1}^2 + p_{y2n_s+1}^2}{2p^2} \right) L_{ss}, \tag{274}$$

with $L_{ss} = \frac{L_s}{2n_s}$ and $p = 1 + \delta_1$.

Step 6.2: Approximation kick for nonlinear multipole components at $0 \leq s \leq \frac{\mathrm{L}}{\mathrm{ndiv}}$ lumped to position $s = \frac{\mathrm{L}}{2\mathrm{ndiv}}$.

Step 6.3: Analytical map for $\frac{\mathrm{L}}{2\mathrm{ndiv}} \leq s \leq \frac{3\mathrm{L}}{2\mathrm{ndiv}}$ by calling subroutine tsolqu

Step 6.4: Approximation kick for nonlinear multipole components at $\frac{\mathrm{L}}{\mathrm{ndiv}} \leq s \leq \frac{2\mathrm{L}}{\mathrm{ndiv}}$ lumped to position $s = \frac{3\mathrm{L}}{2\mathrm{ndiv}}$.

... ...

Step 6.(2*ndiv): Approximation kick for nonlinear multipole components at $L - \frac{\mathrm{L}}{\mathrm{ndiv}} \leq s \leq L$ lumped to position $s = L - \frac{\mathrm{L}}{\mathrm{2ndiv}}$.

Step 6.(2*ndiv+1): Analytical map for $L - \frac{\mathrm{L}}{\mathrm{2ndiv}} \leq s \leq L$ by calling subroutine tsolqu

- Step 7: Quad-like fringe map at the exit, similar as Step 5.

  Source codes:

```
if(mfring .eq. 2 .or. mfring .eq. 3)then
if(f1 .ne. 0.d0 .or. f2 .ne. 0.d0)then
do i=1,np
p=(1.d0+g(i))
a=-f1/p
ea=exp(a)
b= f2/p
pxf=px(i)/ea
pyf=py(i)*ea
z(i)=z(i)-(a*x(i)+b*(1.d0+.5d0*a)*pxf)*px(i) +(a*y(i)+b*(1.d0-.5d0*a)*pyf)*py(i)
x(i)=ea*x(i)+b*px(i)
y(i)=y(i)/ea-b*py(i)
px(i)=pxf
py(i)=pyf
enddo
endif
endif
```

  and the relevant equations:

$$x_2 = x_1 e^{-\frac{f_1}{p/p_0}} + \frac{f_2}{p/p_0} p_{x1}, \tag{275}$$

$$y_2 = y_1 e^{\frac{f_1}{p/p_0}} - \frac{f_2}{p/p_0} p_{y1}, \tag{276}$$

$$p_{x2} = p_{x1} e^{\frac{f_1}{p/p_0}}, \tag{277}$$

$$p_{y2} = p_{y1} e^{-\frac{f_1}{p/p_0}}, \tag{278}$$

$$z_2 = z_1 - \left(-\frac{f_1}{p/p_0} x_1 + \frac{f_2}{p/p_0} \left(1 - \frac{f_1}{2p/p_0} \right) p_{x1} e^{\frac{f_1}{p/p_0}} \right) p_{x1}$$
$$+ \left(-\frac{f_1}{p/p_0} y_1 + \frac{f_2}{p/p_0} \left(1 + \frac{f_1}{2p/p_0} \right) p_{y1} e^{-\frac{f_1}{p/p_0}} \right) p_{y1}. \tag{279}$$

Here f_1 and f_2 are the same as those defined in Step 5.

- Step 7.2: Linear soft-edge focusing at the entrance, similar as Step 5.0

The conditions for activating the linear edge focusing map at the exit are $L > 0$, FRINGE$=$ 2 or 3, FB2$\neq 0$,$K0 \neq 0$ (or $SK0 \neq 0$). The subroutine to be called is *tblfri* defined in source file *tmulti.f* . The equations of motion are almost the same as those defined in Step 5.0, but replacing (K_{0N}, SK_{0N}) and FB1 by $(-K_{0N}, -SK_{0N})$ and FB2, respectively.

