Tracking simulations for cERL-FEL: Detailed comparison of SAD and GPT with theoretical explanations

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Outline

- **Simulation of CSR, LSC and TSC effects using SAD and GPT**
  - * Use magnet layout Daihon20200607
  - * Use initial beam of ideal 3D Gaussian distribution
- **Scaling laws of momentum spread increase due to SC**
- **Summary**

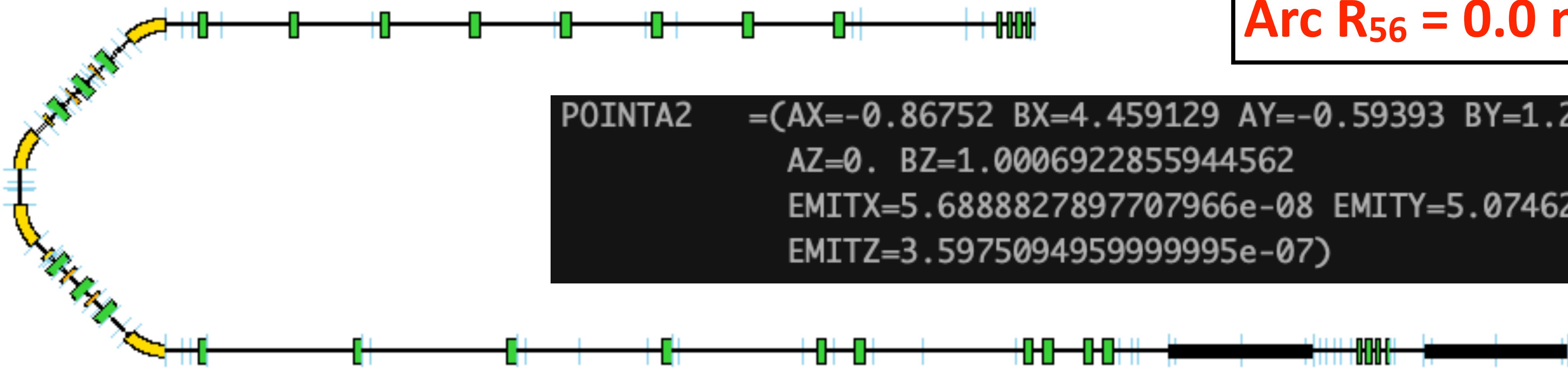
1. Introduction

Layout:

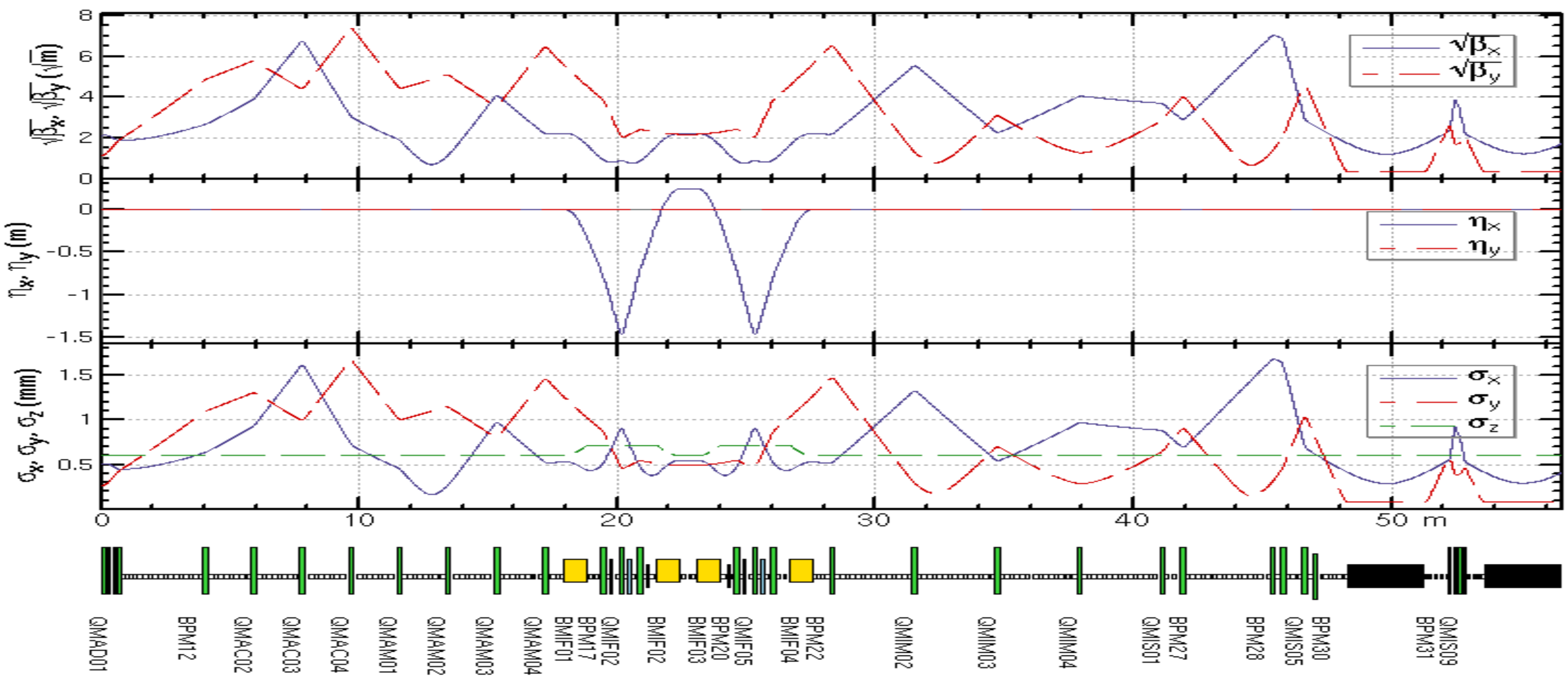
Start from POINTA2 (50 mm before POINTA)

Beam energy 17.5 MeV

Arc R_{56} = 0.0 m



```
POINTA2   =(AX=-0.86752 BX=4.459129 AY=-0.59393 BY=1.273629
            AZ=0. BZ=1.0006922855944562
            EMITX=5.6888827897707966e-08 EMITY=5.0746293858979176e-08
            EMITZ=3.5975094959999995e-07)
```



2. Tracking simulations

On-axis and average LSC models (Ref: https://arxiv.org/abs/2101.04369)

*** On-axis LSC model**

$$Z_{\parallel}(k) = \frac{iZ_0 k}{4\pi\beta^2\gamma^2} e^{\xi_\sigma^2/2} \Gamma\left(0, \xi_\sigma^2/2\right) \qquad \xi_\sigma = \frac{k\sigma}{\beta\gamma} \qquad \sigma = \frac{1}{2}(\sigma_x + \sigma_y)$$

