A Calculation of CSR

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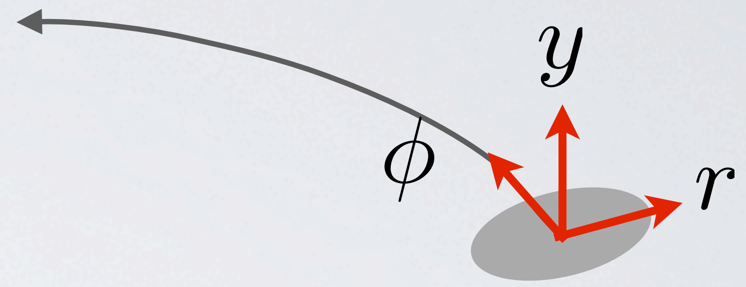
Mini CSR Workshop @ KEK

MAXWELL'S EQUATIONS

$$
\begin{aligned}
\frac{1}{r}\frac{\partial r E_\phi}{\partial r} - \frac{1}{r}\frac{\partial E_r}{\partial \phi} &= -\frac{\partial B_y}{\partial t} \\
\frac{1}{r}\frac{\partial E_y}{\partial \phi} - \frac{\partial E_\phi}{\partial y} &= -\frac{\partial B_r}{\partial t} \\
\frac{\partial E_r}{\partial y} - \frac{\partial E_y}{\partial r} &= -\frac{\partial B_\phi}{\partial t} \\
\frac{1}{r}\frac{\partial r B_\phi}{\partial r} - \frac{1}{r}\frac{\partial B_r}{\partial \phi} &= \mu_0 j_y + \frac{1}{c^2}\frac{\partial E_y}{\partial t} \\
\frac{1}{r}\frac{\partial B_y}{\partial \phi} - \frac{\partial B_\phi}{\partial y} &= \mu_0 j_r + \frac{1}{c^2}\frac{\partial E_r}{\partial t} \\
\frac{\partial B_r}{\partial y} - \frac{\partial B_y}{\partial r} &= \mu_0 j_\phi + \frac{1}{c^2}\frac{\partial E_\phi}{\partial t} \\
\frac{1}{r}\frac{\partial r E_r}{\partial r} + \frac{1}{r}\frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_y}{\partial y} &= \frac{\rho}{\varepsilon_0}
\end{aligned}
$$



$$j_r = j_y = 0, \qquad j_\phi = \rho c$$

$$
\begin{aligned}
\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial r E_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_r}{\partial \phi^2} + \frac{\partial^2 E_r}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 E_r}{\partial t^2} - \frac{2}{r^2}\frac{\partial E_\phi}{\partial \phi} &= \frac{1}{\varepsilon_0}\frac{\partial \rho}{\partial r} \\
\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial r E_\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_\phi}{\partial \phi^2} + \frac{\partial^2 E_\phi}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 E_\phi}{\partial t^2} + \frac{2}{r^2}\frac{\partial E_r}{\partial \phi} &= \frac{1}{\varepsilon_0}\left(\frac{1}{r}\frac{\partial \rho}{\partial \phi} + \frac{1}{c}\frac{\partial \rho}{\partial t}\right)
\end{aligned}
$$

MAXWELL'S EQUATIONS

$$\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial r E_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_r}{\partial \phi^2} + \frac{\partial^2 E_r}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 E_r}{\partial t^2} - \frac{2}{r^2}\frac{\partial E_\phi}{\partial \phi} = \frac{1}{\varepsilon_0}\frac{\partial \rho}{\partial r}$$

$$\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial r E_\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_\phi}{\partial \phi^2} + \frac{\partial^2 E_\phi}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 E_\phi}{\partial t^2} + \frac{2}{r^2}\frac{\partial E_r}{\partial \phi} = \frac{1}{\varepsilon_0}\left(\frac{1}{r}\frac{\partial \rho}{\partial \phi} + \frac{1}{c}\frac{\partial \rho}{\partial t}\right)$$

$$\rho \propto \delta(r-R)\delta(y)\exp\left(ik(R\phi - ct)\right)$$

$$E_{r,\phi} = \left(i\overline{E}_r(\phi), \overline{E}_\phi(\phi)\right)\exp\left(ik(R\phi - ct)\right)$$

$$\overline{E}_r = \overline{E}_r + \overline{E}_{r0} \ ,$$

$$\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial r\overline{E}_{r0}}{\partial r} + \frac{\partial^2 \overline{E}_{r0}}{\partial y^2} = \frac{1}{\varepsilon_0}\frac{\partial \rho}{\partial r}$$

★ Ignore $\frac{\partial^2 \overline{E}}{\partial \phi^2}$ terms (Stupakov-Agoh-Yokoya)

MAXWELL'S EQUATIONS

Then we obtain the first order differential equations for $\overline{E}_{r,\phi}$.

