Theories and simulations of space charge and coherent synchrotron radiation effects

Demin Zhou

Accelerator theory group, Accelerator laboratory, KEK

Acknowledgements:

T. Miyajima, N. Nakamura, M. Shimada, O. Tanaka, N. Yamamoto

ビーム物理研究会, Dec. 08, 2020

Outline

- ➤ **3D theory of space charge (SC) impedance**
- ➤ **Theory and calculation of coherent synchrotron radiation (CSR) impedance**
- ➤ **Simulation of SC and CSR effects for cERL-FEL**
- ➤ **Summary**

1. 3D theory of SC impedance

Fundamental definitions of impedance and wake functions in Cartesian coordinate system:

*** Lorentz force and impulse kick (assume rigid beam):**

$$\vec{\mathcal{F}}(\vec{r}_1, \vec{r}_0; t) = q_1 \left[\vec{\mathcal{E}}(\vec{r}_1, \vec{r}_0; t) + \vec{v} \times \vec{\mathcal{B}}(\vec{r}_1, \vec{r}_0; t) \right] \qquad \overline{\vec{\mathcal{F}}}(\vec{r}_{1\perp}, \vec{r}_{0\perp}; \tau) = \int_{\tau}^{\tau+\frac{L}{v}} dt \; v \vec{\mathcal{F}}(\vec{r}_1, \vec{r}_0; t)$$

*** Wake functions (different from the classical theory):**

$$W_{\|}(\vec{r}_{1\perp}, \vec{r}_{0\perp}; \tau) = -\frac{1}{q_0 q_1} \overline{\mathcal{F}}_{\|}(\vec{r}_{1\perp}, \vec{r}_{0\perp}; \tau) \qquad \vec{W}_{\perp}(\vec{r}_{1\perp}, \vec{r}_{0\perp}; \tau) = \frac{1}{q_0 q_1} \overline{\vec{\mathcal{F}}}_{\perp}(\vec{r}_{1\perp}, \vec{r}_{0\perp}; \tau)$$

*** Impedance:**

$$Z_{\|}(\vec{r}_{1\perp}, \vec{r}_{0\perp}; \omega) = \int_{-\infty}^{\infty} d\tau \; W_{\|}(\vec{r}_{1\perp}, \vec{r}_{0\perp}; \tau) e^{i\omega\tau} \qquad \vec{Z}_{\perp}(\vec{r}_{1\perp}, \vec{r}_{0\perp}; \omega) = \kappa \int_{-\infty}^{\infty} d\tau \; \vec{W}_{\perp}(\vec{r}_{1\perp}, \vec{r}_{0\perp}; \tau) e^{i\omega\tau}$$

*** Relation of impedance and wake functions:**

$$W_{\|}(\vec{r}_{1\perp}, \vec{r}_{0\perp}; \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \; Z_{\|}(\vec{r}_{1\perp}, \vec{r}_{0\perp}; \omega) e^{-i\omega\tau} \qquad W_{\perp}(\vec{r}_{1\perp}, \vec{r}_{0\perp}; \tau) = \frac{1}{2\pi\kappa} \int_{-\infty}^{\infty} d\omega \; Z_{\perp}(\vec{r}_{1\perp}, \vec{r}_{0\perp}; \omega) e^{-i\omega\tau}$$

q_0 is the source particle at $\vec{r}_0 = (x_0, y_0, z_0 \equiv vt)$

q_1 is the test particle at $\vec{r}_1 = (x_1, y_1, z_1 \equiv v(t-\tau))$ $\qquad \kappa = \frac{i}{v/c} = \frac{i}{\beta}$ is a conventional parameter.

*** With the above definitions, the impedance and wakes are functions of the transverse coordinates of the source and test particles.**

*** For the case of cylindrically symmetric structure, the impulse kick can be expanded in the radial and azimuthal directions, giving the classical definitions of impedance and wake functions**

1. 3D theory of SC impedance

Classical theory of space-charge impedance and wake functions [1]:

* Beam radius a in a perfectly conducting beam pipe of b, transverse distribution uniform
* Cylindrically symmetric chamber
* Low-frequency approximation

L: Path length along beam orbit
k: Wavenumber
Z_0: Impedance of vacuum
β: Relative velocity
γ: Lorentz factor
δ(z): Dirac delta function
c: Speed of light

$$\frac{Z_0^{\parallel}}{L} = i\frac{Z_0 k}{4\pi\beta^2\gamma^2}\left(1+2\ln\frac{b}{a}\right) \xrightarrow{\text{Fourier transform}} \frac{W_0'}{L} = \frac{Z_0 c}{4\pi\gamma^2}\left[1+2\ln\frac{b}{a}\right]\delta'(z)$$

$$\frac{Z_{m\neq 0}^{\perp}}{L} = i\frac{Z_0}{2\pi\beta^2\gamma^2 m}\left(\frac{1}{a^{2m}}-\frac{1}{b^{2m}}\right) \longrightarrow \frac{W_{m\neq 0}}{L} = \frac{Z_0 c}{2\pi\gamma^2 m}\left(\frac{1}{a^{2m}}-\frac{1}{b^{2m}}\right)\delta(z) \qquad k \equiv \frac{\omega}{c}$$

* An obvious challenge in the above formulations is that they diverge when $b \to \infty$ or $a \to 0$

A popular 1D longitudinal space-charge impedance model [2]:

* Exact for transverse uniform distribution in free space
* Approximation for on-axis impedance of transverse bi-Gaussian beam
* Correct scaling law at high frequency

$$\frac{Z_{\mathrm{LSC}}(k)}{L} = \frac{iZ_0}{\pi k r_b^2}\left[1-\frac{kr_b}{\gamma}K_1\left(\frac{kr_b}{\gamma}\right)\right] \qquad r_b = 1.747(\sigma_x+\sigma_y)/2$$

* Low-frequency approximation:

r_b: radius of beam cross section
K_1: Modified Bessel function of the second kind
γ_E: Euler constant
$\sigma_{x/y}$: Gaussian beam sizes in x/y direction

$$\frac{Z_{\mathrm{LSC}}(k\to 0)}{L} = -i\frac{Z_0 k}{4\pi\gamma^2}\left(2\ln\frac{kr_b}{2\gamma}+2\gamma_E-1\right)$$

[1] A. Chao et al., Handbook of Accelerator Physics and Engineering, Second edition, p.253
[2] M. Venturini, Phys. Rev. ST Accel. Beams 11 034401 (2008)

1. 3D theory of SC impedance

A 3D transverse space-charge model [3]:

* Longitudinal and transverse Gaussian beam distribution
* Low-frequency approximation assumed

$$\frac{W_{\mathrm{x}}(x,y,z) - iW_{\mathrm{y}}(x,y,z)}{L} = \frac{-iZ_0 c}{2\pi\gamma^2}\psi(z)\frac{\sqrt{\pi}}{2(\sigma_x^2-\sigma_y^2)}\left[w(a+ib) - e^{-B}w(ar+i\frac{b}{r})\right]$$

$$a = \frac{x}{\sqrt{2(\sigma_x^2-\sigma_y^2)}} \qquad b = \frac{y}{\sqrt{2(\sigma_x^2-\sigma_y^2)}} \qquad r = \frac{\sigma_y}{\sigma_x}$$

Complex error function:

$$w(z) = e^{-z^2}\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^z e^{\zeta^2}d\zeta\right)$$

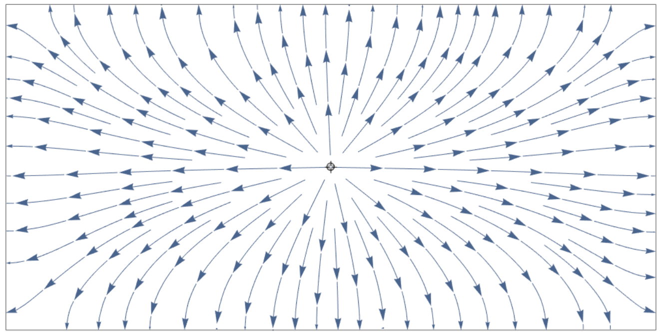
$$B = a^2(1-r^2) + b^2(\frac{1}{r^2}-1) = \frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2} \qquad \psi(z) = \frac{1}{\sqrt{2\pi}\sigma_z}e^{-\frac{z^2}{2\sigma_z^2}}$$

In this talk, I present a 3D theory of space charge impedance and wake functions

* Use Cartesian coordinate system
* Consider rectangular chamber
* Use rigid-beam and impulse approximations [4]

[3] A. Xiao et al., PAC'07, also FERMILAB-CONF-07-702-AD
[4] K. Y. Ng, Physics of Intensity Dependent Beam Instabilities, World Scientific, 2006

1. 3D theory of SC impedance



Mode expansion method

*** Charge and current density expressed by Dirac delta functions:**

$$\varrho(\vec{r},t) = q_0\delta(x-x_0)\delta(y-y_0)\delta(z-z_0) \qquad \vec{\mathcal{J}}(\vec{r},t) = \varrho(\vec{r},t)\vec{v} \qquad z_0 \equiv vt$$