*** Average LSC model (Average over the transverse beam density)**

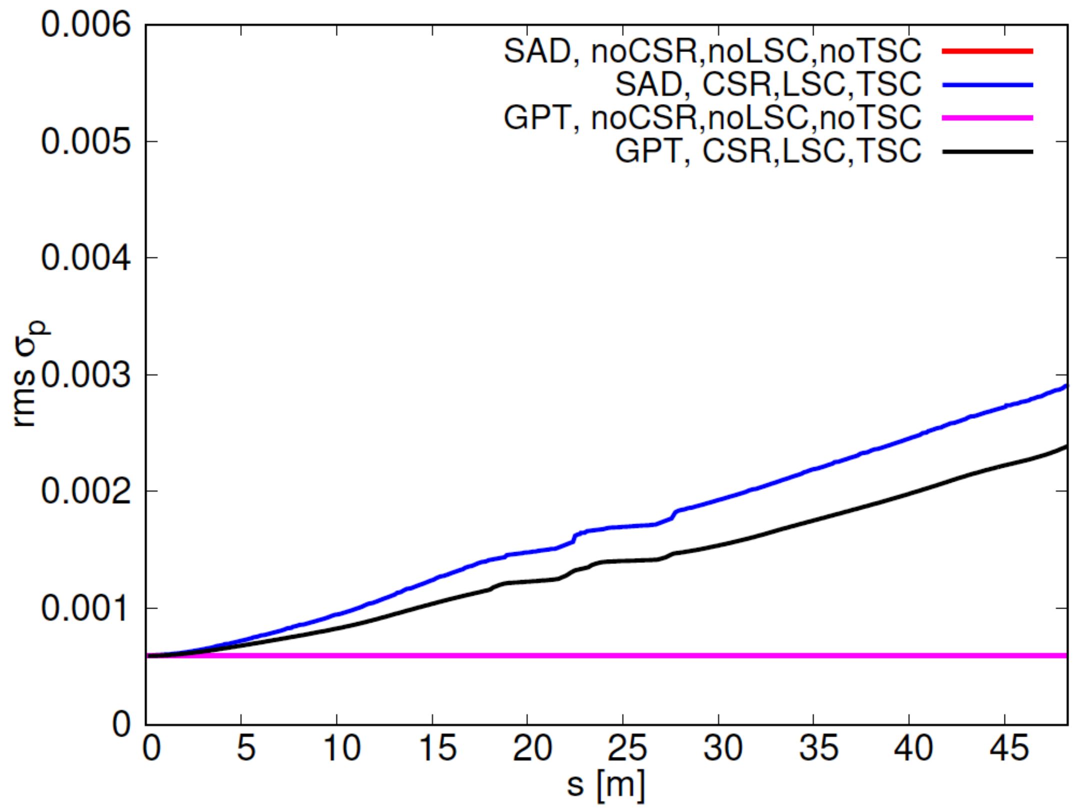
$$Z_{\parallel}(k) = \frac{iZ_0 k}{4\pi\beta^2\gamma^2} e^{\xi_\sigma^2} \Gamma\left(0, \xi_\sigma^2\right)$$

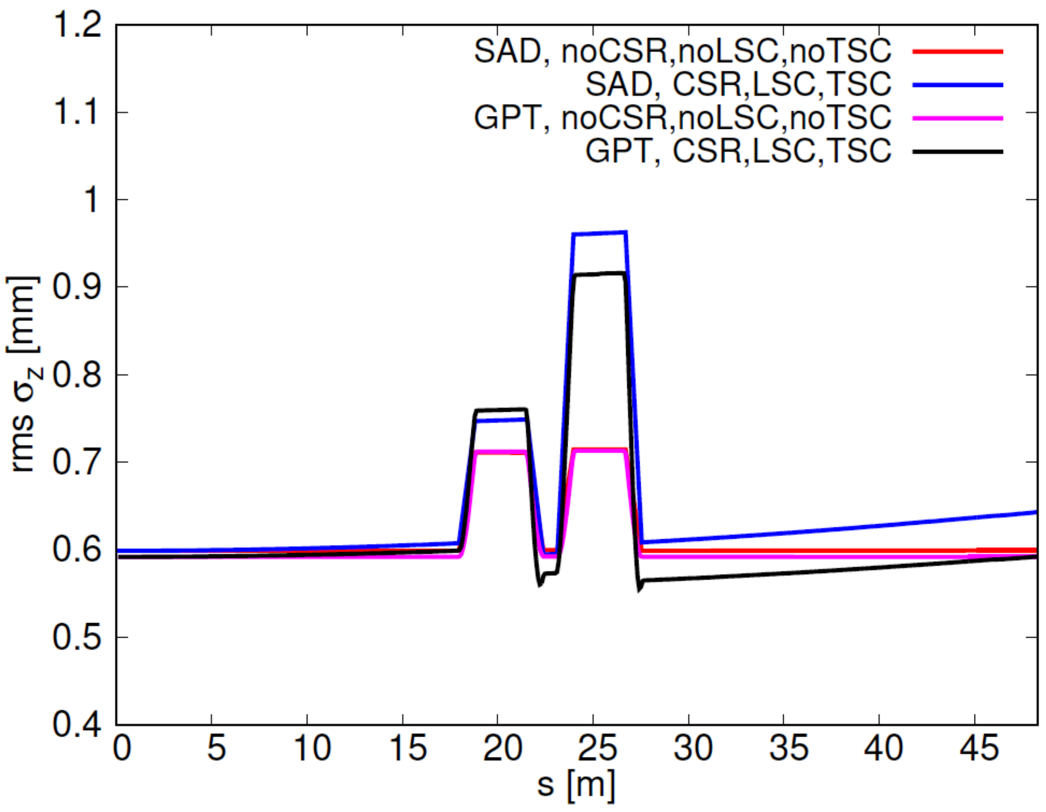
2. Tracking simulations

rms energy spread and bunch length with on-axis LSC model

*** Clear discrepancy in energy spread => 3D LSC model is necessary. Currently SAD simulation use 1D LSC model (same as that used in ELEGANT)**

*** Clear discrepancy in bunch length => Result of difference in energy spread (energy spread translated to bunch length through the arc)**