$$
\begin{aligned}
\frac{\partial \overline{E}_r}{\partial \phi} &= \frac{i}{2(k^2R^2-1)}\left[kR\left(\left(k^2(r^2-R^2)+1\right)(\overline{E}_r+\overline{E}_{r0})+r\frac{\partial}{\partial r}(\overline{E}_r+\overline{E}_{r0})+r^2\left(\frac{\partial^2\overline{E}_r}{\partial r^2}+\frac{\partial^2\overline{E}_r}{\partial y^2}\right)\right)\right. \\
&\quad \left. +\left(k^2(r^2+R^2)-1\right)\overline{E}_\phi+r\frac{\partial\overline{E}_\phi}{\partial r}+r^2\left(\frac{\partial^2\overline{E}_\phi}{\partial r^2}+\frac{\partial^2\overline{E}_\phi}{\partial y^2}\right)\right] \\
\frac{\partial \overline{E}_\phi}{\partial \phi} &= \frac{i}{2(k^2R^2-1)}\left[kR\left(\left(k^2(r^2-R^2)+1\right)\overline{E}_\phi+r\frac{\partial\overline{E}_\phi}{\partial r}+r^2\left(\frac{\partial^2\overline{E}_\phi}{\partial r^2}+\frac{\partial^2\overline{E}_\phi}{\partial y^2}\right)\right)\right. \\
&\quad \left. +\left(k^2(r^2+R^2)-1\right)(\overline{E}_r+\overline{E}_{r0})+r\frac{\partial}{\partial r}(\overline{E}_r+\overline{E}_{r0})+r^2\left(\frac{\partial^2\overline{E}_r}{\partial r^2}+\frac{\partial^2\overline{E}_r}{\partial y^2}\right)\right]
\end{aligned}
$$

SOLVER

$$\frac{d\boldsymbol{f}}{d\phi} = A\boldsymbol{f} + \boldsymbol{b}\ , \qquad \boldsymbol{f} = (\overline{E}_r, \overline{E}_\phi)\ ,$$

$$\boldsymbol{f}(\phi) = \boldsymbol{f}_0 \exp(A\phi) + \boldsymbol{b} \int_0^\phi \exp\left(A(\phi' - \phi)\right) d\phi'$$

A: Spatial differentiation matrix with boundary conditions

$\boldsymbol{b}$: Driving source term by $\overline{E}_{r0}$.

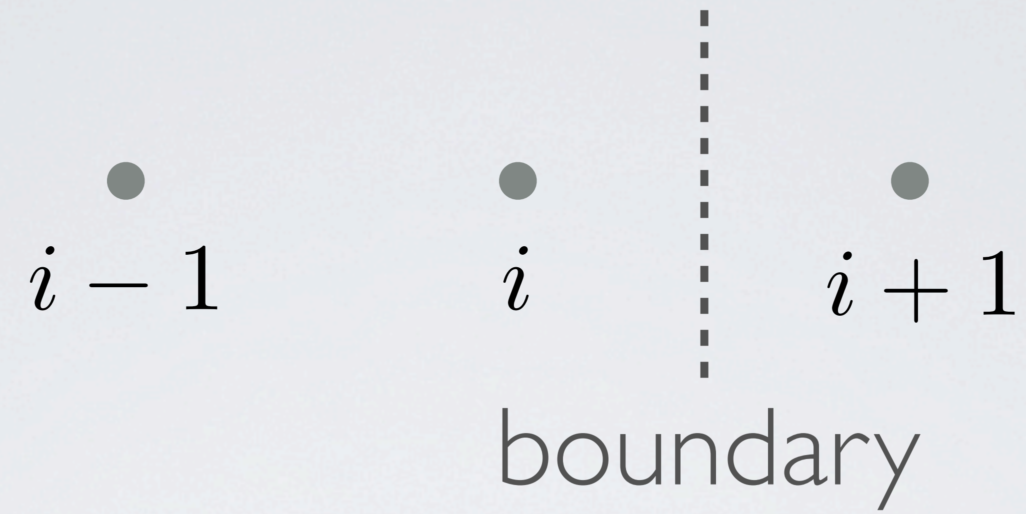
The exponent is evaluated by the eigen system of A .

The cross section of the beam pipe must be uniform along ϕ .

The mesh size for A is varied with k under the condition:

$$(\Delta x, \Delta y) = \frac{(R/k^2)^{1/3}}{(M_x, M_y)}\ , \qquad M_{x,y} \gtrsim (4, 1)$$