*** Delta functions expressed in forms of Fourier transform and summation of orthonormal eigenfunctions of rectangular waveguide(full width = a, full height = b):**

$$\delta(z-vt) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik(z-vt)}\, dk$$

$$\delta(\vec{r}_\perp - \vec{r}_{0\perp}) = \delta(x-x_0)\delta(y-y_0) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\phi_{mn\nu}(\vec{r}_\perp)\,\phi_{mn\nu}(\vec{r}_{0\perp}) \qquad \nu = x,\ y,\ \text{or } z$$

$$\phi_{mnx}(\vec{r}_\perp) = \frac{2}{\sqrt{(1+\delta_{m0})\,ab}}C_x(x)S_y(y)$$

$$\phi_{mny}(\vec{r}_\perp) = \frac{2}{\sqrt{(1+\delta_{n0})\,ab}}S_x(x)C_y(y)$$

$$\phi_{mnz}(\vec{r}_\perp) = \frac{2}{\sqrt{ab}}S_x(x)S_y(y)$$

$$C_x(x) \equiv \cos\left(k_x(x+a/2)\right)$$

$$S_x(x) \equiv \sin\left(k_x(x+a/2)\right)$$

$$C_y(y) \equiv \cos\left(k_y(y+b/2)\right)$$

$$S_y(y) \equiv \sin\left(k_y(y+b/2)\right)$$

$$k_x = m\pi/a \quad k_y = n\pi/b$$

*** When applied to Maxwell's equations, the expansions are selected according to the boundary conditions.**

1. 3D theory of SC impedance

Mode expansion method

*** Inhomogeneous Helmholtz equations for vector and scalar potentials in frequency domain:**

$$\nabla^2 \vec{A} + k^2 \vec{A} = -\mu_0 \vec{J} \qquad \nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\epsilon_0} \qquad \Phi = \frac{c}{ik} \nabla \cdot \vec{A}$$

*** Magnetic induction and electric field:**

$$\vec{B} = \nabla \times \vec{A} \qquad \vec{E} = ikc\vec{A} - \nabla\Phi = ikc\vec{A} - \frac{c}{ikc}\nabla\nabla\cdot\vec{A} \qquad k \equiv \frac{\omega}{c}$$

*** Explicit expression of vector potential with boundary conditions satisfied:**

$$\vec{A}(\vec{r}, k) = \mu_0 q_0 \beta^2 \gamma^2 \vec{i}_z \sum_{m,n\geq 0} \frac{\phi_{mnz}(\vec{r}_\perp)\,\phi_{mnz}(\vec{r}_{0\perp})}{k^2 + \beta^2\gamma^2(k_x^2 + k_y^2)} e^{ikz/\beta}$$

*** The magnetic induction and electric field can be calculated using the above vector potential. Then using the previous definitions of impedance, we can obtain an explicit form of impedance:**

$$\frac{Z_\parallel(k)}{L} = \frac{4i\mu_0 kc}{ab} \sum_{m,n\geq 0} \frac{\phi'_{mnz}(\vec{r}_{1\perp})\,\phi'_{mnz}(\vec{r}_{0\perp})}{k^2 + \beta^2\gamma^2(k_x^2 + k_y^2)}$$

$$\frac{Z_x(k)}{L} = \frac{-4\mu_0 \beta c\kappa}{ab} \sum_{m,n\geq 0} \frac{k_x \phi'_{mnx}(\vec{r}_{1\perp})\,\phi'_{mnz}(\vec{r}_{0\perp})}{k^2 + \beta^2\gamma^2(k_x^2 + k_y^2)}$$

$$\frac{Z_y(k)}{L} = \frac{-4\mu_0 \beta c\kappa}{ab} \sum_{m,n\geq 0} \frac{k_y \phi'_{mny}(\vec{r}_{1\perp})\,\phi'_{mnz}(\vec{r}_{0\perp})}{k^2 + \beta^2\gamma^2(k_x^2 + k_y^2)}$$

$$\phi'_{mnx}(x, y) = C_x(x) S_y(y)$$
$$\phi'_{mny}(x, y) = S_x(x) C_y(y)$$
$$\phi'_{mnz}(x, y) = S_x(x) S_y(y)$$

1. 3D theory of SC impedance

Mode expansion method

*** Point-charge (Green's function) wake functions:**

$$\frac{W_{\parallel}(z)}{L} = \frac{2Z_0 c}{ab}\mathrm{sgn}(z) \sum_{m,n\geq 0} \phi'_{mnz}\left(\vec{r}_{1\perp}\right) \phi'_{mnz}\left(\vec{r}_{0\perp}\right) e^{-\gamma k_c |z|}$$

$$\frac{W_x(z)}{L} = \frac{-2Z_0 c}{\gamma ab} \sum_{m,n\geq 0} \frac{k_x}{k_c} \phi'_{mnx}\left(\vec{r}_{1\perp}\right) \phi'_{mnz}\left(\vec{r}_{0\perp}\right) e^{-\gamma k_c |z|} \qquad k_c = \sqrt{k_x^2 + k_y^2}$$

$$\frac{W_y(z)}{L} = \frac{-2Z_0 c}{\gamma ab} \sum_{m,n\geq 0} \frac{k_y}{k_c} \phi'_{mny}\left(\vec{r}_{1\perp}\right) \phi'_{mnz}\left(\vec{r}_{0\perp}\right) e^{-\gamma k_c |z|}$$

*** The space-charge impedance and wake functions expressed by summation of trigonometric functions converge slowly. With powerful parallel computers, this method is still useful in practical simulations of space charge effects [5].**
*** In the following I discuss their applications to calculate various forms of space charge impedance and wakes in free space or with Gaussian beam distribution.**
*** A useful identity:**

$$\frac{1}{k^2 + \beta^2\gamma^2(k_x^2 + k_y^2)} = \int_0^\infty e^{-\left[k^2+\beta^2\gamma^2(k_x^2+k_y^2)\right]t} dt$$

*** Also the relation between longitudinal and transverse impedances:**

$$Z_x(k) = \frac{-i\beta\kappa}{k}\frac{\partial Z_{\parallel}(k)}{\partial x_1} \qquad Z_y(k) = \frac{-i\beta\kappa}{k}\frac{\partial Z_{\parallel}(k)}{\partial y_1}$$

[5] J. Qiang, PRAB 21, 054201 (2018)

1. 3D theory of SC impedance

Impedance and wake functions in free space

*** For free space, take the limit of $a \to \infty \quad b \to \infty$. The summation over m and n can be replaced by integration:**

$$\sum_{m,n\geq 0} \to \frac{ab}{\pi^2}\int_0^\infty dk_x \int_0^\infty dk_y$$

*** The integration over kx and ky can be done analytically:**

$$\frac{Z_\parallel(k)}{L} = \frac{i\mu_0 kc}{4\pi\beta^2\gamma^2}\int_0^\infty dt \frac{1}{t} e^{-\frac{(x_1-x_0)^2+(y_1-y_0)^2}{t}} e^{-\frac{k^2 t}{4\beta^2\gamma^2}}$$

$$\frac{Z_x(k)}{L} = \frac{\mu_0 c\kappa(x_1-x_0)}{2\pi\beta\gamma^2}\int_0^\infty dt \frac{1}{t^2} e^{-\frac{(x_1-x_0)^2+(y_1-y_0)^2}{t}} e^{-\frac{k^2 t}{4\beta^2\gamma^2}}$$

$$\frac{Z_y(k)}{L} = \frac{\mu_0 c\kappa(y_1-y_0)}{2\pi\beta\gamma^2}\int_0^\infty dt \frac{1}{t^2} e^{-\frac{(x_1-x_0)^2+(y_1-y_0)^2}{t}} e^{-\frac{k^2 t}{4\beta^2\gamma^2}}$$

*** The integration over t can also be done, resulting in forms by Bessel functions of the second kind:**

$$\frac{Z_\parallel(k)}{L} = \frac{i\mu_0 kc}{2\pi\beta^2\gamma^2} K_0\left(\frac{k}{\beta\gamma}\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}\right)$$

$$\frac{Z_x(k)}{L} = \frac{\mu_0 kc\kappa(x_1-x_0)}{2\pi\beta^2\gamma^3\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}} K_1\left(\frac{k}{\beta\gamma}\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}\right)$$

$$\frac{Z_y(k)}{L} = \frac{\mu_0 kc\kappa(y_1-y_0)}{2\pi\beta^2\gamma^3\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}} K_1\left(\frac{k}{\beta\gamma}\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}\right)$$

*** There is an obvious singularity for** $x_1 \to x_0$ and $y_1 \to y_0$

1. 3D theory of SC impedance

Impedance and wake functions in free space

*** Point-charge (Green's function) wake functions in free space:**

$$\frac{W_\parallel(z)}{L}=\frac{\mu_0 c^2\gamma}{4\pi}\frac{z}{[(x_1-x_0)^2+(y_1-y_0)^2+\gamma^2z^2]^{3/2}}$$

$$\frac{W_x(z)}{L}=\frac{\mu_0 c^2}{4\pi\gamma}\frac{x_1-x_0}{[(x_1-x_0)^2+(y_1-y_0)^2+\gamma^2z^2]^{3/2}}$$

$$\frac{W_y(z)}{L}=\frac{\mu_0 c^2}{4\pi\gamma}\frac{y_1-y_0}{[(x_1-x_0)^2+(y_1-y_0)^2+\gamma^2z^2]^{3/2}}$$