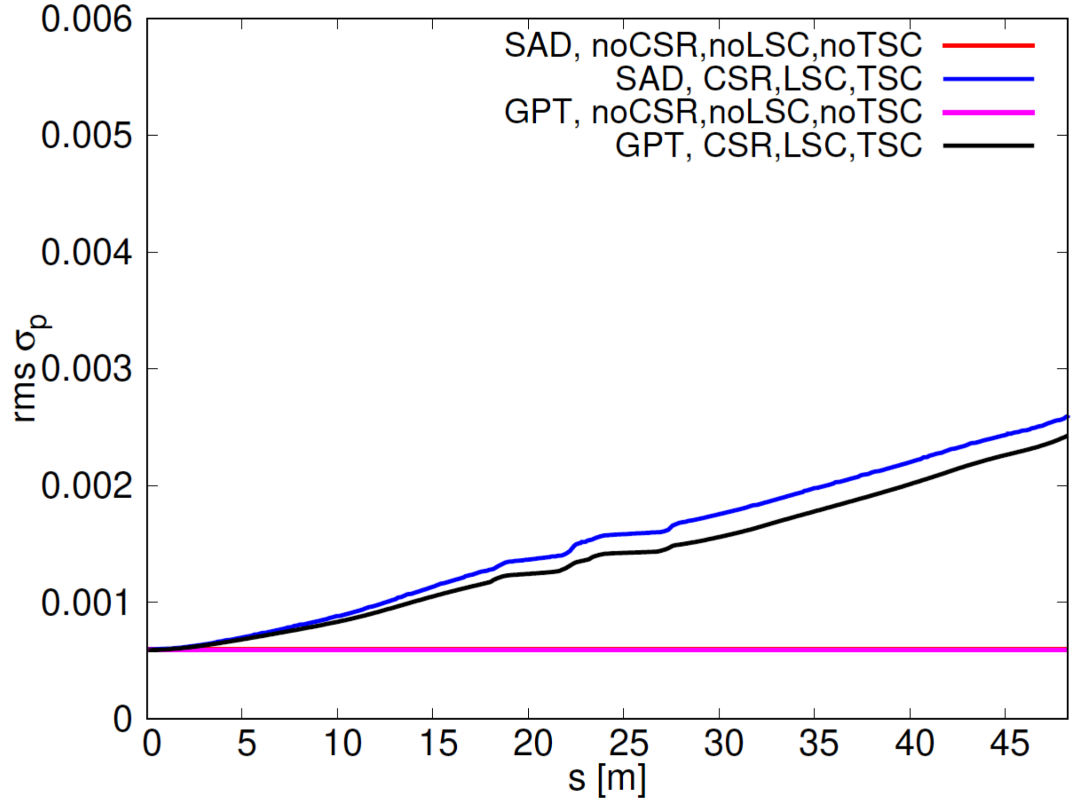


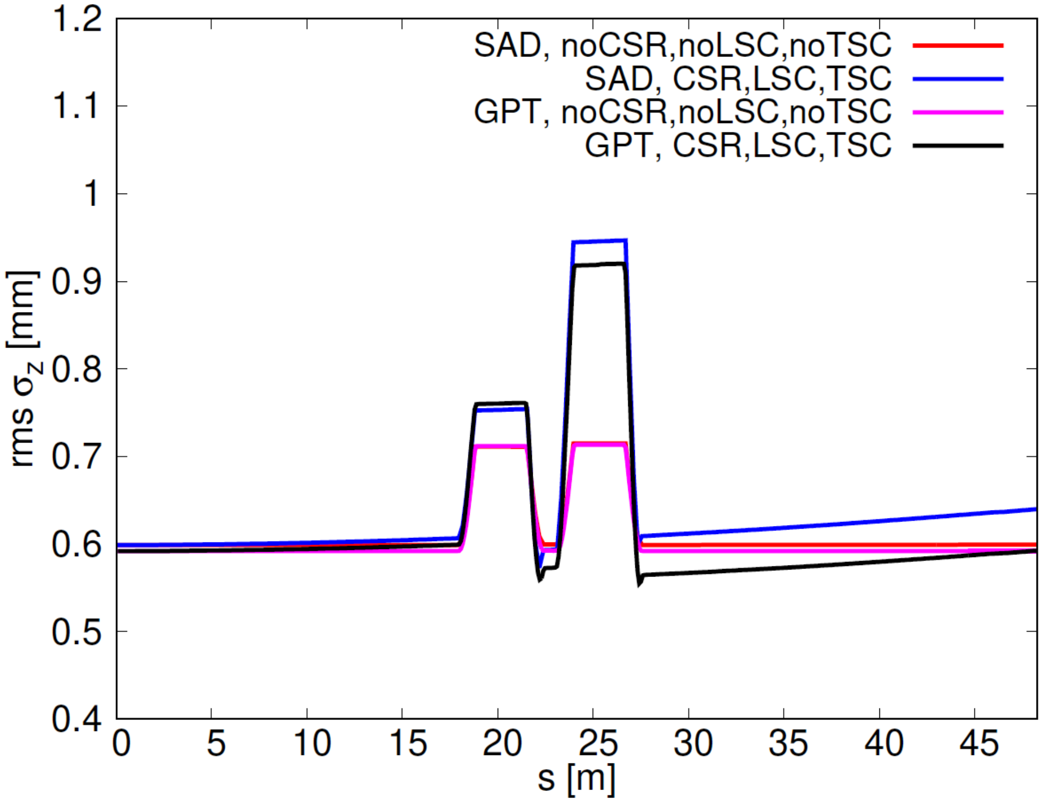
2. Tracking simulations

rms energy spread and bunch length with average LSC model

*** Smaller discrepancy in energy spread => Average model is better. Remaining discrepancy might be due to 3D effects**

*** Discrepancy in bunch length => Related to energy spread and CSR (CSR models are different between SAD and GPT)?**

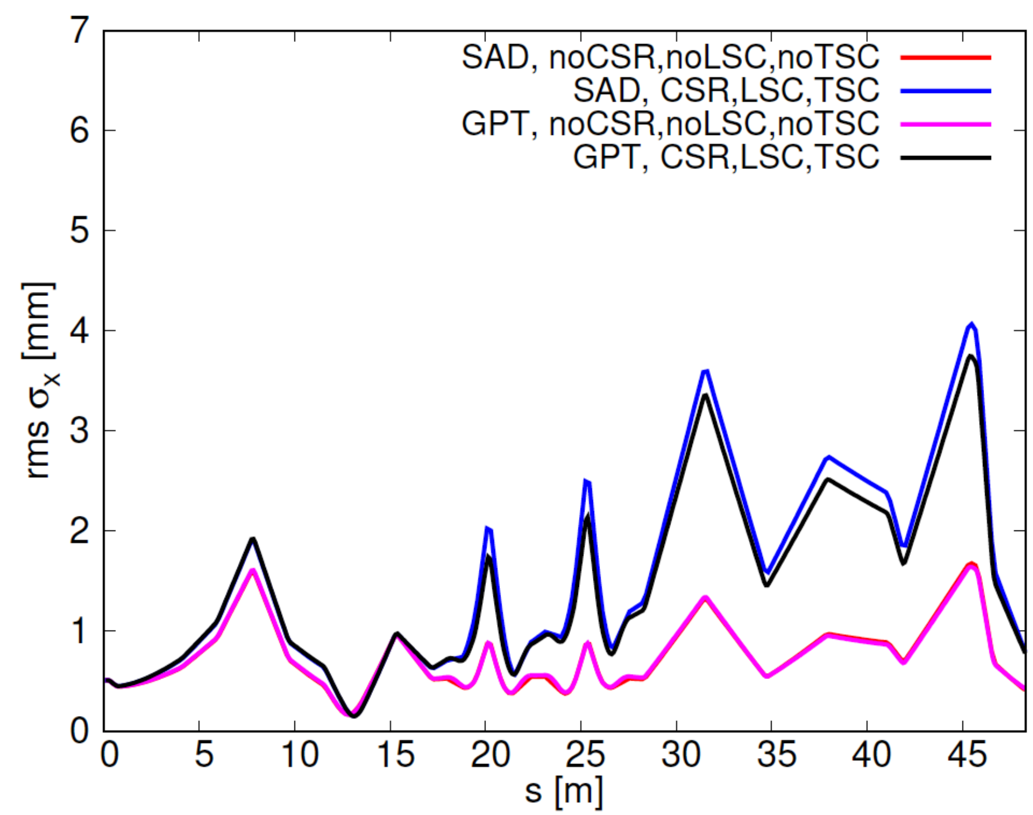