Implementation of the boundary condition

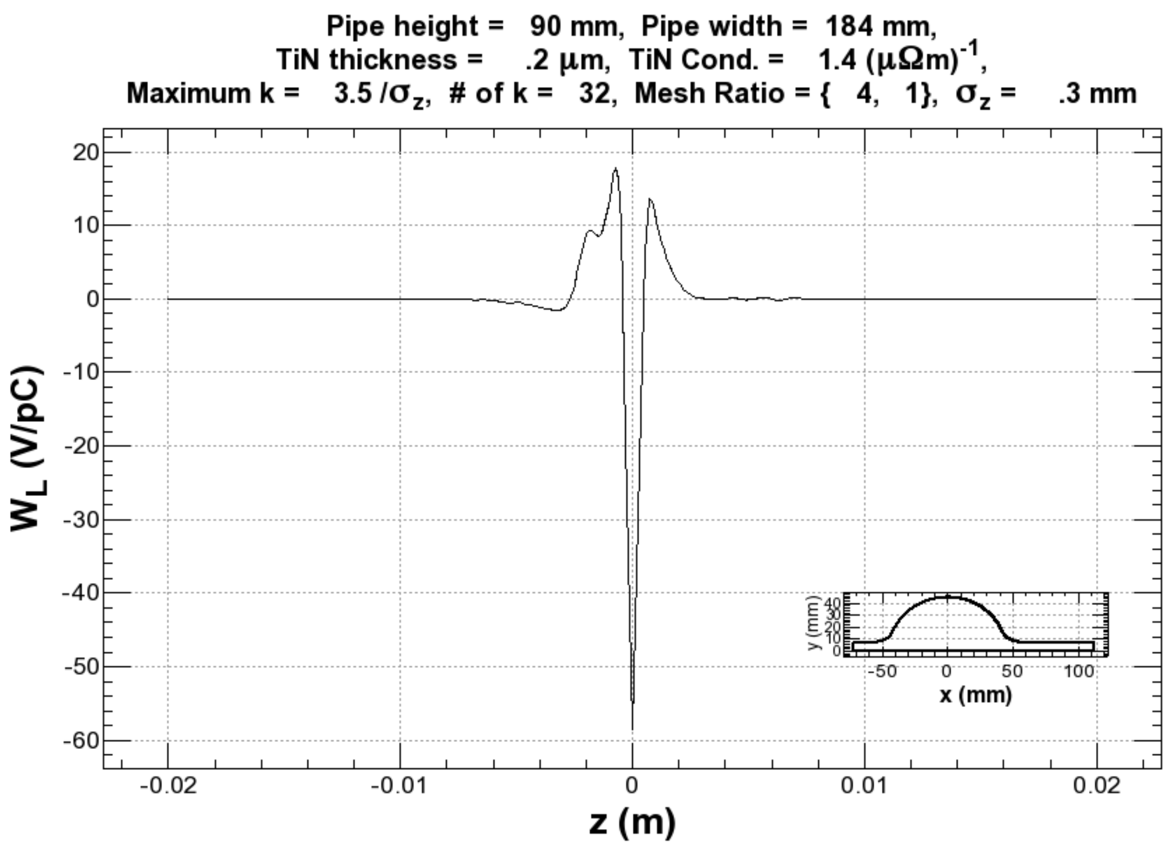


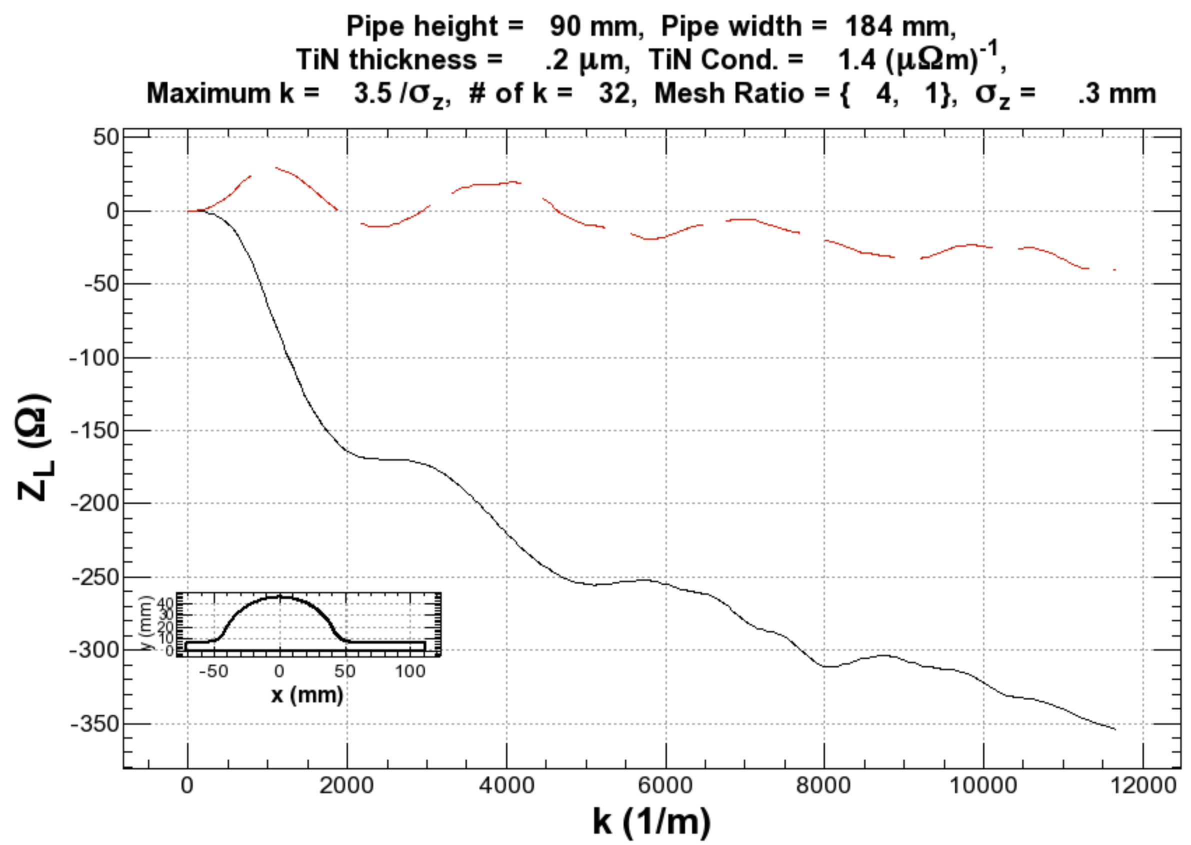
$$f_{\text{boundary}} = 0: \qquad f_{i+1} = -f_{i-1}, \qquad f_i'' = \frac{f_{i-1} - 3f_i}{2}$$

$$f'_{\text{boundary}} = 0: \qquad f_{i+1} = f_i, \qquad f_i'' = \frac{f_{i-1} - f_i}{2}$$

- Are these right choice?