*** Apparently they are Green's functions of a point charge in the lab frame.**
*** Then we can use the Green-functions of the point charge to calculate the impedance and wakes of various distributions. For example, the on-axis longitudinal SC impedance of a round beam with uniform distribution [2] can be easily obtained:**

$$\frac{Z_\parallel(k)}{L}=\frac{i\mu_0 kc}{2\pi\beta^2\gamma^2}K_0\left(\frac{k}{\beta\gamma}\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}\right)$$

$$x_1=y_1=0$$

$$x_0=r\cos\theta,\ \ y_0=\sin\theta$$

$$\frac{1}{\pi r_b^2}\int_0^{r_b}\frac{Z_\parallel}{L}rdrd\theta \longrightarrow \frac{Z_{\mathrm{LSC}}(k)}{L}=\frac{iZ_0}{\pi k r_b^2}\left[1-\frac{kr_b}{\gamma}K_1\left(\frac{kr_b}{\gamma}\right)\right]$$

*** This is a validation of the previous formulae for space-charge impedance and wakes in Green-function forms.**

1. 3D theory of SC impedance

Impedance and wake functions in free space

*** Bi-Gaussian transverse distribution in free space:**

$$\rho(x,y)=\frac{1}{2\pi\sigma_x\sigma_y}e^{-\frac{(x-x_c)^2}{2\sigma_x^2}-\frac{(y-y_c)^2}{2\sigma_y^2}}$$

$$\frac{Z_{\parallel}(k)}{L}=\frac{i\mu_0 kc}{4\pi\beta^2\gamma^2}\int_0^\infty dt\frac{1}{(t+2\sigma_x^2)^{1/2}(t+2\sigma_y^2)^{1/2}}e^{-\frac{(x_1-x_c)^2}{t+2\sigma_x^2}-\frac{(y_1-y_c)^2}{t+2\sigma_y^2}}e^{-\frac{k^2t}{4\beta^2\gamma^2}}$$

$$\frac{Z_x(k)}{L}=\frac{\mu_0 c\kappa}{2\pi\beta\gamma^2}\int_0^\infty dt'\frac{x_1-x_c}{(t+2\sigma_x^2)^{3/2}(t+2\sigma_y^2)^{1/2}}e^{-\frac{(x_1-x_c)^2}{t+2\sigma_x^2}-\frac{(y_1-y_c)^2}{t+2\sigma_y^2}}e^{-\frac{k^2t}{4\beta^2\gamma^2}}$$

$$\frac{Z_y(k)}{L}=\frac{\mu_0 c\kappa}{2\pi\beta\gamma^2}\int_0^\infty dt\frac{y_1-y_c}{(t+2\sigma_x^2)^{1/2}(t+2\sigma_y^2)^{3/2}}e^{-\frac{(x_1-x_c)^2}{t+2\sigma_x^2}-\frac{(y_1-y_c)^2}{t+2\sigma_y^2}}e^{-\frac{k^2t}{4\beta^2\gamma^2}}$$

*** The above equations are not integrable in general, but provide a 3D model of space-charge impedance useful in practical simulations of SC effects. We can rewrite them as:**

$$\frac{Z_{\parallel}(k)}{L}=\frac{i\mu_0 c}{4\pi\beta^2\gamma^2}F_z(x_1,y_1,r,k)$$

$$\frac{Z_x(k)}{L}=\frac{\mu_0 c\kappa(x_1-x_c)}{4\pi\beta\gamma^2\sigma_x^2}F_x(x_1,y_1,r,k)$$

$$\frac{Z_y(k)}{L}=\frac{\mu_0 c\kappa(y_1-y_c)}{4\pi\beta\gamma^2\sigma_x^2}F_y(x_1,y_1,r,k)$$

$$F_z(x,y,r,k)=k\int_0^\infty dt\frac{1}{(t+1)^{1/2}(t+r^2)^{1/2}}e^{-\frac{(x-x_c)^2}{2\sigma_x^2(t+1)}-\frac{(y-y_c)^2}{2\sigma_x^2(t+r^2)}}e^{-\frac{k^2\sigma_x^2t}{2\beta^2\gamma^2}}$$

$$F_x(x,y,r,k)=\int_0^\infty dt\frac{1}{(t+1)^{3/2}(t+r^2)^{1/2}}e^{-\frac{(x-x_c)^2}{2\sigma_x^2(t+1)}-\frac{(y-y_c)^2}{2\sigma_x^2(t+r^2)}}e^{-\frac{k^2\sigma_x^2t}{2\beta^2\gamma^2}}$$

$$F_y(x,y,r,k)=\int_0^\infty dt\frac{1}{(t+1)^{1/2}(t+r^2)^{3/2}}e^{-\frac{(x-x_c)^2}{2\sigma_x^2(t+1)}-\frac{(y-y_c)^2}{2\sigma_x^2(t+r^2)}}e^{-\frac{k^2\sigma_x^2t}{2\beta^2\gamma^2}}$$

$$r=\sigma_x/\sigma_y$$

*** We can check how they are related with the existing models in the literature.**

1. 3D theory of SC impedance

Impedance and wake functions in free space

*** For axis-symmetric beam with Gaussian transverse distribution, we can find the on-axis impedance:**

$$F_z(0,0,1,k) = k e^{\frac{k^2\sigma^2}{2\beta^2\gamma^2}} \Gamma\left(\frac{k^2\sigma^2}{2\beta^2\gamma^2}\right) \qquad \sigma_x = \sigma_y = \sigma$$

*** This is exactly Eq.(15) in [2]:**

$$Z(k) = -i\frac{Z_0}{\pi\gamma\sigma}\frac{\xi_\sigma}{4}e^{\xi_\sigma^2/2}\mathrm{Ei}\left(-\frac{\xi_\sigma^2}{2}\right) = i\frac{Z_0}{\pi\gamma\sigma}\frac{\xi_\sigma}{4}e^{\xi_\sigma^2/2}\Gamma\left(\frac{\xi_\sigma^2}{2}\right) \qquad \xi_\sigma = \frac{k\sigma}{\gamma}$$

*** Take the low frequency approximation for the transverse impedance:**

$$F_x(x,y,r,k) = \int_0^\infty dt \frac{1}{(t+1)^{3/2}(t+r^2)^{1/2}} e^{-\frac{(x-x_c)^2}{2\sigma_x^2(t+1)} - \frac{(y-y_c)^2}{2\sigma_x^2(t+r^2)}}$$

$$F_y(x,y,r,k) = \int_0^\infty dt \frac{1}{(t+1)^{1/2}(t+r^2)^{3/2}} e^{-\frac{(x-x_c)^2}{2\sigma_x^2(t+1)} - \frac{(y-y_c)^2}{2\sigma_x^2(t+r^2)}} \qquad k \ll k_{th}$$

*** After change of variables, this gives the 3D transverse space-charge model proposed in [3]:**

$$\frac{W_x(x,y,z) - iW_y(x,y,z)}{L} = \frac{-iZ_0c}{2\pi\gamma^2}\psi(z)\frac{\sqrt{\pi}}{2(\sigma_x^2-\sigma_y^2)}\left[w(a+ib) - e^{-B}w(ar+i\frac{b}{r})\right]$$

*** The application condition of the above model should be:**

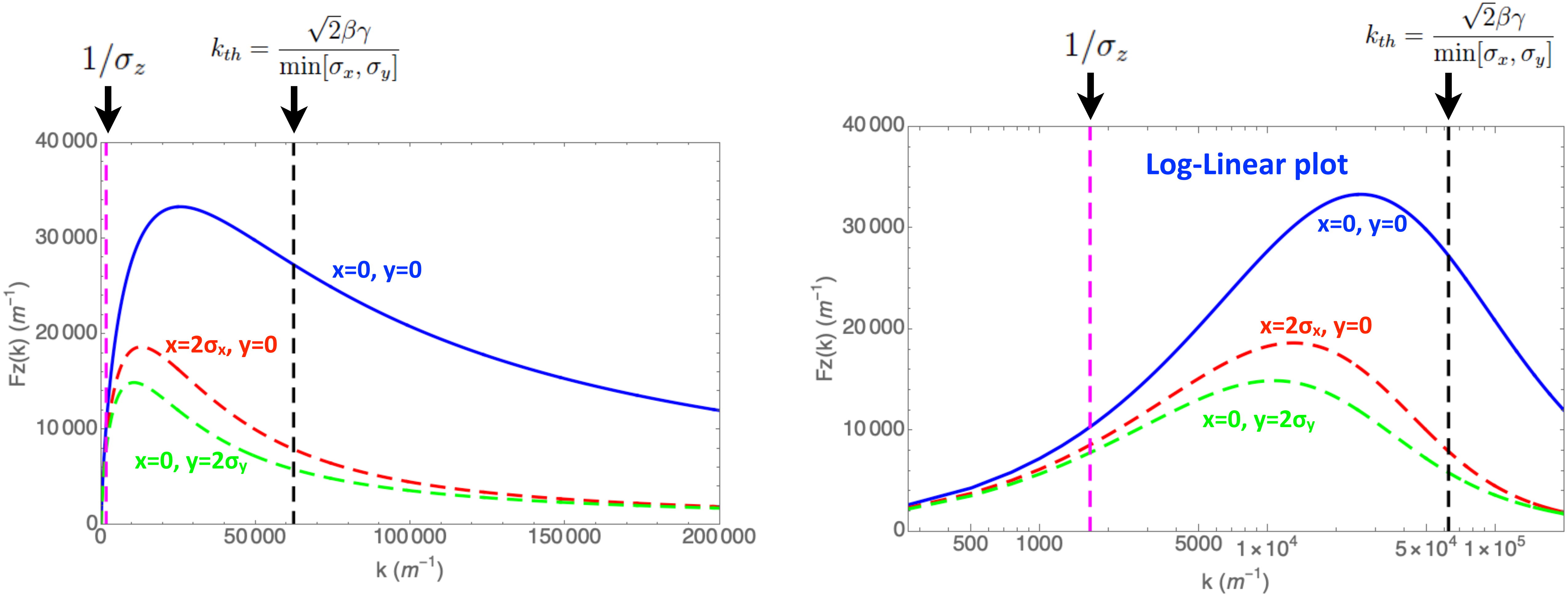
$$\gamma\sigma_z \gg \sigma_\perp \qquad \sigma_\perp = \mathrm{Min}[\sigma_x, \sigma_y] \qquad k_{th} = \frac{\sqrt{2}\beta\gamma}{\min[\sigma_x,\sigma_y]}$$