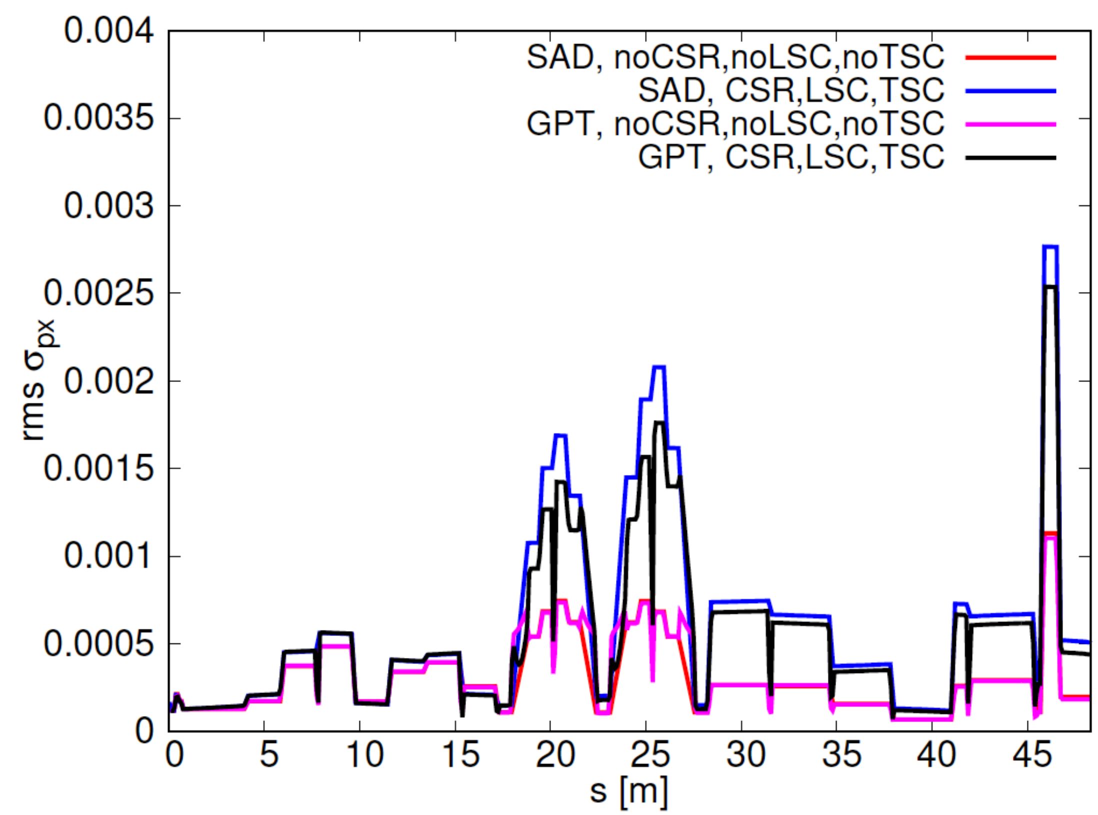


2. Tracking simulations

rms horizontal beam size and momentum spread with on-axis LSC model

* Good agreement in general
* After the arc, difference appear to be remarkable. Can be correlated with the difference in energy spread





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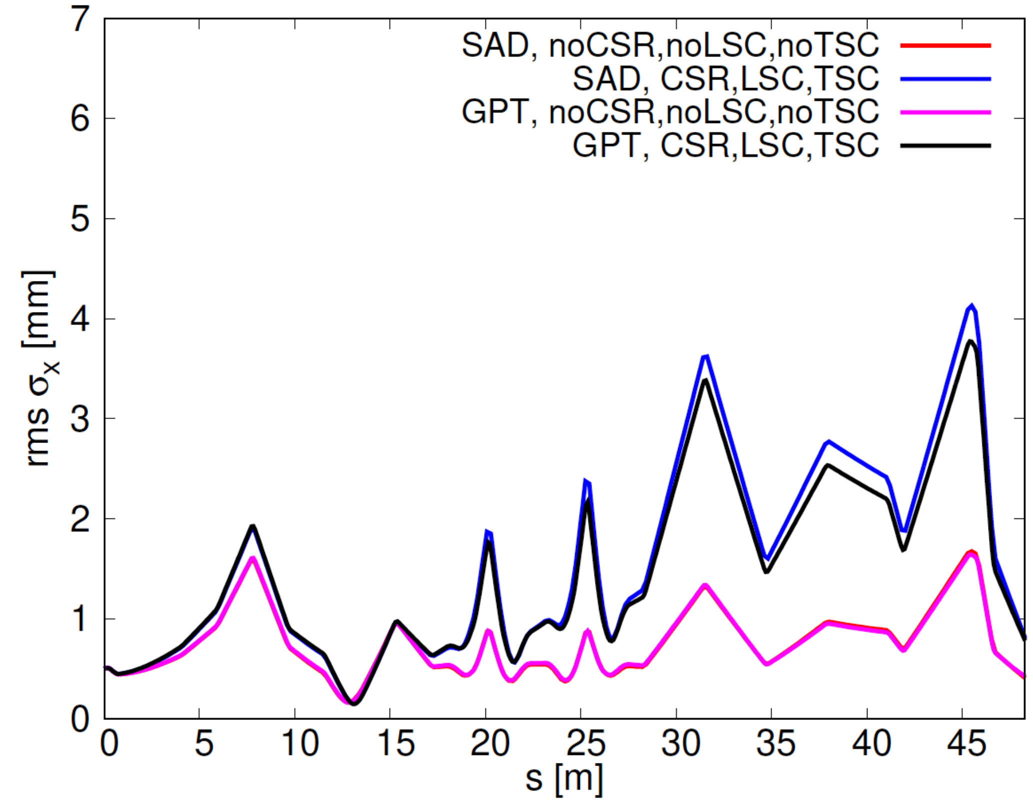
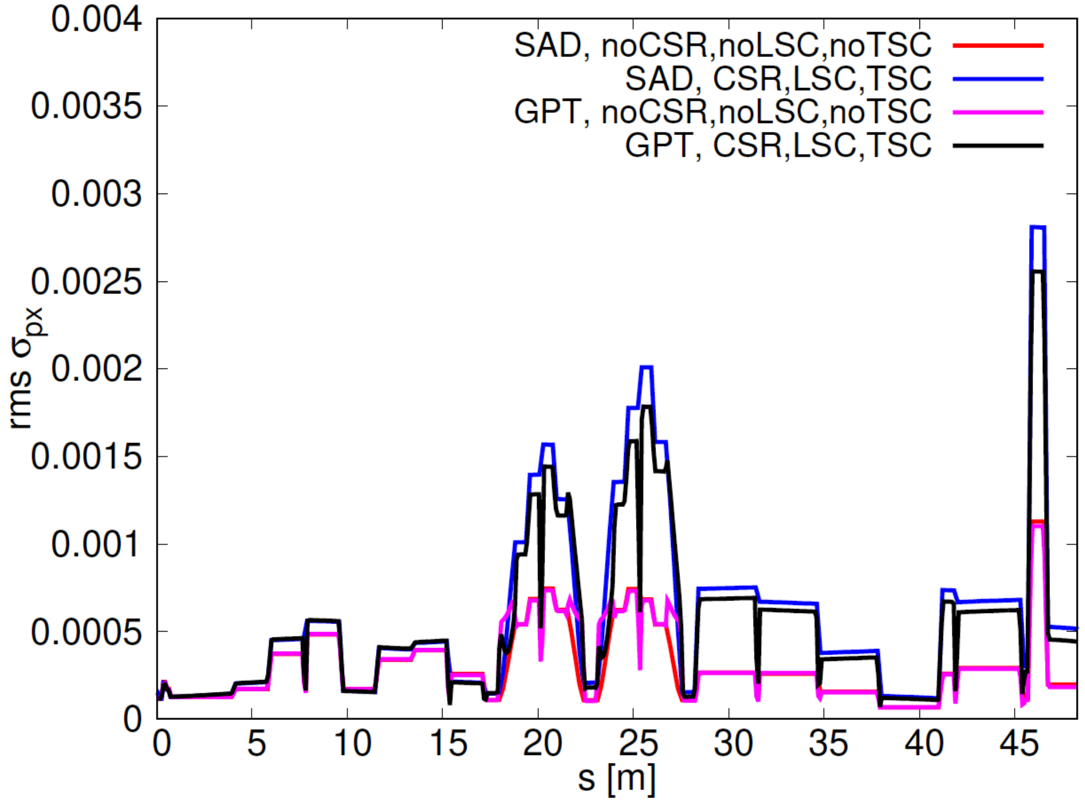