First, we define parameters

$$\Delta x_{fx} = -\frac{K_{0N} F_{B1}^2}{24L}, \tag{280}$$

$$\Delta y_{fx} = \frac{K_{0N}^2 F_{B1}}{24L^2}, \tag{281}$$

$$\Delta y_{fax} = \frac{2K_{0N}^2}{3F_{B1} L^2}, \tag{282}$$

$$\Delta y_{fy} = -\frac{SK_{0N} F_{B1}^2}{24L}, \tag{283}$$

$$\Delta x_{fy} = \frac{SK_{0N}^2 F_{B1}}{24L^2}, \tag{284}$$

$$\Delta x_{fay} = \frac{2SK_{0N}^2}{3F_{B1} L^2}, \tag{285}$$

where F_{B1}=FB1, K_{0N} = Re[akr(0)}], and SK_{0N} = Im[akr(0)}]. The equations of motion are (in canonical coordinates):

$$x_2 = x_1 + \frac{\delta_1}{1+\delta_1}\Delta x_{fx}, \tag{286}$$

$$y_2 = y_1 - \frac{\delta_1}{1+\delta_1}\Delta y_{fy}, \tag{287}$$

$$p_{x2} = p_{x1} + \frac{x_2}{1+\delta_1}\left(\Delta x_{fy} - \Delta x_{fay} x_2^2\right), \tag{288}$$

$$p_{y2} = p_{y1} + \frac{y_2}{1+\delta_1}\left(\Delta y_{fx} - \Delta y_{fax} y_2^2\right), \tag{289}$$

$$\begin{aligned} z_2 =& z_1 + \frac{1}{(1+\delta_1)^2}(\Delta x_{fx} p_{x2} - \Delta y_{fy} p_{y2}) + \frac{x_2^2}{(1+\delta_1)^2}\left(\frac{1}{2}\Delta x_{fy} - \frac{1}{4}\Delta x_{fay} x_2^2\right) \\ &+ \frac{y_2^2}{(1+\delta_1)^2}\left(\frac{1}{2}\Delta y_{fx} - \frac{1}{4}\Delta y_{fax} y_2^2\right). \end{aligned} \tag{290}$$

- Step 8: Map for nonlinear Maxwellian fringe field at the exit, similar as Step 4.

  Source codes:

  ```
  if(fringe .and. mfring .ne. 1 .and. al .gt. 0.d0)then
  do n=nmmax,0,-1
  if(dofr(n))then
  call ttfrins(np,x,px,y,py,z,g,n*2+2,-akr(n),al,bzs)
  endif
  enddo
  endif
  ```

- Step 9: Rotation of particle coordinates at the exit, inverse of Step 2.

  If $\theta_2 \neq 0$, then do the following transformations:

  Source codes:

  ```
  x0=x(i)
  x(i)=cost*x0+sint*y(i)
  y(i)=-sint*x0+cost*y(i)
  ```

```
px0=px(i)
px(i)=cost*px0+sint*py(i)
py(i)=-sint*px0+cost*py(i)
```

and the relevant equations:

$$x_2 = x_1 \cos(\theta_2) + y_1 \sin(\theta_2), \tag{291}$$

$$y_2 = -x_1 \sin(\theta_2) + y_1 \cos(\theta_2), \tag{292}$$

$$p_{x2} = p_{x1} \cos(\theta_2) + p_{y1} \sin(\theta_2), \tag{293}$$

$$p_{y2} = -p_{x1} \sin(\theta_2) + p_{y1} \cos(\theta_2). \tag{294}$$

- Step 10: Map for solenoid fringe at the exit, similar as Step 1.

  Source codes:

```
pr=(1.d0+g(i))
px(i)=px(i)-fx/pr
py(i)=py(i)-fy/pr
x(i)=x(i)+dx
y(i)=y(i)+dy
```

  and relevant equations:

$$x_2 = x_1 + dx, \tag{295}$$

$$y_2 = y_1 + dy, \tag{296}$$

$$p_{x2} = p_{x1} - \frac{1}{2p/p_0} B_z dy, \tag{297}$$

$$p_{y2} = p_{y1} + \frac{1}{2p/p_0} B_z dx, \tag{298}$$

  with B_z =bz, dx =dx=DX, and dy =dy=DY. Note that dx =0 for all MULT elements in SuperKEKB lattices.

Remained questions:

1) Since QC* and EC* overlap with each other, why both QC* and EC* have finite lengths?

2) Are the DX, DY, DZ, CHI1, CHI2, and CHI3 defined in SOL elements transferred to succeeding elements?

3) Is the definition of akk correct?

3 Maps for emittance calculation

3.1 General definitions

In emittance calculation, SAD uses canonical coordinates $(x, p_x, y, p_y, z, \delta)$ with $p_x = P_x/P_0$, $p_y = P_y/P_0$, and $\delta = (P-P_0)/P_0$ with $P_0 =$ MOMENTUM= $\gamma\_0m\_0v\_0$.