*** Again we find good consistency with the existing theories.**

1. 3D theory of SC impedance

Impedance and wake functions in free space

*** Application of the 3D LSC model to cERL-FEL:**
Beam parameters after the RF cavity: σ_x=0.45 mm, σ_y=0.68 mm, σ_z=0.6 mm, γ=19.6
*** The first finding is that the off-axis particles feel weaker wake kicks, compared with the on-axis particles.**



Fz uses the formula on page.11

1. 3D theory of SC impedance

Impedance and wake functions in free space

*** Application of the 3D LSC model to cERL-FEL:**
Beam parameters after the RF cavity: σ_x=0.45 mm, σ_y=0.68 mm, σ_z=0.6 mm, γ=19.6

*** Compare a few simple 1D LSC models with the exact model (Fz of page.11):**
Model-1: Simplified model from page.12:

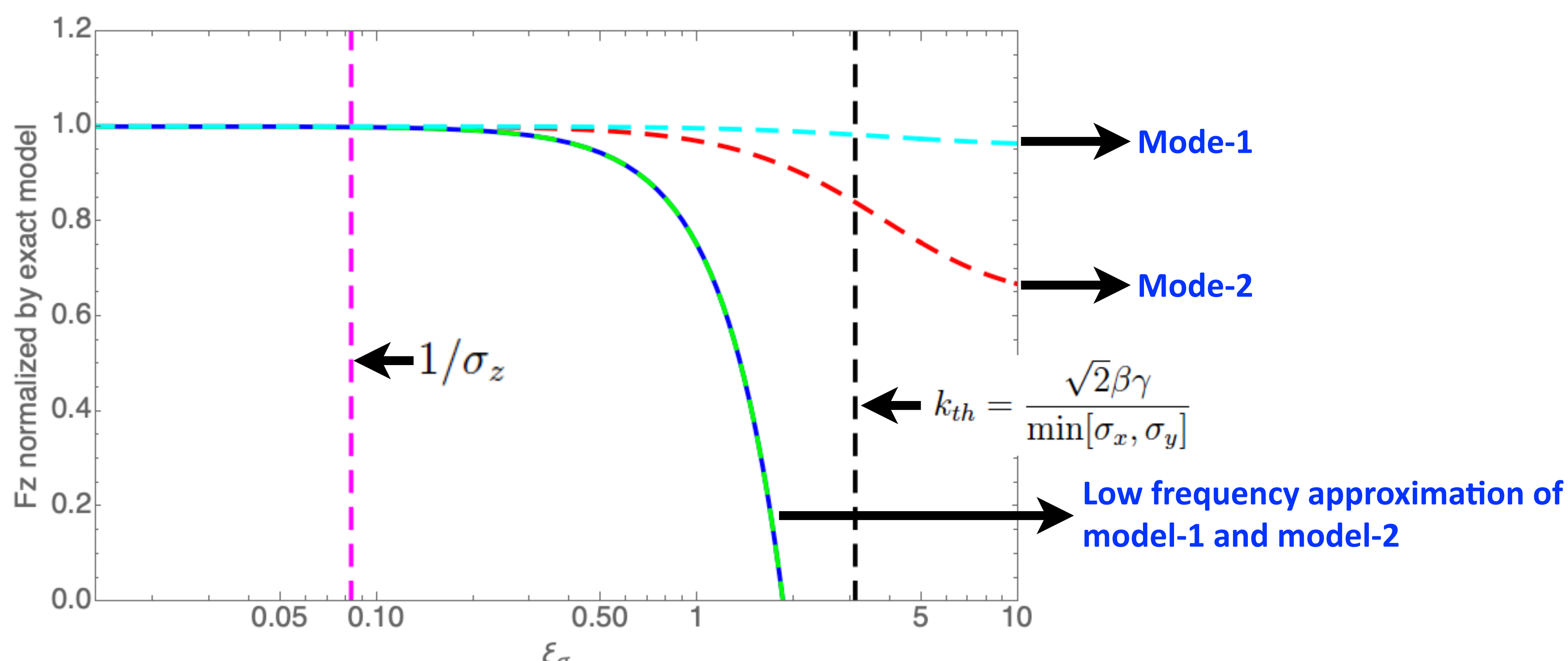
$$F_z(0,0,r,k) = ke^{\frac{k^2(\sigma_x+\sigma_y)^2}{8\beta^2\gamma^2}}\Gamma\left(\frac{k^2(\sigma_x+\sigma_y)^2}{8\beta^2\gamma^2}\right) \longrightarrow F_z(0,0,r,k) \approx -k\left(\ln\frac{k^2(\sigma_x+\sigma_y)^2}{8\beta^2\gamma^2}+\gamma_E\right)$$

Model-2: Simplified model from page.10 (Implemented in ELEGANT code)

$$F_z(k) = \frac{4k}{\xi_\sigma^2}\left[1-\xi_\sigma K_1(\xi_\sigma)\right] \longrightarrow F_z(k) \approx -k\left(\ln\frac{k^2r_b^2}{4\beta^2\gamma^2}+2\gamma_E-1\right) \quad \xi_\sigma = \frac{k\sigma}{\gamma}$$

$$r_b \approx 1.747(\sigma_x+\sigma_y)/2$$

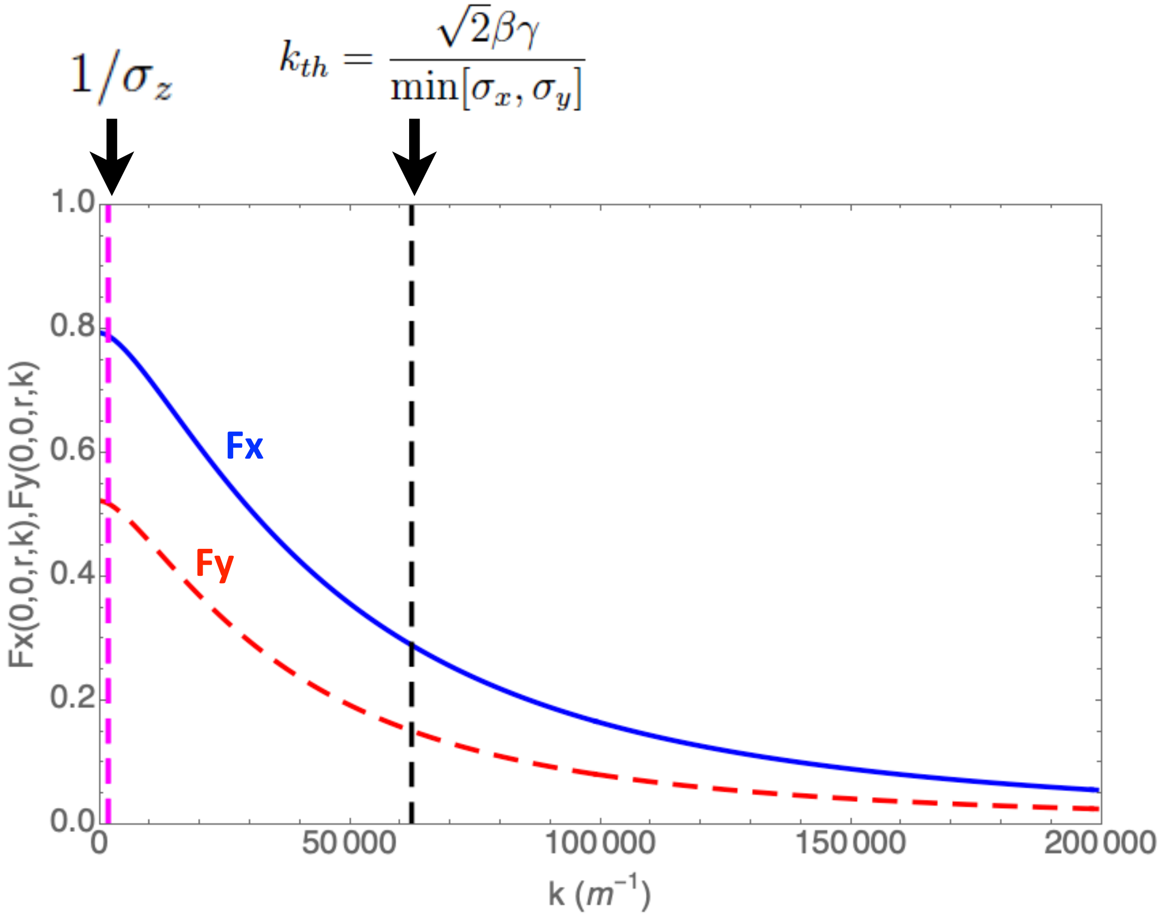
*** Obviously model-1 is better than model-2 (?)**