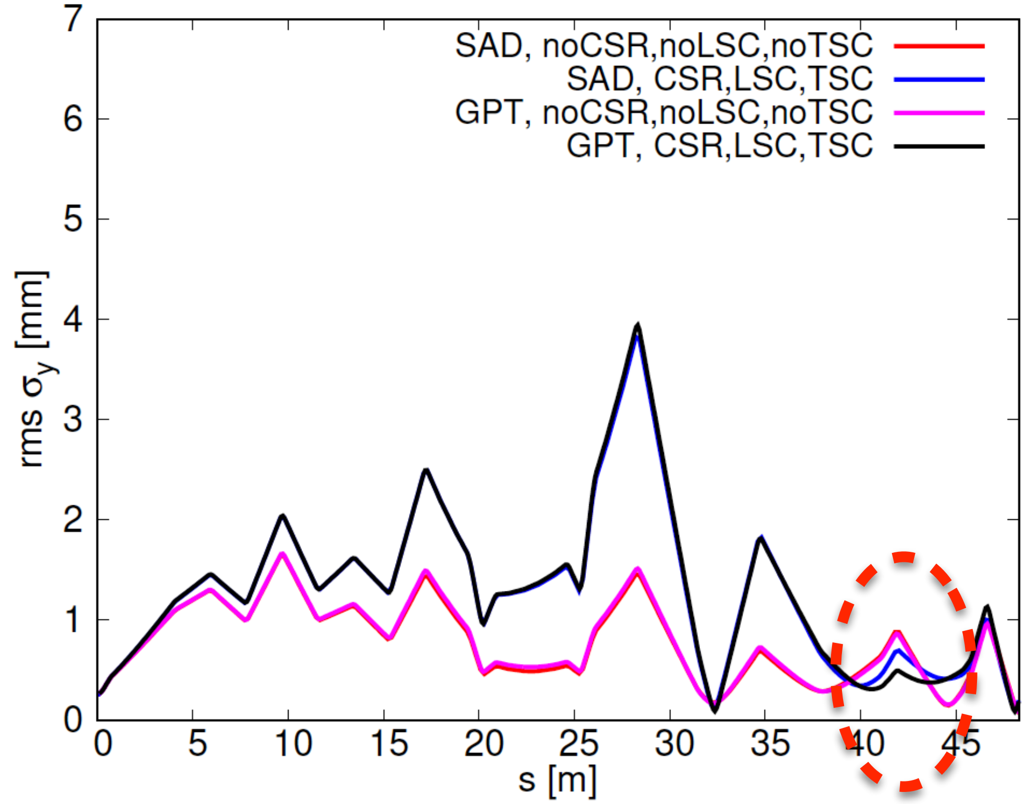


2. Tracking simulations

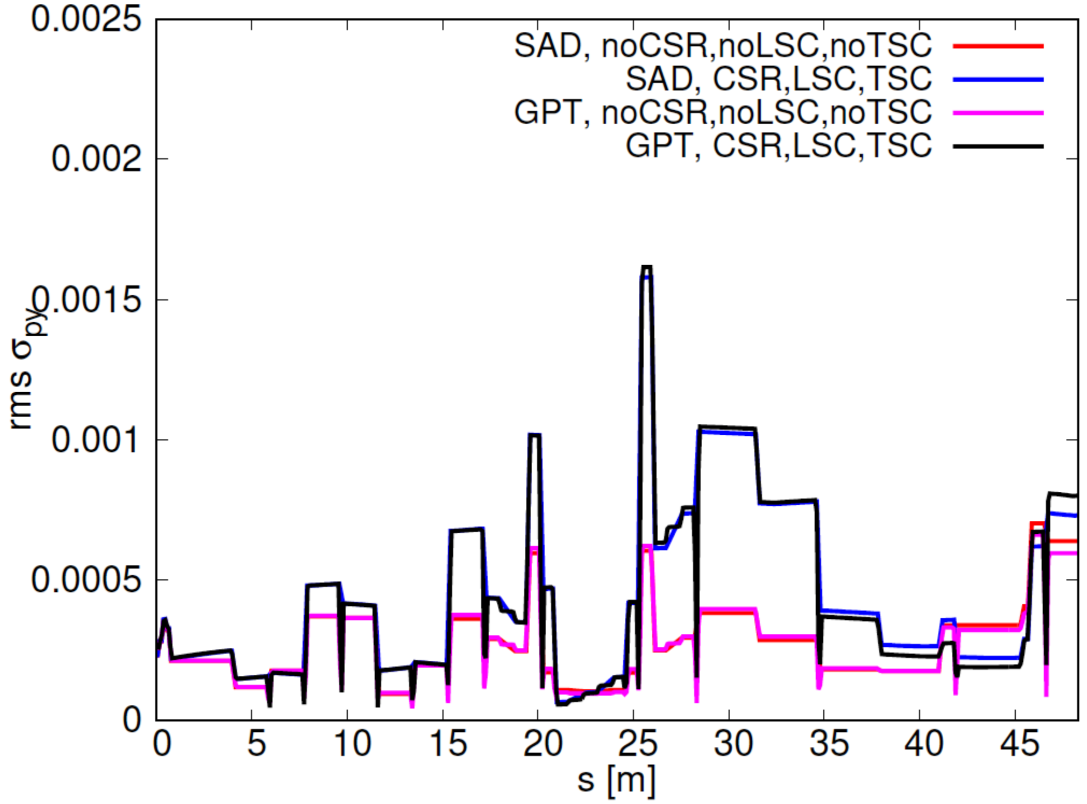
rms vertical beam size and momentum spread with on-axis LSC model

*** Good agreement in general**

*** The arc does no make difference since it does not create coupling in Y and Z directions.**



A little strange!