The closed orbit at the entrance of an element is $(\Delta x, \Delta p_x, \Delta y, \Delta p_y, \Delta z, \Delta\delta)$. The reference particle has $p_0 = \frac{P_0}{m_0 c}$.

3.2 Map for DRIFT

The map is defined in file *tdrife.f*. The parameters of a DRIFT element include L, B_z, K_0, and SK_0. Suppose the 6×6 transfer matrix is M.

Case of $L = 0$: If $L = 0$, the transformation only acts on the closed orbit:

$$\Delta p_{x2} = \Delta p_{x1} - K_0, \qquad \Delta p_{y2} = \Delta p_{y1} - SK_0. \tag{299}$$

Case of $L \neq 0, B_z = 0, K_0 = 0, SK_0 = 0$: The transfer matrix is

$$M = \begin{pmatrix} 1 & m_{12} & 0 & m_{14} & 0 & m_{16} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & m_{32} & 1 & m_{34} & 0 & m_{36} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & m_{52} & 0 & m_{54} & 1 & m_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{300}$$

where

$$m_{12} = \frac{L}{\left[(1+\Delta\delta_1)^2 - \Delta p_{x1}^2 - \Delta p_{y1}^2\right]^{1/2}} + \frac{\Delta p_{x1}^2 L}{\left[(1+\Delta\delta_1)^2 - \Delta p_{x1}^2 - \Delta p_{y1}^2\right]^{3/2}}, \tag{301}$$

$$m_{14} = m_{32} = \frac{\Delta p_{x1}\Delta p_{y1} L}{\left[(1+\Delta\delta_1)^2 - \Delta p_{x1}^2 - \Delta p_{y1}^2\right]^{3/2}}, \tag{302}$$

$$m_{16} = m_{52} = -\frac{\Delta p_{x1}(1+\Delta\delta_1) L}{\left[(1+\Delta\delta_1)^2 - \Delta p_{x1}^2 - \Delta p_{y1}^2\right]^{3/2}}, \tag{303}$$

$$m_{34} = \frac{L}{\left[(1+\Delta\delta_1)^2 - \Delta p_{x1}^2 - \Delta p_{y1}^2\right]^{1/2}} + \frac{\Delta p_{y1}^2 L}{\left[(1+\Delta\delta_1)^2 - \Delta p_{x1}^2 - \Delta p_{y1}^2\right]^{3/2}}, \tag{304}$$

$$m_{36} = m_{54} = -\frac{\Delta p_{y1}(1+\Delta\delta_1)L}{\left[(1+\Delta\delta_1)^2 - \Delta p_{x1}^2 - \Delta p_{y1}^2\right]^{3/2}}, \tag{305}$$

$$m_{56} = \frac{\left(\Delta p_{x1}^2 + \Delta p_{y1}^2\right) L}{\left[(1+\Delta\delta_1)^2 - \Delta p_{x1}^2 - \Delta p_{y1}^2\right]^{3/2}} + \frac{L\sqrt{1+p_0^2}}{\left[1+p_0^2(1+\Delta\delta_1)^2\right]^{3/2}}. \tag{306}$$

If there is no closed orbit distortion at the entrance of the element, e.g. $(\Delta x_1, \Delta p_{x1}, \Delta y_1, \Delta p_{y1}, \Delta z_1, \Delta\delta_1) = (0,0,0,0,0,0)$, the transfer matrix Eq.(300) is reduced to

$$M = \begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{1+p_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{307}$$

Finally, the closed orbit is updated by

$$\Delta x_2 = \Delta x_1 + \frac{\Delta p_{x1} L}{\left[(1+\Delta\delta_1)^2 - \Delta p_{x1}^2 - \Delta p_{y1}^2\right]^{1/2}}, \tag{308}$$

$$\Delta y_2 = \Delta y_1 + \frac{\Delta p_{y1} L}{\left[(1+\Delta\delta_1)^2 - \Delta p_{x1}^2 - \Delta p_{y1}^2\right]^{1/2}}, \tag{309}$$

$$\Delta z_2 = \Delta z_1 - L\left[\frac{1+\Delta\delta_1}{\sqrt{(1+\Delta\delta_1)^2 - \Delta p_x^2 - \Delta p_y^2}} - \frac{(1+\Delta\delta_1)\sqrt{1+p_0^2}}{\sqrt{1+p_0^2(1+\Delta\delta_1)^2}}\right]. \tag{310}$$

3.3 Map for MULT

The map for MULT element is defined in file *tmulte.f*.The input parameters include L, $K_{0:n}$, B_z, ANGLE, FB1, FB2, ...