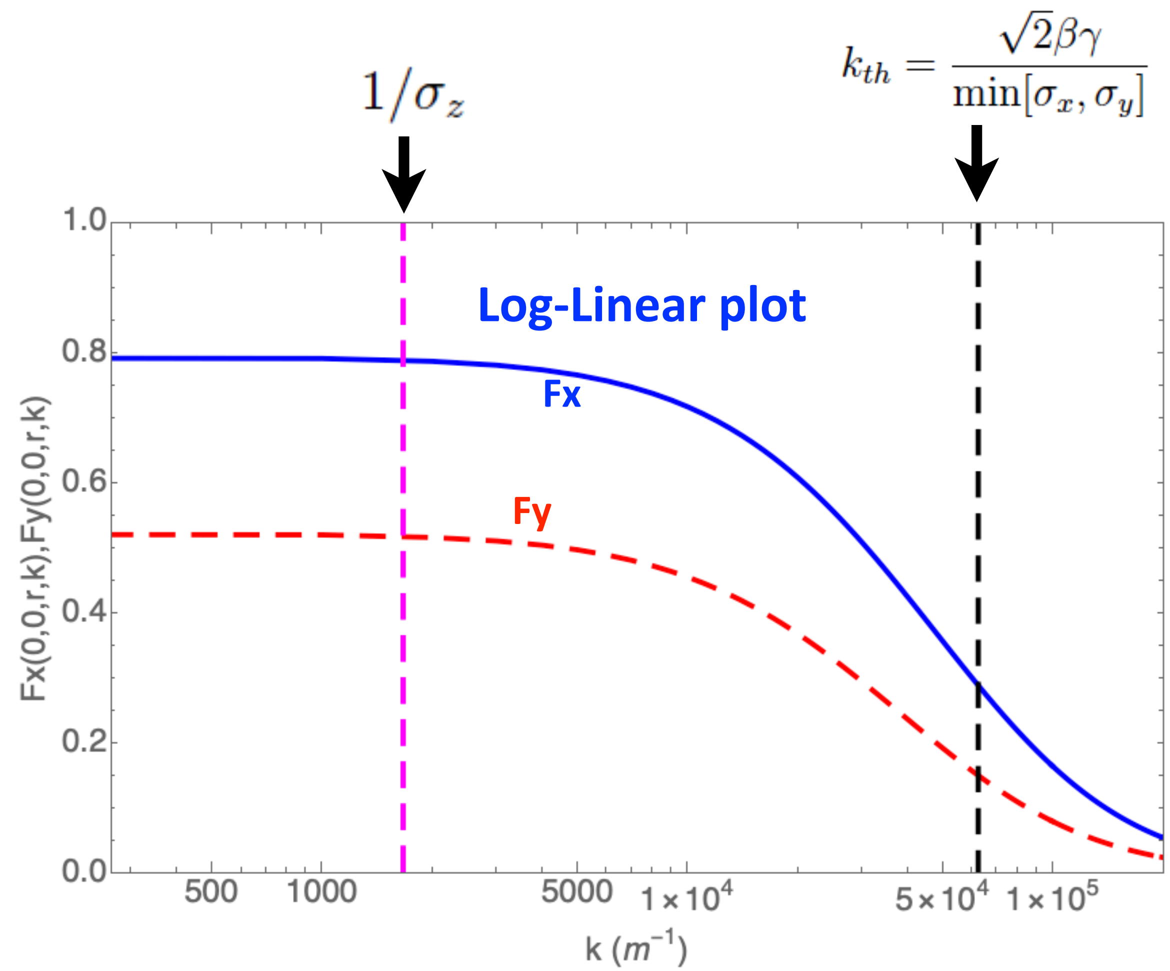


1. 3D theory of SC impedance

Impedance and wake functions in free space

* Application of the 3D TSC model to cERL-FEL:
  Beam parameters after the RF cavity: σ_x=0.45 mm, σ_y=0.68 mm, σ_z=0.6 mm, γ=19.6
* The low frequency approximation give constant TSC impedance. But it is only valid for the case of $1/\sigma_z$ << k_{th}. The microbunching in z-direction makes the condition of use more strict.





Fx and Fy uses the formula on page.11

2. Theory and calculation of CSR impedance

Theory of CSR impedance

*** The 1D steady-state CSR impedance model with parallel-plates shielding [6]:**

$$Z_{pp}(k) = \frac{2\pi^2 n Z_0}{\beta}\frac{\rho}{h}\sum_{p\geq 1}^{p=\mathrm{odd}}\Lambda_p\left[\beta^2 J_n'(\gamma_p\rho)\left[J_n'(\gamma_p\rho)+iY_n'(\gamma_p\rho)\right]+\left(\frac{\alpha_p}{\gamma_p}\right)^2 J_n(\gamma_p\rho)\left[J_n(\gamma_p\rho)+iY_n(\gamma_p\rho)\right]\right]$$

$$\alpha_p = \frac{p\pi}{h} \qquad \gamma_p = \sqrt{\frac{n^2\beta^2}{\rho^2}-\alpha_p^2} = \sqrt{k^2-\alpha_p^2} \qquad n = \frac{\rho k}{\beta}$$

J_n, Y_n: Bessel functions of the first and second kind
h is full height of parallel plates

*** Computing of Bessel functions with large order and large argument is difficult. Usually it is approximated by the expression using Airy functions [7]:**

$$Z_{pp}(k) \approx \frac{2\pi^2 n Z_0}{\beta}\frac{\rho}{h}\left(\frac{2}{n}\right)^{4/3}\sum_{p\geq 1}^{p=\mathrm{odd}}\Lambda_p\left[Ai'(X)\left[Ai'(X)-iBi'(X)\right]+\left(\frac{\alpha_p}{\kappa^2}\right)^2 Ai(X)\left[Ai(X)-iBi(X)\right]\right]$$

$$X = \left(\frac{n}{2}\right)^{2/3}x^2 \qquad x = \sqrt{1-\frac{\beta^2}{k^2}(k^2-\alpha_p^2)} \qquad \kappa = \left(\frac{2\beta k^2}{\rho}\right)^{1/3}$$

Ai, Bi: Airy function of the first and second kind

*** Taking the limit of $\rho \to \infty$, we can obtain the 1D LSC impedance with parallel-plates shielding:**

$$\frac{Z_{pp}^{\mathrm{LSC}}(k)}{2\pi\rho} = \frac{iZ_0 k}{\beta\gamma h}\sum_{p\geq 1}^{p=\mathrm{odd}}\Lambda_p\frac{1}{\sqrt{k^2+\beta^2\gamma^2\alpha_p^2}}$$

*** The above equation can be derived from the LSC model on page.7.**
*** In theory, CSR impedance contains the impedance of synchrotron radiation and space charge. This is a kind of interference between CSR and SC.**

[6] J.B. Murphy et al., PA, 1997, Vol. 57, pp.9-64
[7] T. Agoh, PRST-AB 12, 094402 (2009)

2. Theory and calculation of CSR impedance

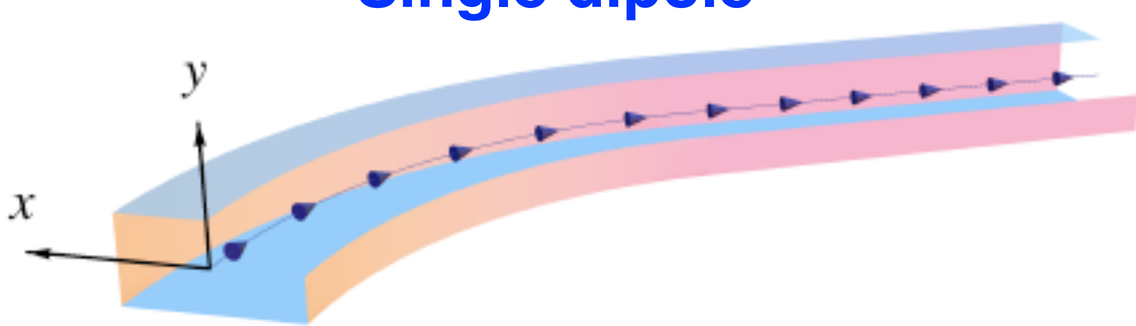
Calculation of CSR impedance

*** In practical simulations of CSR effects, several issues need to be taken into account:**
1) Chamber shielding; 2) s-dependent orbit curvature; 3) s-dependent beam distribution; 4) Resistive wall; 5) Space charge.
*** These issues are properly treated in my CSRZ code [8]: A code of CSR impedance calculation in frequency domain (based on T. Agoh and K. Yokoya's method [9])**