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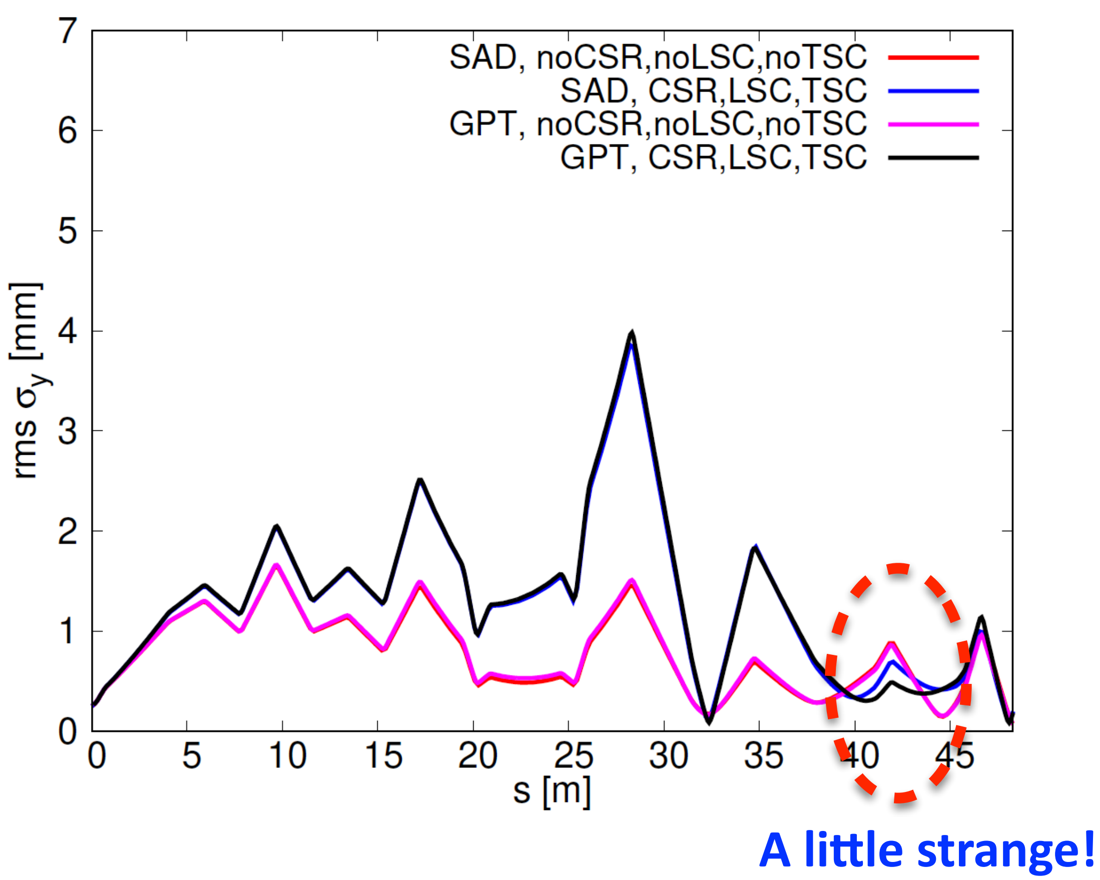
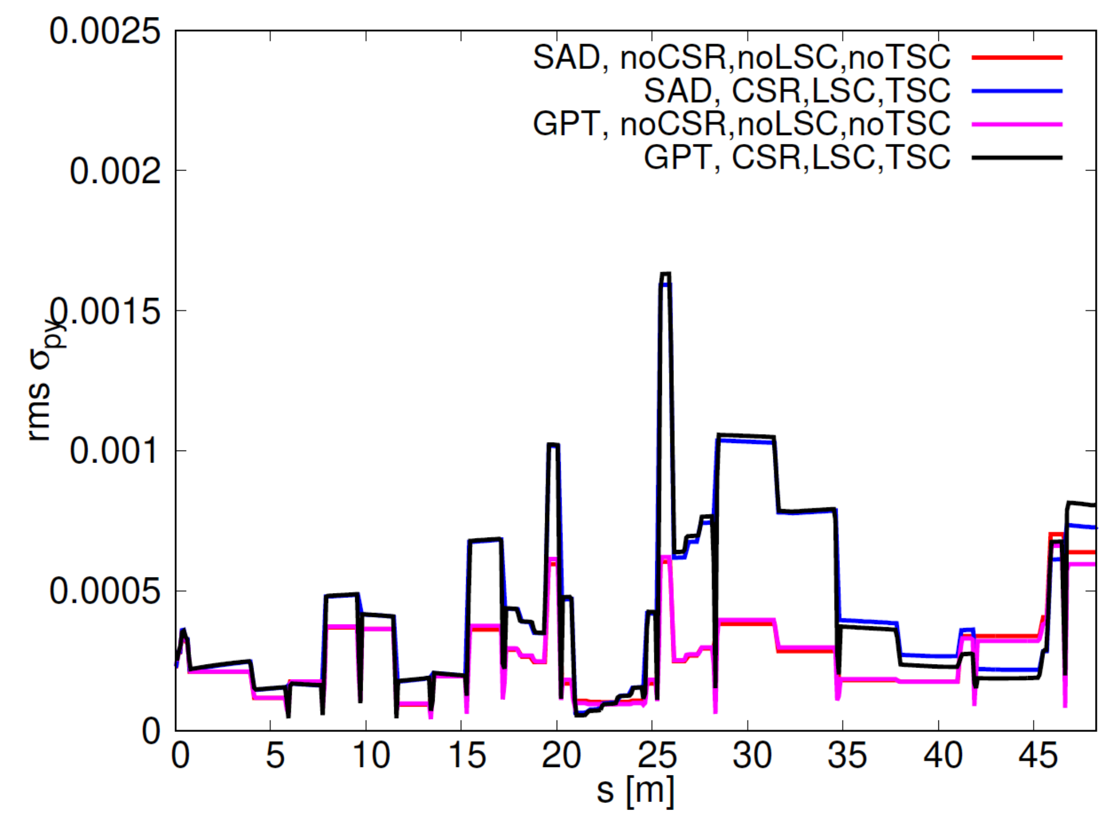




3. Theory of SC effects

Scaling law for energy spread increase due to LSC [Refer to my paper under preparation]

*** Energy spread due to LSC with path length L along the beam line:**

$$\sigma_p^{\mathrm{LSC}} = \frac{3^{-\frac{3}{4}} N r_e \overline{g} L}{\sqrt{2\pi}\gamma^3 \sigma_z^2}$$

$$\overline{g} = 3\ln 2 - 2 - 2\ln\left(\alpha_x + \alpha_y\right)$$

$$\alpha_x = \sigma_x/(\gamma\sigma_z) \quad \alpha_y = \sigma_y/(\gamma\sigma_z)$$

This formulation is only valid for $\alpha_x \ll 1$ and $\alpha_y \ll 1$.

*** More general equation:**

$$\sigma_p'(s) = \frac{\partial\sigma_p(s)}{\partial s} = \frac{3^{-\frac{3}{4}} N r_e \overline{g}(s)}{\sqrt{2\pi}\gamma^3(s)\sigma_z^2\,(s)}$$

$$\sigma_p(s) = \sqrt{\sigma_{p0}^2 + \left[\int_0^s \sigma_p'(s')ds'\right]^2}$$

*** Predictions of LSC driven energy spread:**

**** Linearly proportional to bunch population**

**** Roughly linear proportional to length of beam line**

**** Inverse cubic power law for beam energy**

**** Inverse-square law for bunch length**

3. Theory of SC effects

Scaling law for energy spread increase due to LSC [Refer to my paper under preparation]