Map for linear edge focusing at the entrance: The flags DISFRIN and FRINGE are used to switch on or off the fringe effects of a magnet. When FRINGE=1 or 3, and K0≠0 (or SK0≠0), the linear edge focusing map will be activated by calling subroutine *tbfrme*. SAD defines a parameter F_{B1}=FB1 to describe the linear fringe length for the K0 component at the entrance. Assume that $L \neq 0$, and suppose that the closed orbit at the entrance of the MULT element is $(\Delta x_1, \Delta p_{x1}, \Delta y_1, \Delta p_{y1}, \Delta z_1, \Delta\delta_1)$, we first define some parameters:

$$\Delta x_{fx} = \frac{K_0 F_{B1}^2}{24L}, \tag{311}$$

$$\Delta y_{fx} = \frac{K_0^2 F_{B1}}{24L^2}, \tag{312}$$

$$\Delta y_{fax} = \frac{2K_0^2}{3F_{B1}L^2}, \tag{313}$$

$$\Delta y_{fy} = \frac{SK_0 F_{B1}^2}{24L}, \tag{314}$$

$$\Delta x_{fy} = \frac{SK_0^2 F_{B1}}{24L^2}, \tag{315}$$

$$\Delta x_{fay} = \frac{2SK_0^2}{3F_{B1}L^2}. \tag{316}$$

If $\Delta x_{fx} \neq 0$ or $\Delta y_{fy} \neq 0$, the soft edge fringe field is assumed. First, the following transformations are performed on COD and transfer matrix:

$$\Delta p_{x2} = \Delta p_{x1} + \frac{(\Delta x_1 + \frac{\Delta\delta_1}{1+\Delta\delta_1}\Delta x_{fx})}{1+\Delta\delta_1}(\Delta x_{fy} - \Delta x_{fay}(\Delta x_1 + \frac{\Delta\delta_1}{1+\Delta\delta_1}\Delta x_{fx})^2), \tag{317}$$

$$\Delta p_{y2} = \Delta p_{y1} + \frac{(\Delta y_1 - \frac{\Delta\delta_1}{1+\Delta\delta_1}\Delta y_{fy})}{1+\Delta\delta_1}\left[\Delta y_{fx} - \Delta y_{fax}(\Delta x_1 - \frac{\Delta\delta_1}{1+\Delta\delta_1}\Delta y_{fy})^2\right], \tag{318}$$

$$\Delta x_2 = \Delta x_1 + \frac{\Delta\delta_1}{1+\Delta\delta_1}\Delta x_{fx}, \tag{319}$$

$$\Delta y_2 = \Delta y_1 - \frac{\Delta\delta_1}{1+\Delta\delta_1}\Delta y_{fy}, \tag{320}$$

$$\begin{aligned}\Delta z_2 =& \Delta z_1 + \frac{1}{(1+\Delta\delta_1)^2}[\Delta p_{x2}\Delta x_{fx} - \Delta p_{y2}\Delta y_{fy} \\ &+ \Delta x_2^2(\frac{\Delta x_{fy}}{2} - \frac{\Delta x_{fay}}{4}\Delta x_2^2) + \Delta y_2^2(\frac{\Delta y_{fx}}{2} - \frac{\Delta y_{fax}}{4}\Delta y_2^2)],\end{aligned} \tag{321}$$

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & m_{16} \\ m_{21} & 1 & 0 & 0 & 0 & m_{26} \\ 0 & 0 & 1 & 0 & 0 & m_{36} \\ 0 & 0 & m_{43} & 1 & 0 & m_{46} \\ m_{51} & m_{52} & m_{53} & m_{54} & 1 & m_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{322}$$

with

$$m_{16} = \frac{1}{(1+\Delta\delta_1)^2}\Delta x_{fx}, \tag{323}$$

$$m_{21} = \frac{1}{1+\Delta\delta_1}\left(\Delta x_{fy} - 3\Delta x_{fy}\Delta x_2^2\right), \tag{324}$$

$$m_{26} = -\frac{\Delta x_2}{(1+\Delta\delta_1)^2}\left(\Delta x_{fy} - \Delta x_{fay}\Delta x_2^2\right) - \frac{3\Delta x_2^2}{1+\Delta\delta_1}m_{16}, \tag{325}$$