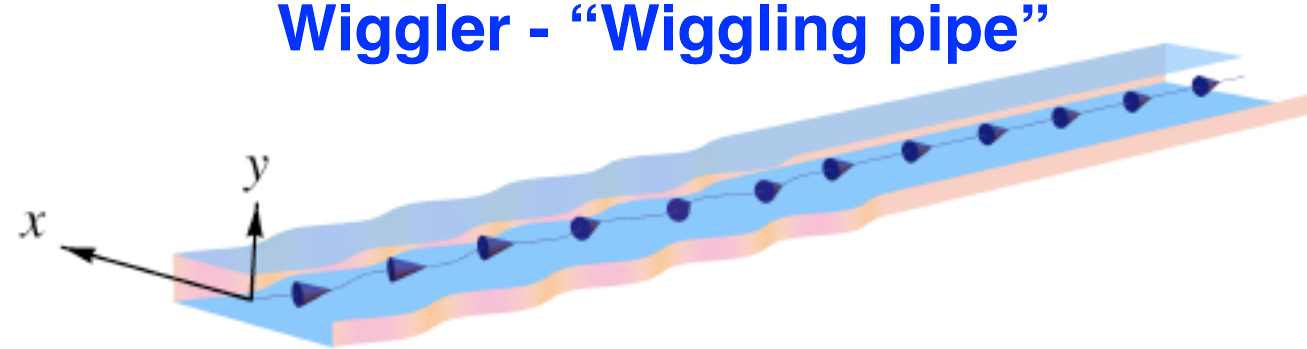
$$\vec{E}_\perp = \vec{E}^r_\perp + \vec{E}^b_\perp \quad \rightarrow \quad \frac{\partial \vec{E}^r_\perp}{\partial s} = \frac{i}{2k}\left[\nabla^2_\perp \vec{E}^r_\perp + 2k^2\left(\underbrace{\frac{x}{R(s)}}_{\text{Orbit curvature}} - \underbrace{\frac{1}{2\gamma^2}}_{\text{Space charge}}\right)\left(\vec{E}^r_\perp + \vec{E}^b_\perp\right)\right]$$

Examples:

Single dipole



Wiggler - "Wiggling pipe"



[8] D. Zhou, et al., Jpn. J. Appl. Phys. 51 (2012) 016401
[9] T. Agoh and K. Yokoya, PRST-AB 7, 054403 (2004)

2. Theory and calculation of CSR impedance

Calculation of CSR impedance

* Application of the 1D steady-state CSR model to cERL-FEL:

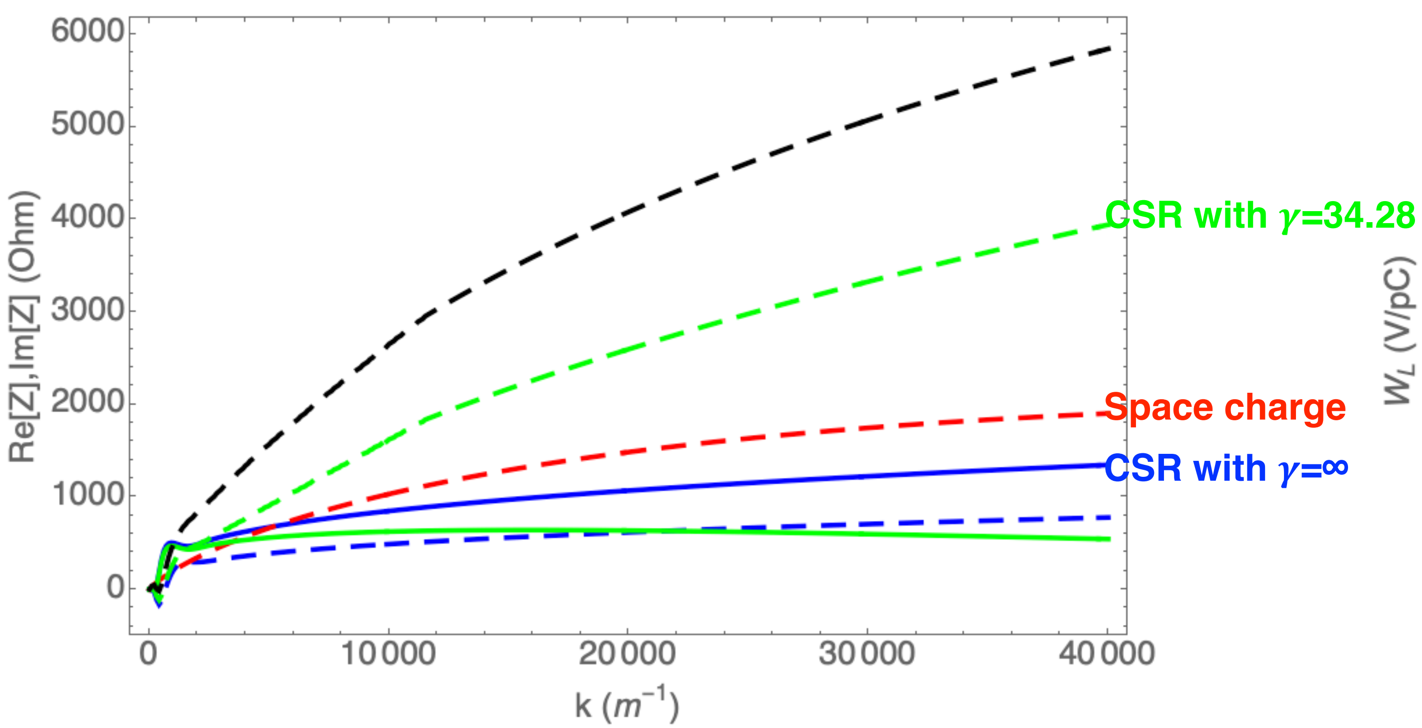
Beam parameters in the arc: σ_x=0.46 mm, σ_y=0.23 mm, σ_z=0.6 mm, γ=34.28

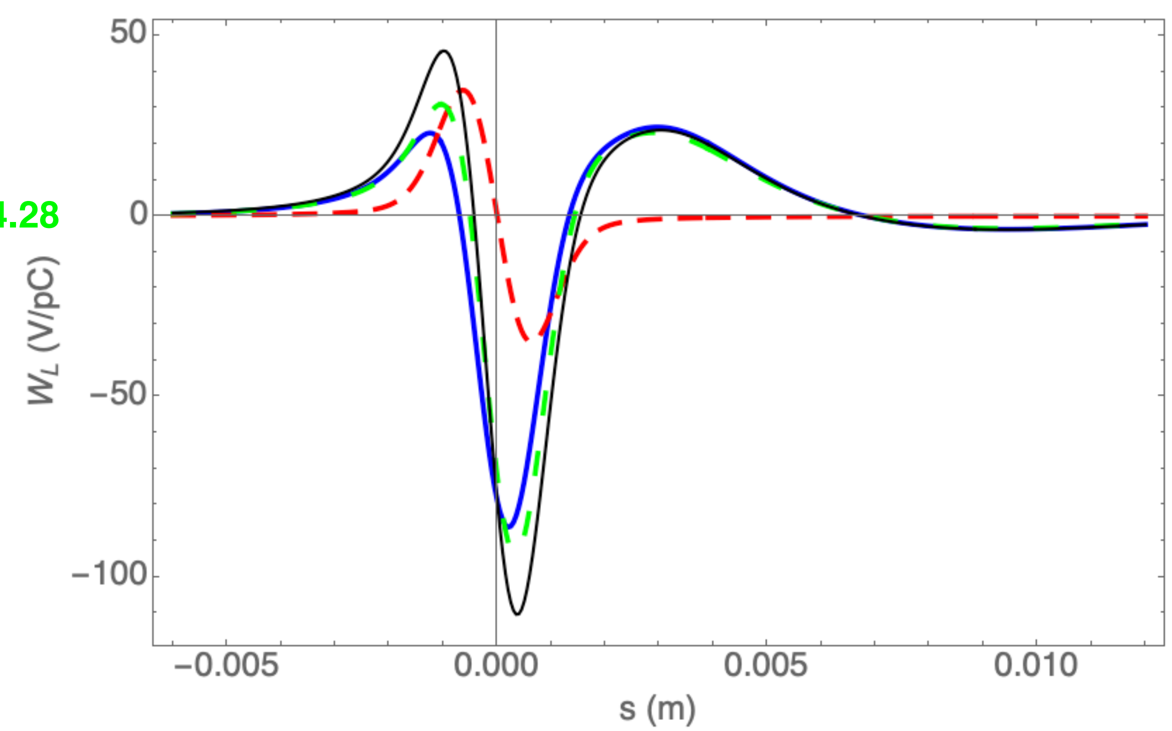
Maget: L_{dipole}=0.85 m, R_{dipole}=1.08 m, a=70 mm(Full chamber width), b=40 mm(Full Chamber height)

* We compare four type of impedances:

a) CSR with $\gamma=\infty$ (Blue lines); b) CSR with γ=34.28 (Green lines); c) Space charge (Red lines); d) Sum of a) and c) (Black lines)

* From both impedance and wakes, we can see that there is interference between CSR and SC: Simply treating CSR and SC separately can be wrong.