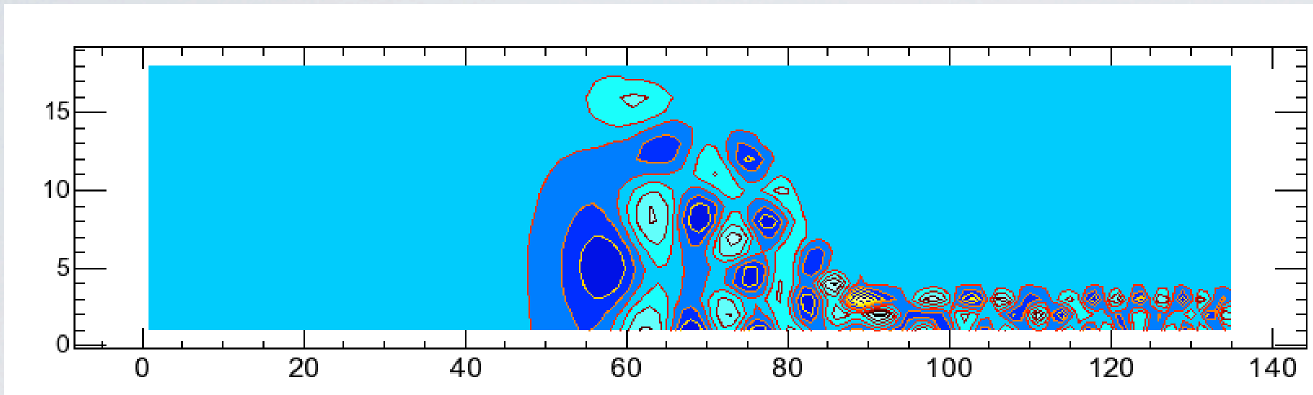
Results for KEKB antechamber



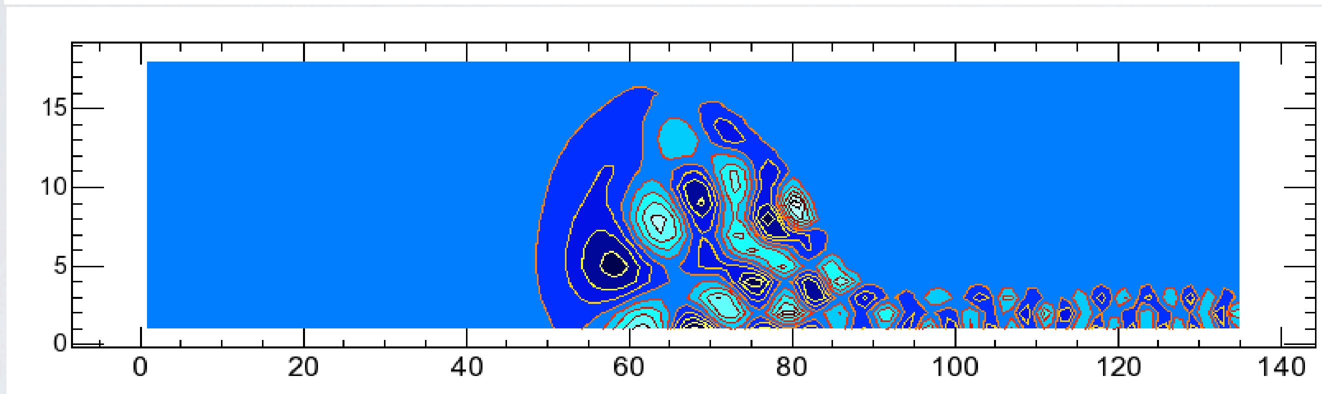


ρ = 16.3 m, L = 0.89 m, ∞ drift, 11/7/2010

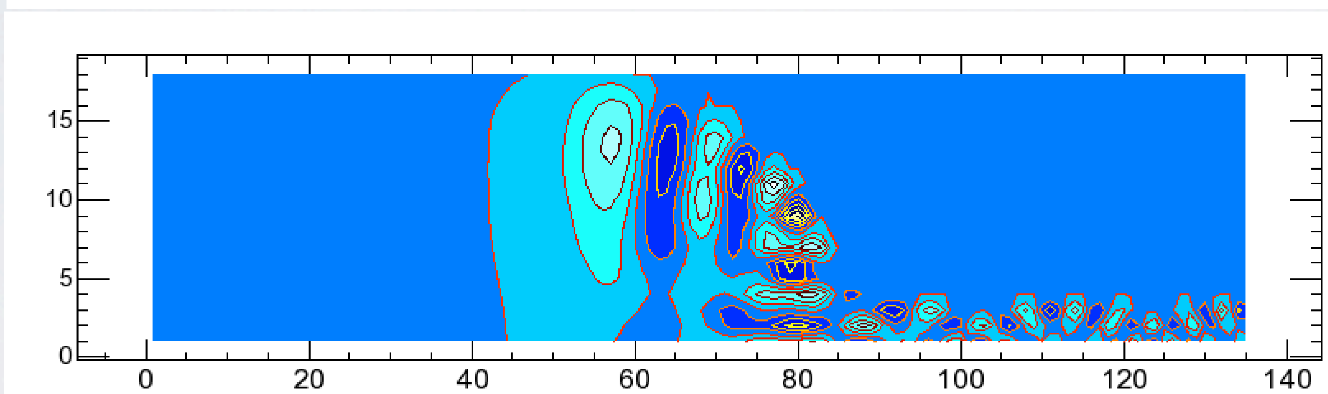
Some eigen modes



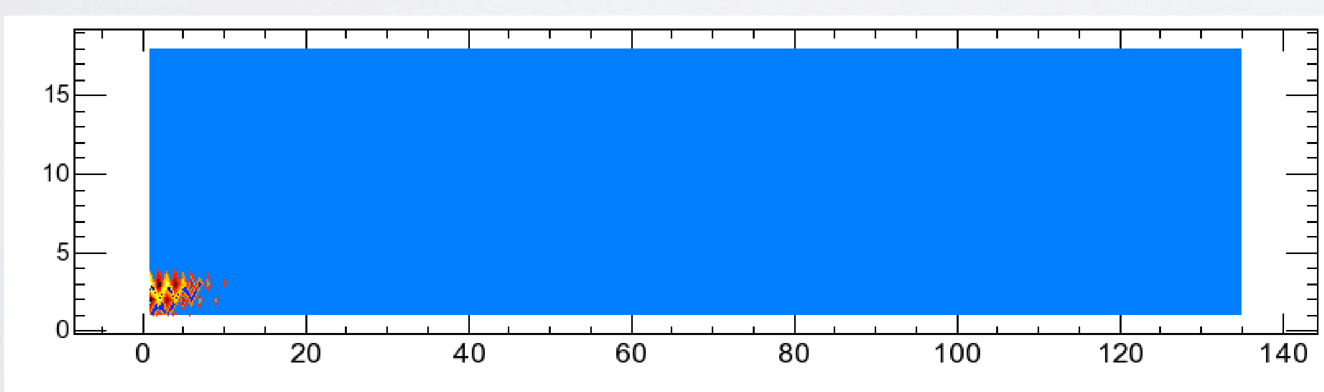