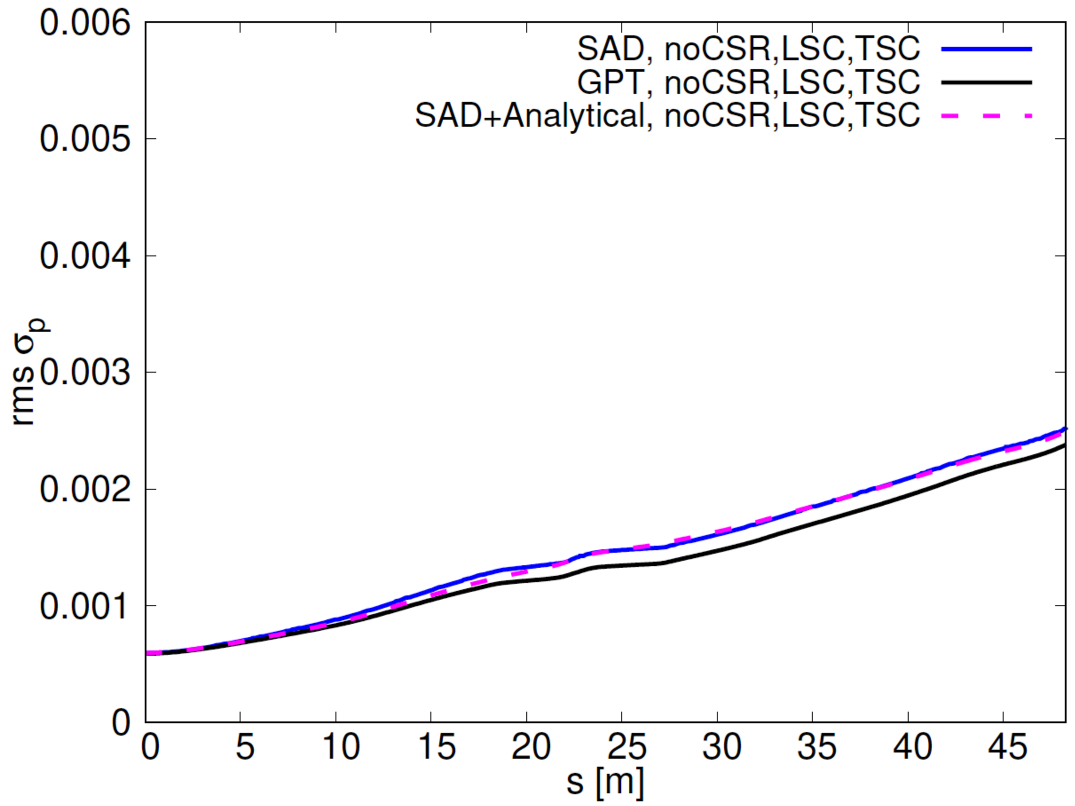
*** Apply to cERL-FEL (magenta line in the figure):**

$$\sigma_p'(s) = \frac{\partial \sigma_p(s)}{\partial s} = \frac{3^{-\frac{3}{4}} N r_e \overline{g}(s)}{\sqrt{2\pi}\gamma^3(s)\sigma_z^2(s)}$$

$$\sigma_p(s) = \sqrt{\sigma_{p0}^2 + \left[\int_0^s \sigma_p'(s')ds'\right]^2}$$

*** Very good agreement:**

**** In the equation, beam sizes taken from SAD simulations**



3. Theory of SC effects

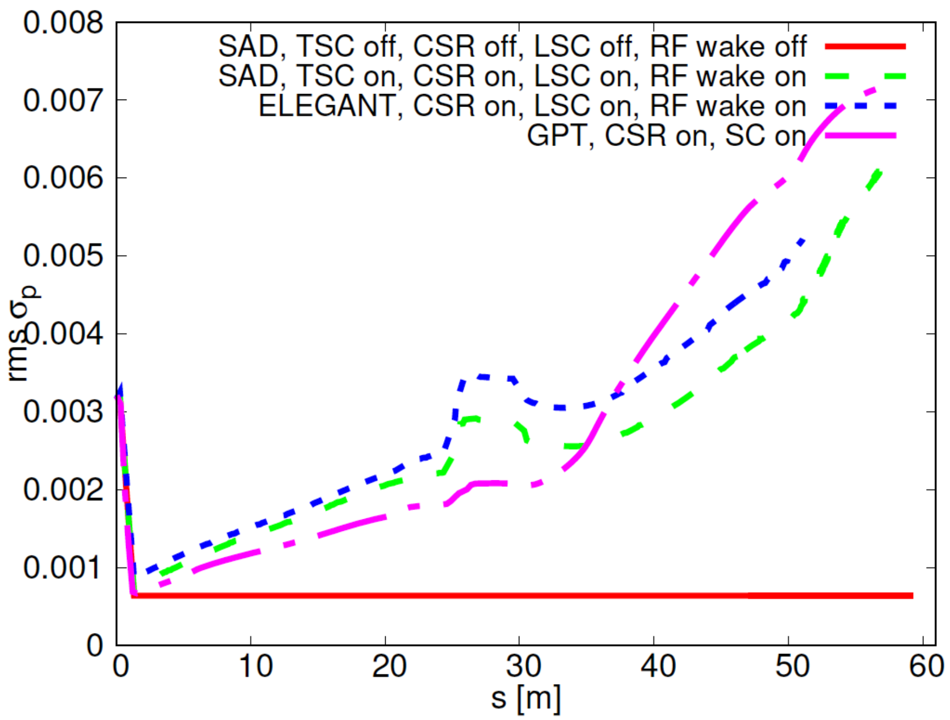
Scaling law for energy spread increase due to LSC [Refer to my paper under preparation]

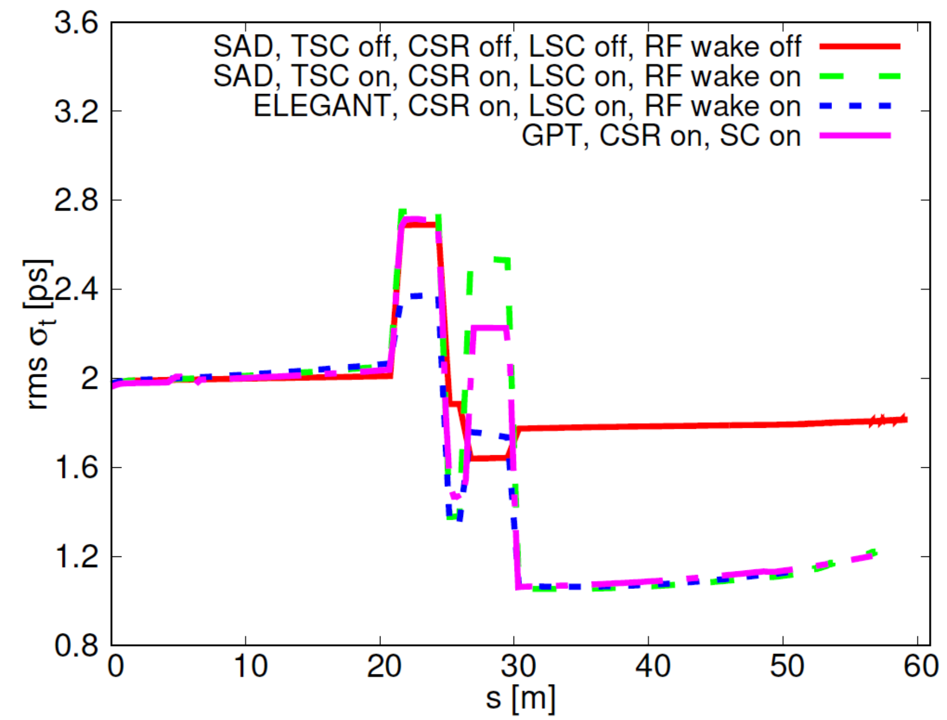
*** Apply to cERL-FEL (magenta line in the figure):**

$$\sigma_p'(s) = \frac{\partial \sigma_p(s)}{\partial s} = \frac{3^{-\frac{3}{4}} N r_e \overline{g}(s)}{\sqrt{2\pi}\gamma^3(s)\sigma_z^2(s)} \qquad \sigma_p(s) = \sqrt{\sigma_{p0}^2 + \left[\int_0^s \sigma_p'(s')ds'\right]^2}$$

*** The theory explains the extra increase of energy spread after the arc with R_{56}=0.217 configuration:**

**** Shorter bunch length causes faster increase of energy spread**





From D. Zhou's talk in cERL-FEL meeting, Oct. 01, 2020

3. Theory of SC effects

Scaling law for momentum spread increase due to TSC [Refer to my paper under preparation]

*** Momentum spread due to TSC in a drift:**

$$\sigma'_{p_x}(s) = \frac{\partial \sigma_{p_x}(s)}{\partial s} = \frac{3^{-\frac{1}{4}}\sqrt{\ln\frac{4}{3}}\,N r_e \overline{g}(s)}{\sqrt{2\pi}\gamma^3(s)\sigma_z(s)\,\sigma(s)} \qquad \sigma = (\sigma_x + \sigma_y)/2$$

*** In a quadrupole, there is another term to be counted.**

*** To be benchmarked with simulations.**

4. Theory of beam envelope equation

So far the simulations and theory I used are based on the assumption of 3D Gaussian beams (The longitudinal distribution is not necessarily to be Gaussian). Initial Gaussian distribution used is also a reason for good agreement between SAD and GPT benchmark simulations

Question: How good is this assumption?