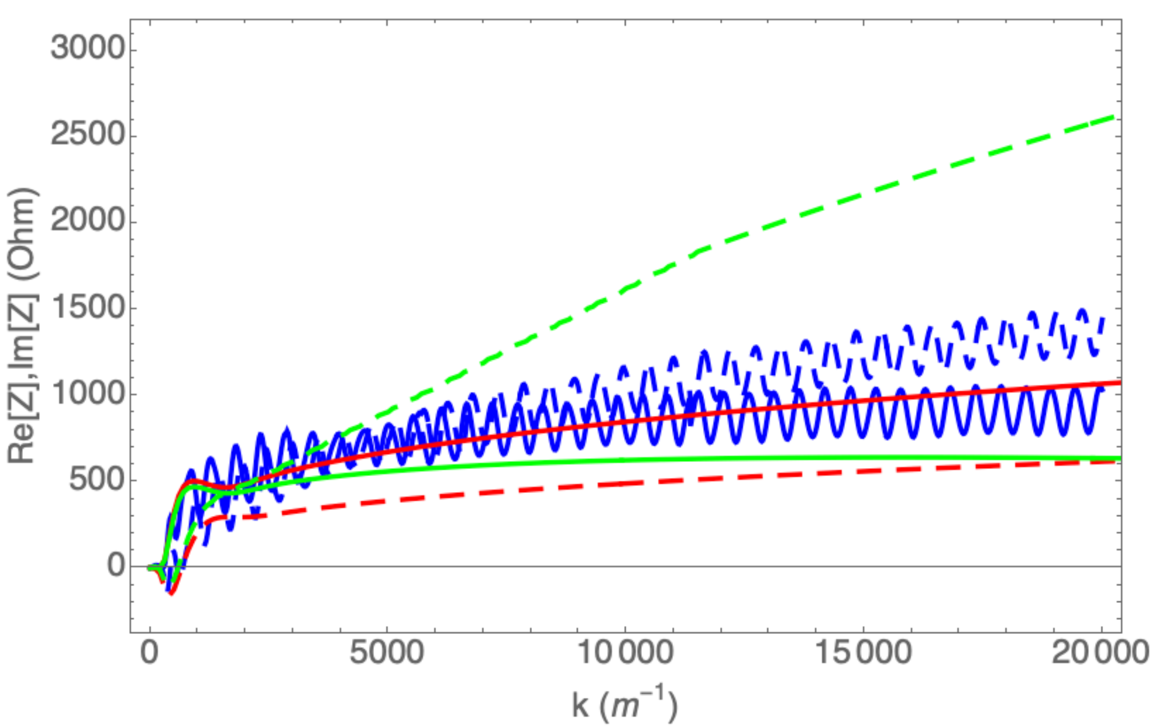
2. Theory and calculation of CSR impedance

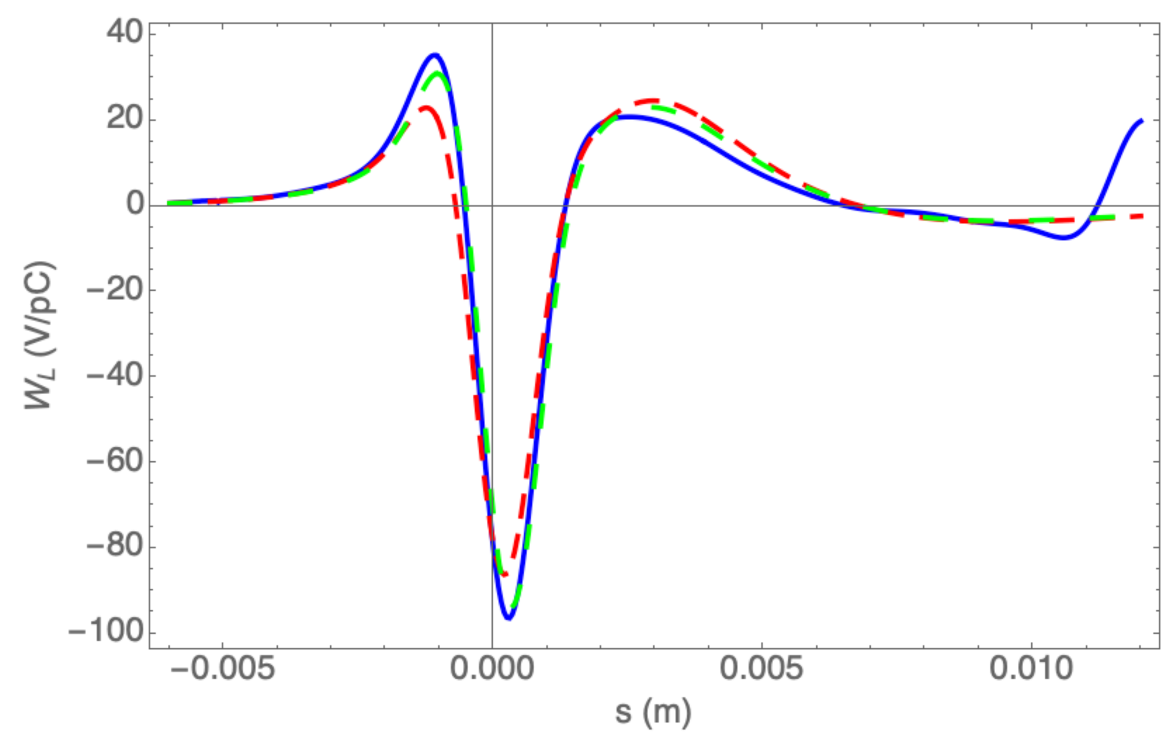
Calculation of CSR impedance

* Application of CSRZ code to cERL-FEL and compare with steady-state models:
  Beam parameters in the arc: σ_x=0.46 mm, σ_y=0.23 mm, σ_z=0.6 mm, γ=34.28
  Maget: L_{dipole}=0.85 m, R_{dipole}=1.08 m, a=70 mm(Full chamber width), b=40 mm(Full Chamber height)
* We compare three type of impedances:

a) CSRZ with γ=34.28 (Blue lines); b) Steady-state CSR with γ=34.28 (Green lines); c) Steady-state CSR with $\gamma=\infty$ (Red lines)

* From both impedance and wakes, numerically calculated CSR impedance with space charge included can still be quite different from the prediction of analytical theory.





3. Simulation of SC and CSR effects for cERL-FEL

Setup of tracking simulations using SAD [10]

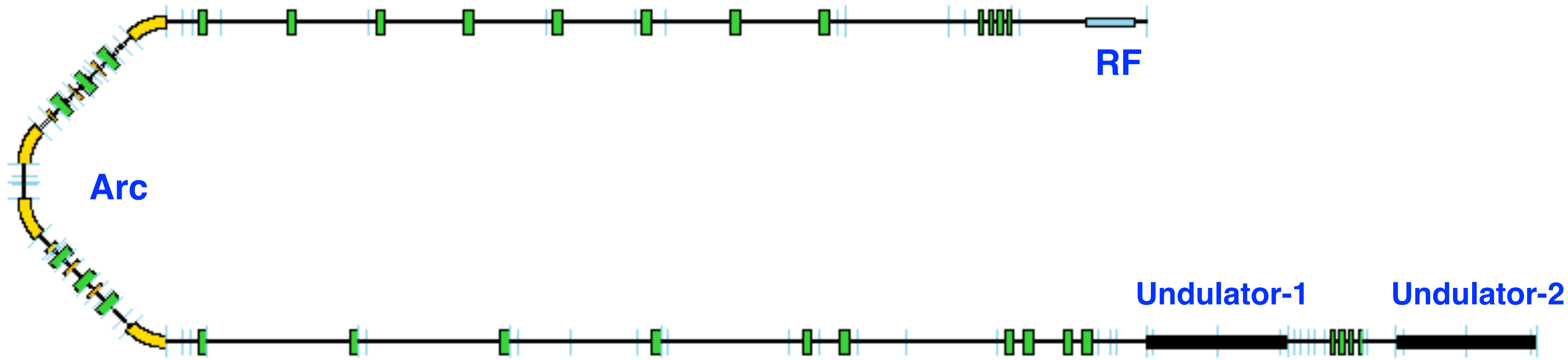
*** Beam parameters:**

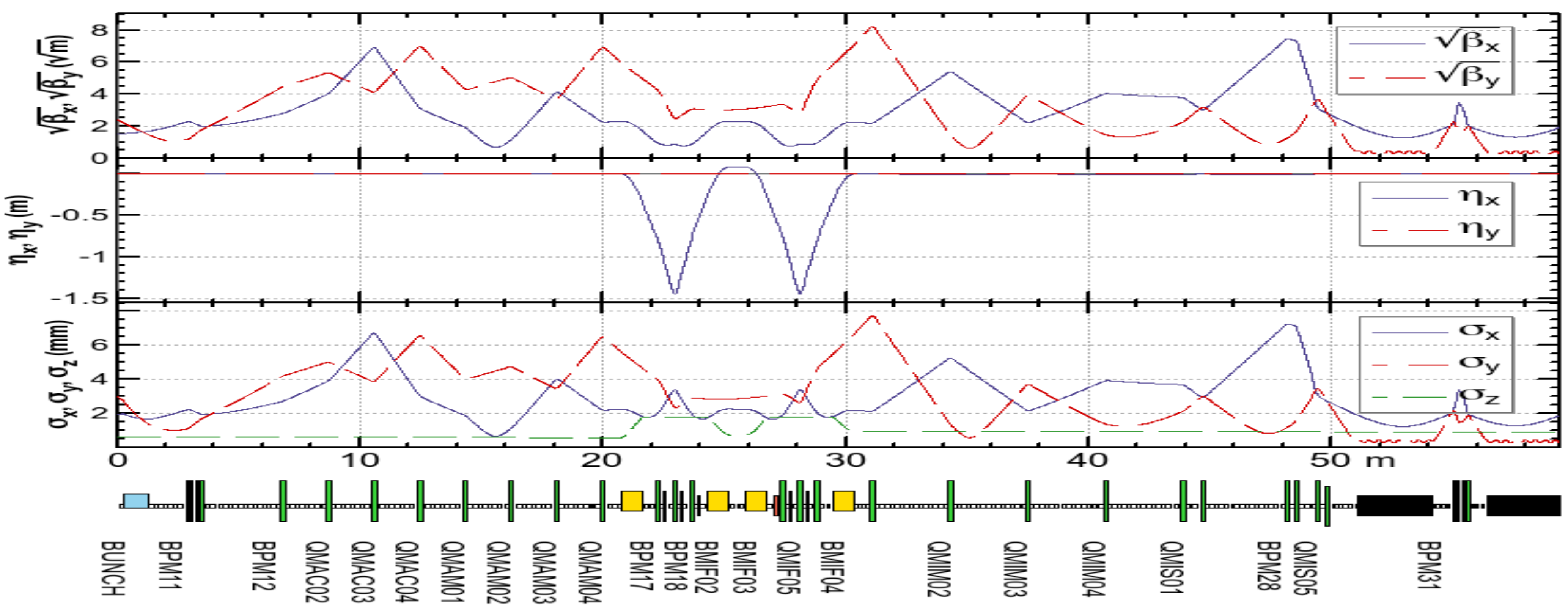
$E_{injection}$=10 MeV, E_{beam}=17.5 MeV (V_{RF}=8.31 MV, ϕ_{RF}=79°)