w = 187 /mm



w = 521 /mm



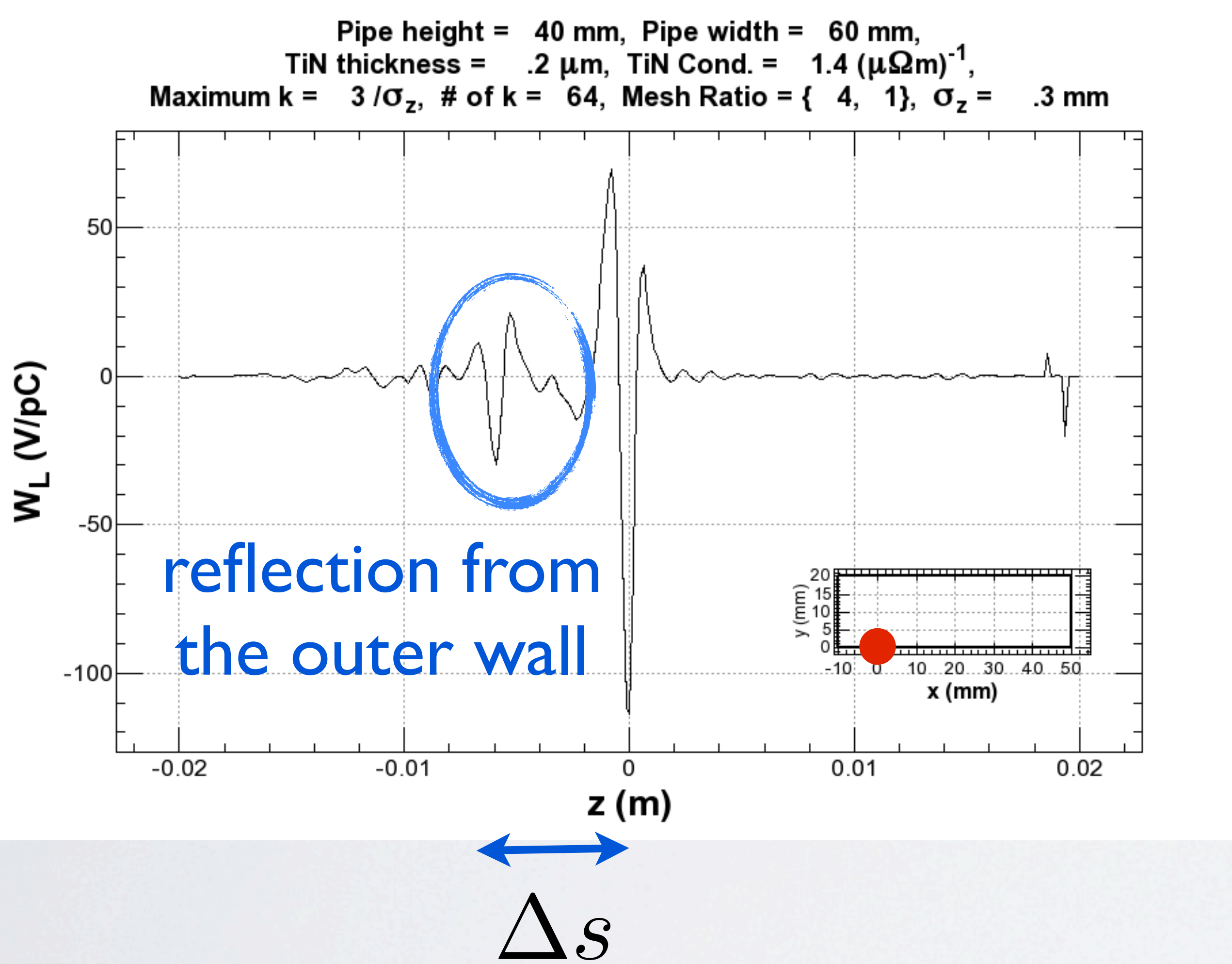
w = 595 /mm

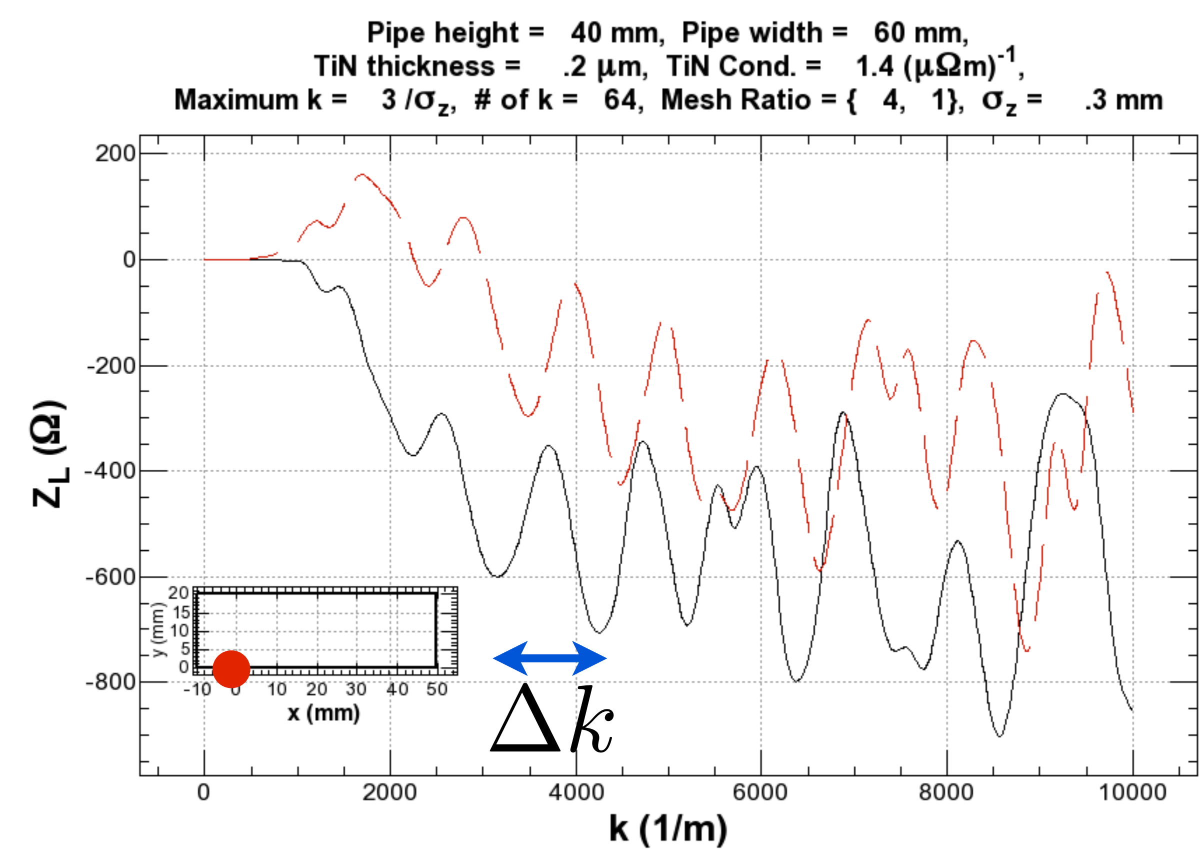


w = 940 /µm

ρ = 16.3 m, k = 10 /mm, 11/7/2010

Results with an asymmetric pipe

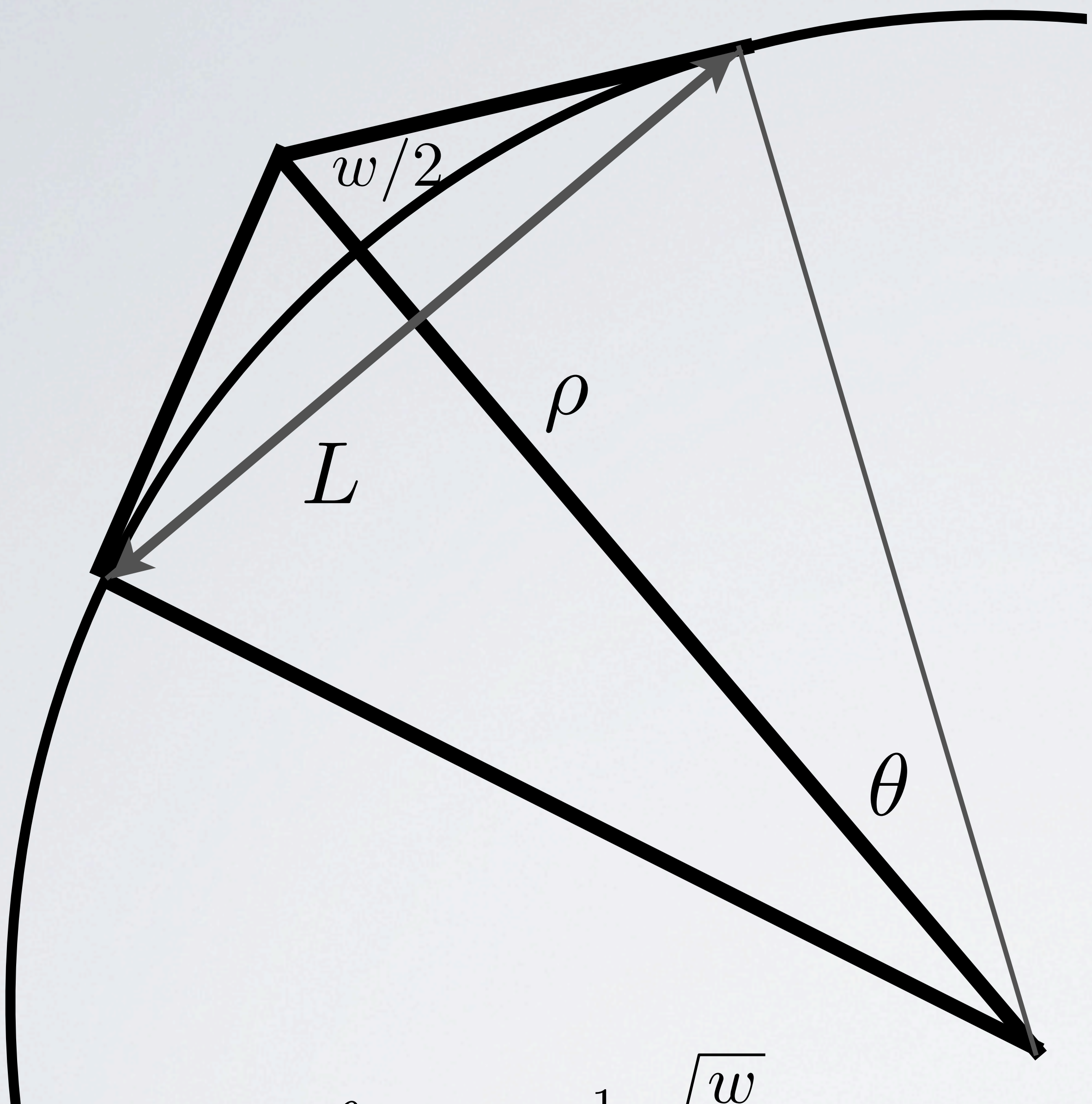