$$m_{36} = -\frac{1}{(1+\Delta\delta_1)^2}\Delta y_{fy}, \tag{326}$$

$$m_{43} = \frac{1}{1+\Delta\delta_1}\left(\Delta y_{fx} - 3\Delta y_{fx}\Delta y_2^2\right), \tag{327}$$

$$m_{46} = -\frac{\Delta y_2}{(1+\Delta\delta_1)^2}\left(\Delta y_{fx} - \Delta y_{fax}\Delta y_2^2\right), \tag{328}$$

$$m_{51} = -m_{26} + m_{21}m_{16}, \tag{329}$$

$$m_{52} = m_{16}, \tag{330}$$

$$m_{53} = -m_{46} + m_{43}m_{36}, \tag{331}$$

$$m_{54} = m_{36}, \tag{332}$$

$$\begin{aligned} m_{56} =& \frac{1}{(1+\Delta\delta_1)^2}[\Delta x_{fx} m_{26} - \Delta y_{fy} m_{46} + (\Delta y_{fx}\Delta y_2 - \Delta y_{fax}\Delta y_2^2)\Delta y_2 m_{36} \\ &+ (\Delta x_{fy}\Delta x_2 - \Delta x_{fay}\Delta x_2^2)\Delta x_2 m_{16}] - \frac{2}{1+\Delta\delta_1}(\Delta z_2 - \Delta z_1). \end{aligned} \tag{333}$$

If FB1=0, the above transformations are not necessarily to be performed. But due to the Maxwellian condition of magnetic field at the edge, there still is fringe effect. The transfer map due to so-called Maxwellian edge effect in hard-edge model is described by

$$M_2 = \begin{pmatrix} m_{11} & 0 & m_{13} & 0 & 0 & m_{16} \\ m_{21} & m_{22} & m_{23} & m_{24} & 0 & m_{26} \\ m_{31} & 0 & m_{33} & 0 & 0 & m_{36} \\ m_{41} & m_{42} & m_{43} & m_{44} & 0 & m_{46} \\ m_{51} & m_{52} & m_{53} & m_{54} & 1 & m_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{334}$$

with

$$m_{11} = 1 + \frac{K0 \times SK0}{(1+\Delta\delta_2)(K0^2 + SK0^2)L}(SK0\Delta x_2 + K0\Delta y_2), \tag{335}$$

$$m_{13} = \frac{K0^2}{(1+\Delta\delta_2)(K0^2 + SK0^2)L}(SK0\Delta x_2 + K0\Delta y_2), \tag{336}$$

$$m_{16} = -\frac{K0}{2(1+\Delta\delta_2)^2(K0^2 + SK0^2)L}(SK0\Delta x_2 + K0\Delta y_2)^2, \tag{337}$$

$$m_{21} = -\frac{SK0^2}{(1+\Delta\delta_2)(K0^2 + SK0^2)L}(K0\Delta p_{x2} - SK0\Delta p_{y2}), \tag{338}$$

$$m_{22} = 1 - \frac{K0 \times SK0}{(1+\Delta\delta_2)(K0^2 + SK0^2)L}(SK0\Delta x_2 + K0\Delta y_2), \tag{339}$$

$$m_{23} = -\frac{K0 \times SK0}{(1+\Delta\delta_2)(K0^2 + SK0^2)L}(K0\Delta p_{x2} - SK0\Delta p_{y2}), \tag{340}$$

$$m_{24} = \frac{SK0^2}{(1+\Delta\delta_2)(K0^2 + SK0^2)L}(SK0\Delta x_2 + K0\Delta y_2), \tag{341}$$

$$m_{26} = \frac{SK0}{(1+\Delta\delta_2)^2(K0^2 + SK0^2)L}(K0\Delta p_{x2} - SK0\Delta p_{y2})(SK0\Delta x_2 + K0\Delta y_2), \tag{342}$$

$$m_{31} = -\frac{SK0^2}{(1+\Delta\delta_2)(K0^2 + SK0^2)L}(SK0\Delta x_2 + K0\Delta y_2), \tag{343}$$

$$m_{33} = 1 - \frac{K0 \times SK0}{(1+\Delta\delta_2)(K0^2 + SK0^2)L}(SK0\Delta x_2 + K0\Delta y_2), \tag{344}$$