If this is a good assumption, we can derive beam envelope equation using the space-charge forces of Gaussian beams.

4. Theory of beam envelope equation

A revisit of beam envelope equation

II. BEAM ENVELOPE EQUATION WITH SPACE CHARGE

The beam envelope equations with space charge are [5]

$$\sigma_x'' + K_x(s)\sigma_x = \frac{\epsilon_x^2}{\sigma_x^3} + \frac{< xf_x >}{\sigma_x}, \tag{9a}$$

$$\sigma_y'' + K_y(s)\sigma_y = \frac{\epsilon_y^2}{\sigma_y^3} + \frac{< yf_y >}{\sigma_y}, \tag{9b}$$

$<>$ means average over the beam density

$$< xf_x >= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdydz\rho(z)\psi(x,y)xf_x(x,y,z). \tag{10}$$

With nonzero dispersion, the above equations should be extended to include dispersion functions [6, 7]. The space charge force is defined as

$$f_x = \frac{1}{\gamma m_0 \beta^2 c^2} F_x. \tag{11}$$

4. Theory of beam envelope equation

A revisit of beam envelope equation

For beams with transverse Gaussian distribution, there is[8, 9]

$$F_x(x,y) = \frac{Ne^2 Z_0 c}{2\pi\gamma^2}\rho(z)\int_0^\infty \frac{x}{(t+2\sigma_x^2)^{3/2}(t+2\sigma_y^2)^{1/2}} e^{-\frac{x^2}{t+2\sigma_x^2}-\frac{y^2}{t+2\sigma_y^2}}. \quad (12)$$

This space charge force is valid for $\alpha_{x,y} = \sigma_{x,y}/(\gamma\sigma_z) \ll 1$. The ELEGANT code used the on-axis force to model the linear TSC effects:

$$F_x(0,0) = \frac{Ne^2 Z_0 c}{2\pi\gamma^2}\frac{x\rho(z)}{\sigma_x(\sigma_x+\sigma_y)}. \quad (13)$$

Linear TSC model used in ELEGANT
=> Overestimate TSC effects

With the above definitions, we can find:

$$< xf_x >= \frac{Nr_e\sigma_x}{2\sqrt{\pi}\beta^2\gamma^3\sigma_z(\sigma_x+\sigma_y)}. \quad (14)$$

Apply Eq. (14) to Eq. (9a), we can obtain:

$$\sigma_x'' + K_x(s)\sigma_x = \frac{\epsilon_x^2}{\sigma_x^3} + \frac{I}{I_0\beta^3\gamma^3(\sigma_x+\sigma_y)}, \quad (15)$$

with $I_0 = ec/r_e$, and $I = Ne/(2\sqrt{\pi}\sigma_z/(\beta c) = Ne/(2\sqrt{\pi}\sigma_t)$. Here I is a bunched beam with Gaussian distribution and also the average force of $< xf_x >$ is considered. The scaling factor $2\sqrt{\pi}$ is slightly different from that in the literature.

For a quick check of TSC effects at cERL-FEL, we use some typical beam parameters: $\beta\gamma\epsilon_x = 2$ μm, $\overline{\sigma}_x = 1.4$ mm, $\overline{\sigma}_y = 1.3$ mm, $\overline{\sigma}_z = 0.65$ mm, $\gamma = 34.2$. Then we can obtain $\frac{\epsilon_x^2}{\overline{\sigma}_x^3} \approx 1.3\times10^{-6}$ m^{-1}, and $\frac{<xf_x>}{\overline{\sigma}_x} \approx 4.2\times10^{-6}$ m^{-1}. Since $\frac{<xf_x>}{\overline{\sigma}_x} > \frac{\epsilon_x^2}{\overline{\sigma}_x^3}$, we can conclude that TSC is important in cERL-FEL.

4. Theory of beam envelope equation

A revisit of beam envelope equation (references)

[5] T. P. Wangler, Multiparticle dynamics with space charge, in *RF Linear Accelerators* (John Wiley & Sons, Ltd, 2008) Chap. 9, pp. 282–340.

[6] H. Okamoto and S. Machida, Particle beam resonances driven by dispersion and space charge, Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment **482**, 65 (2002).

[7] A. Khan, O. Boine-Frankenheim, F. Hug, and C. Stoll, Beam matching with space charge in energy recovery linacs, Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment **948**, 162822 (2019).

[8] D. Zhou and Y. Nie, Wake fields and impedance of space charge in cartesian coordinate system (2021), arXiv:2101.04369 [physics.acc-ph].

[9] Taufik and K. Takayama, Beam-core evolution equation and space-charge limit, Physics Letters A **383**, 125855 (2019).

4. Theory of beam envelope equation

From ELEGANT manual:

This element simulates transverse space charge (SC) kicks using K.Y. Ng's formula [24].

The linear SC force is given by:

$$\Delta x' = \frac{K_{sc} L e^{-z^2/(2\sigma_z^2)}}{\sqrt{2\pi}\sigma_z} \frac{x}{\sigma_x(\sigma_x+\sigma_y)}$$

$$\Delta y' = \frac{K_{sc} L e^{-z^2/(2\sigma_z^2)}}{\sqrt{2\pi}\sigma_z} \frac{y}{\sigma_y(\sigma_x+\sigma_y)}$$

where $K_{sc} = \frac{2Nr_e}{\gamma^3\beta^2}$, L is the integrating length, $\sigma_{x,y,z}$ are rms beam size.

The non-linear SC force is given by:

$$\Delta x' = \frac{K_{sc} L e^{-z^2/(2\sigma_z^2)}}{2\sigma_z\sqrt{\sigma_x^2-\sigma_y^2}} Im\left[w\left(\frac{x+iy}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right) - e^{-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}} w\left(\frac{x\frac{\sigma_y}{\sigma_x}+iy\frac{\sigma_x}{\sigma_y}}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right)\right]$$

$$\Delta y' = \frac{K_{sc} L e^{-z^2/(2\sigma_z^2)}}{2\sigma_z\sqrt{\sigma_x^2-\sigma_y^2}} Re\left[w\left(\frac{x+iy}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right) - e^{-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}} w\left(\frac{x\frac{\sigma_y}{\sigma_x}+iy\frac{\sigma_x}{\sigma_y}}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right)\right]$$

where $w(z)$ is the complex error function

$$w(z) = e^{-z^2}\left[1 + \frac{2i}{\sqrt{\pi}}\int_0^z e^{\zeta^2} d\zeta\right]$$

5. Summary

* Bench mark of SAD and GPT

** Using averaged LSC 1D model shows better agreement with GPT simulations

* Theory of SC effects

** Simple scaling laws were found (Any similar theory in the literature?)

** Reproduce the simulations well (Condition: bunches close to Gaussian distribution)

** Sigma-matrix equation can be used for simulation and optics matching with space charge)?