- Δs and Δk agree with the path difference (next page) between the reflection.
- Also the modulation ratio of Z roughly agrees with the ratio of interference lengths:

$$L/(2L_0 - L) = 2.55/(82 \times 4 - 2.55) = 0.47$$

ρ = 16.3 m, L = 4 m, ∞ drift, 11/7/2010



$$\rho = 16.3 \text{ m}, w = 100 \text{ mm}:$$

$$\downarrow$$

$$L = 2.55 \text{ m},$$

$$\Delta s = 5.2 \text{ mm},$$

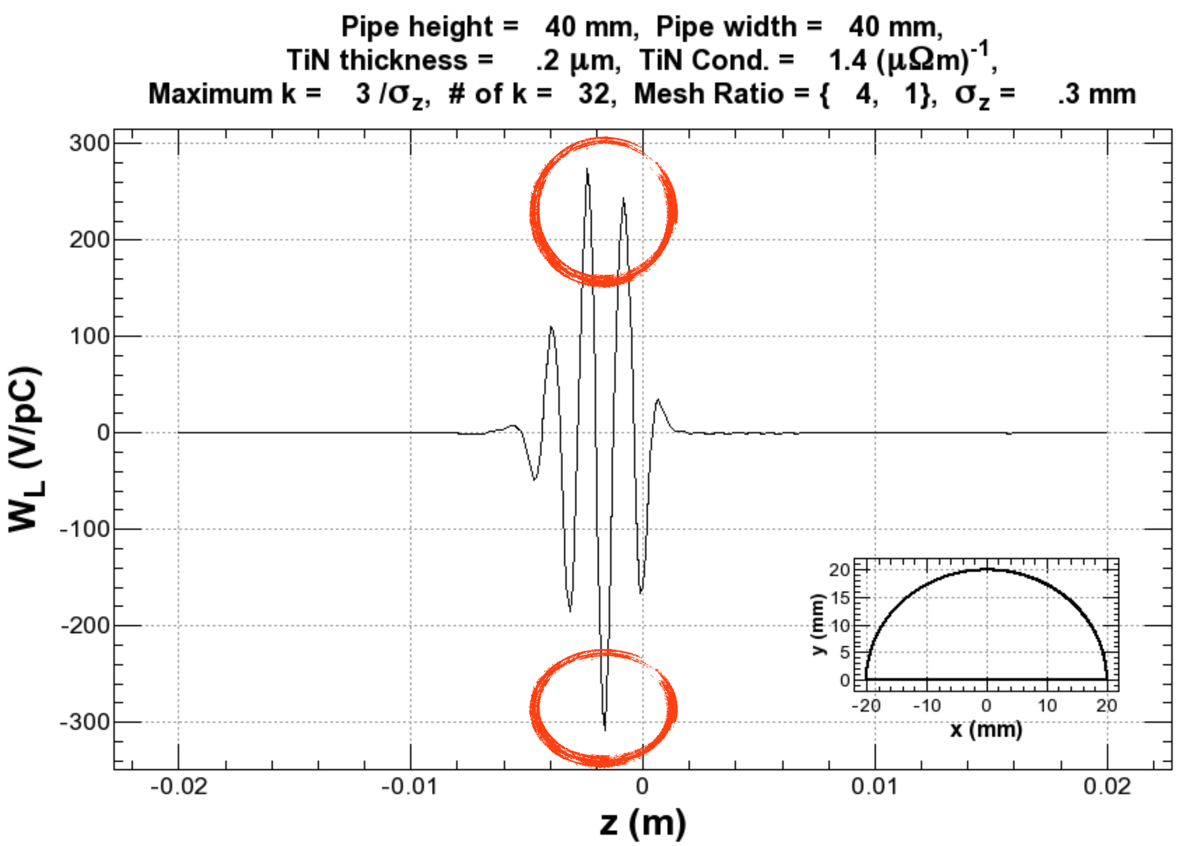
$$\Delta k = \frac{2\pi}{\Delta s} = 1210 \ m^{-1}$$