$$m_{36} = \frac{SK0}{2(1+\Delta\delta_2)^2(K0^2 + SK0^2)L}(SK0\Delta x_2 + K0\Delta y_2)^2, \tag{345}$$

$$m_{41} = -\frac{K0 \times SK0}{(1+\Delta\delta_2)(K0^2 + SK0^2)L}(K0\Delta p_{x2} - SK0\Delta p_{y2}), \tag{346}$$

$$m_{42} = -\frac{K0^2}{(1+\Delta\delta_2)(K0^2 + SK0^2)L}(SK0\Delta x_2 + K0\Delta y_2), \tag{347}$$

$$m_{43} = -\frac{K0^2}{(1+\Delta\delta_2)(K0^2 + SK0^2)L}(K0\Delta p_{x2} - SK0\Delta p_{y2}), \tag{348}$$

$$m_{44} = 1 + \frac{K0 \times SK0}{(1+\Delta\delta_2)(K0^2 + SK0^2)L}(SK0\Delta x_2 + K0\Delta y_2), \quad (349)$$

$$m_{46} = \frac{K0}{(1+\Delta\delta_2)^2(K0^2 + SK0^2)L}(K0\Delta p_{x2} - SK0\Delta p_{y2})(SK0\Delta x_2 + K0\Delta y_2),$$
(350)

$$m_{51} = -\frac{SK0}{(1+\Delta\delta_2)^2(K0^2 + SK0^2)L}(K0\Delta p_{x2} - SK0\Delta p_{y2})(SK0\Delta x_2 + K0\Delta y_2),$$
(351)

$$m_{52} = -\frac{K0}{2(1+\Delta\delta_2)^2(K0^2 + SK0^2)L}(SK0\Delta x_2 + K0\Delta y_2)^2, \quad (352)$$

$$m_{53} = -\frac{K0}{(1+\Delta\delta_2)^2(K0^2 + SK0^2)L}(K0\Delta p_{x2} - SK0\Delta p_{y2})(SK0\Delta x_2 + K0\Delta y_2),$$
(353)

$$m_{54} = \frac{SK0}{2(1+\Delta\delta_2)^2(K0^2 + SK0^2)L}(SK0\Delta x_2 + K0\Delta y_2)^2, \quad (354)$$

$$m_{56} = \frac{1}{(1+\Delta\delta_2)^3(K0^2 + SK0^2)L}(K0\Delta p_{x2} - SK0\Delta p_{y2})(SK0\Delta x_2 + K0\Delta y_2)^2.$$
(355)

The transfermations on COD are:

$$\Delta x_3 = \Delta x_2 + \frac{K0}{2(1+\Delta\delta_2)(K0^2 + SK0^2)L}(SK0\Delta x_2 + K0\Delta y_2)^2, \quad (356)$$

$$\Delta p_{x3} = \Delta p_{x2} - \frac{SK0}{(1+\Delta\delta_2)(K0^2 + SK0^2)L}(K0\Delta p_{x2} - SK0\Delta p_{y2})(SK0\Delta x_2 + K0\Delta y_2),$$
(357)

$$\Delta y_3 = \Delta y_2 - \frac{SK0}{2(1+\Delta\delta_2)(K0^2+SK0^2)L}(SK0\Delta x_2 + K0\Delta y_2)^2, \quad (358)$$

$$\Delta p_{y3} = \Delta p_{y2} - \frac{K0}{(1+\Delta\delta_2)(K0^2+SK0^2)L}(K0\Delta p_{x2} - SK0\Delta p_{y2})(SK0\Delta x_2 + K0\Delta y_2),$$
(359)

$$\Delta z_3 = \Delta z_2 - \frac{1}{2(1+\Delta\delta_2)^2(K0^2+SK0^2)L}(K0\Delta p_{x2} - SK0\Delta p_{y2})(SK0\Delta x_2 + K0\Delta y_2)^2.$$
(360)

Map for linear edge focusing at the exit: The map for the linear edge focusing at the exit is very similar to that at the etrance. The conditions for activating this map are $L \neq 0$, FRINGE=2 or 3, and K0$\neq$ 0 (or SK0$\neq$ 0). The same subroutine *tbfrme*will be called, but replacing $(K0, SK0)$ and FB1 by $(-K0, -SK0)$ and FB2, respectively. Here FB2 is the linear fringe length at the exit. Another difference from the map at the entrance is that: the order of maps for the soft and hard edge fringe are exchanged. That is, at the exit, the map for hard-edge fringe effect will be performed first.

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