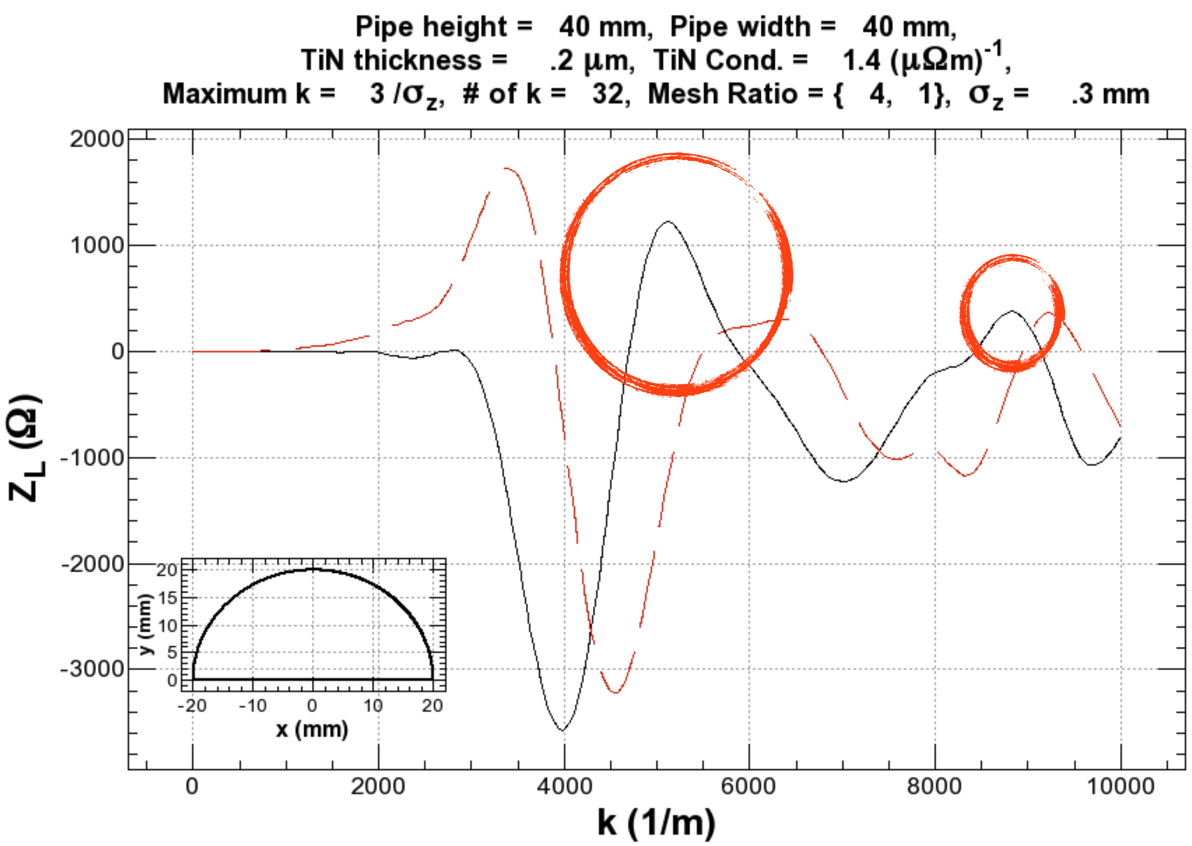
$$\theta \approx \tan^{-1}\sqrt{\frac{w}{\rho}}$$

$$L = 2\rho\theta \approx 2\sqrt{\rho w}$$

$$\Delta s = 2\rho(\tan\theta - \theta) \approx 2\rho\frac{\theta^3}{3} \approx \frac{2w^{3/2}}{3\rho^{1/2}}$$

Unphysical results with a round pipe





- Converges to an unphysical result, even for M ≥ 128.
- The reason has not been identified.

ρ = 16.3 m, L = 4 m, ∞ drift, 11/7/2010

Discussions

- The eigen mode method may have some merits:
  - Capability to handle arbitral shape of the beam pipe.
  - Saving computation for a repetitive arrangement.
- But it has demerits:
  - Heavy computation, if finer mesh is necessary.
  - Not suitable for varying cross section.