*** Initial beam distribution: simulated from injector by GPT code**

Unnormalized rms emittance: ε_x=8.5e-8 m, ε_y=7.9e-8 m, ε_z=1.9e-6 m

Twiss parameters: $\beta_{x/y/z}$=2.3/5.8/0.2 m, $\alpha_{x/y/z}$=-0.01/1.8/0





[10] Collaborate with T. Miyajima, N. Nakamura and M. Shimada

3. Simulation of SC and CSR effects for cERL-FEL

Setup of tracking simulations using SAD

*** CSR impedance by CSRZ. CSR wake lumped at the exit of each dipole magnet.**

*** LSC impedance model [2] using SAD script:**

$$\frac{Z_{\mathrm{LSC}}(k)}{L} = \frac{iZ_0}{\pi k r_b^2}\left[1 - \frac{kr_b}{\gamma}K_1\left(\frac{kr_b}{\gamma}\right)\right] \qquad r_b = 1.747(\sigma_x + \sigma_y)/2$$

*** TSC impedance model [3] using SAD script:**

$$\frac{W_{\mathrm{x}}(x,y,z) - iW_{\mathrm{y}}(x,y,z)}{L} = \frac{-iZ_0 c}{2\pi\gamma^2}\psi(z)\frac{\sqrt{\pi}}{2(\sigma_x^2 - \sigma_y^2)}\left[w(a+ib) - e^{-B}w(ar + i\frac{b}{r})\right]$$

$$a = \frac{x}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} \qquad b = \frac{y}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} \qquad r = \frac{\sigma_y}{\sigma_x}$$

$$B = a^2(1 - r^2) + b^2(\frac{1}{r^2} - 1) = \frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2}$$

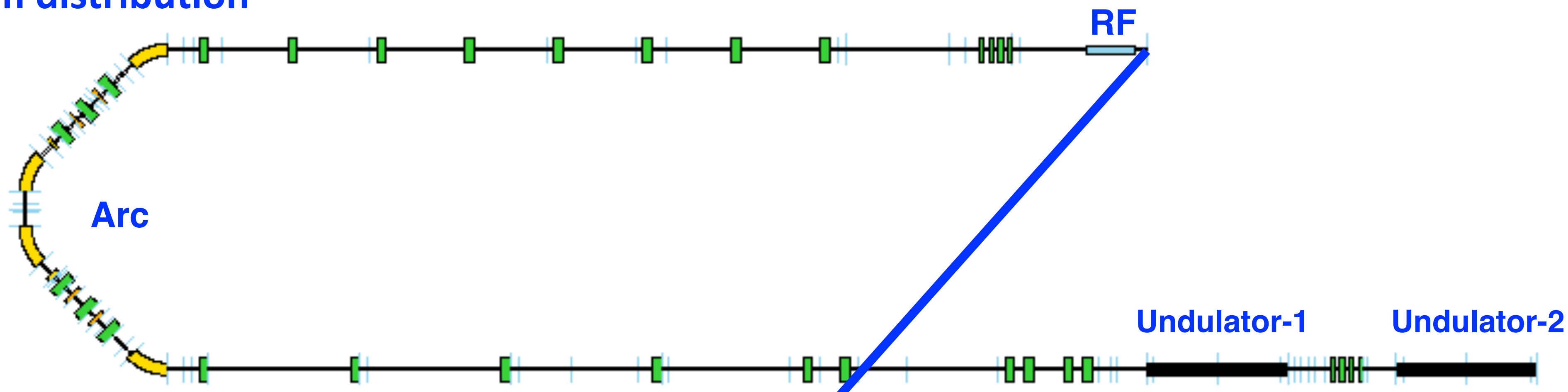
ψ(z): Longitudinal arbitrary density.

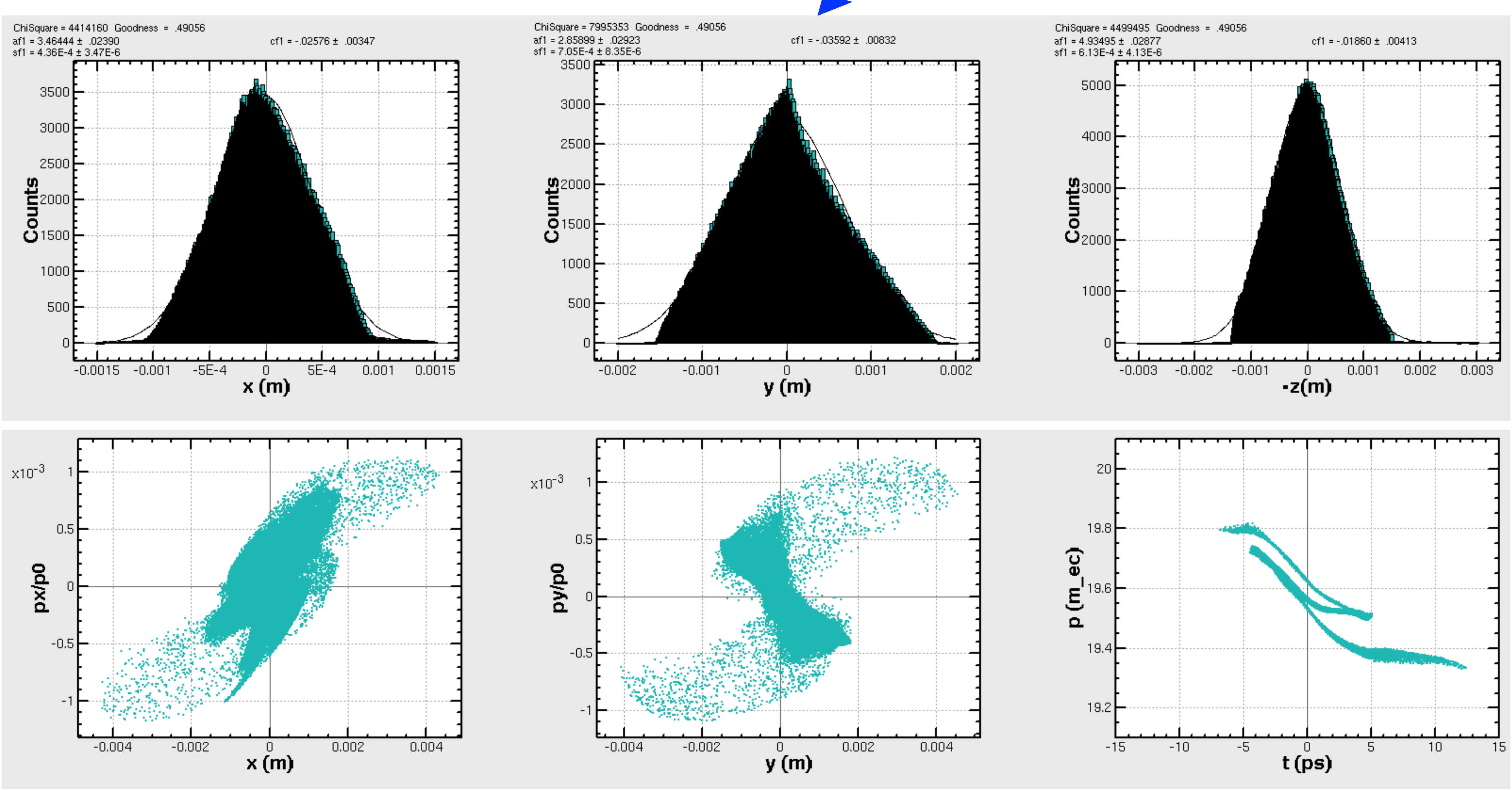
*** Directly using the 3D SC model formulated on page.11 is under plan.**

3. Simulation of SC and CSR effects for cERL-FEL

Tracking simulations using SAD

*** Initial beam distribution**

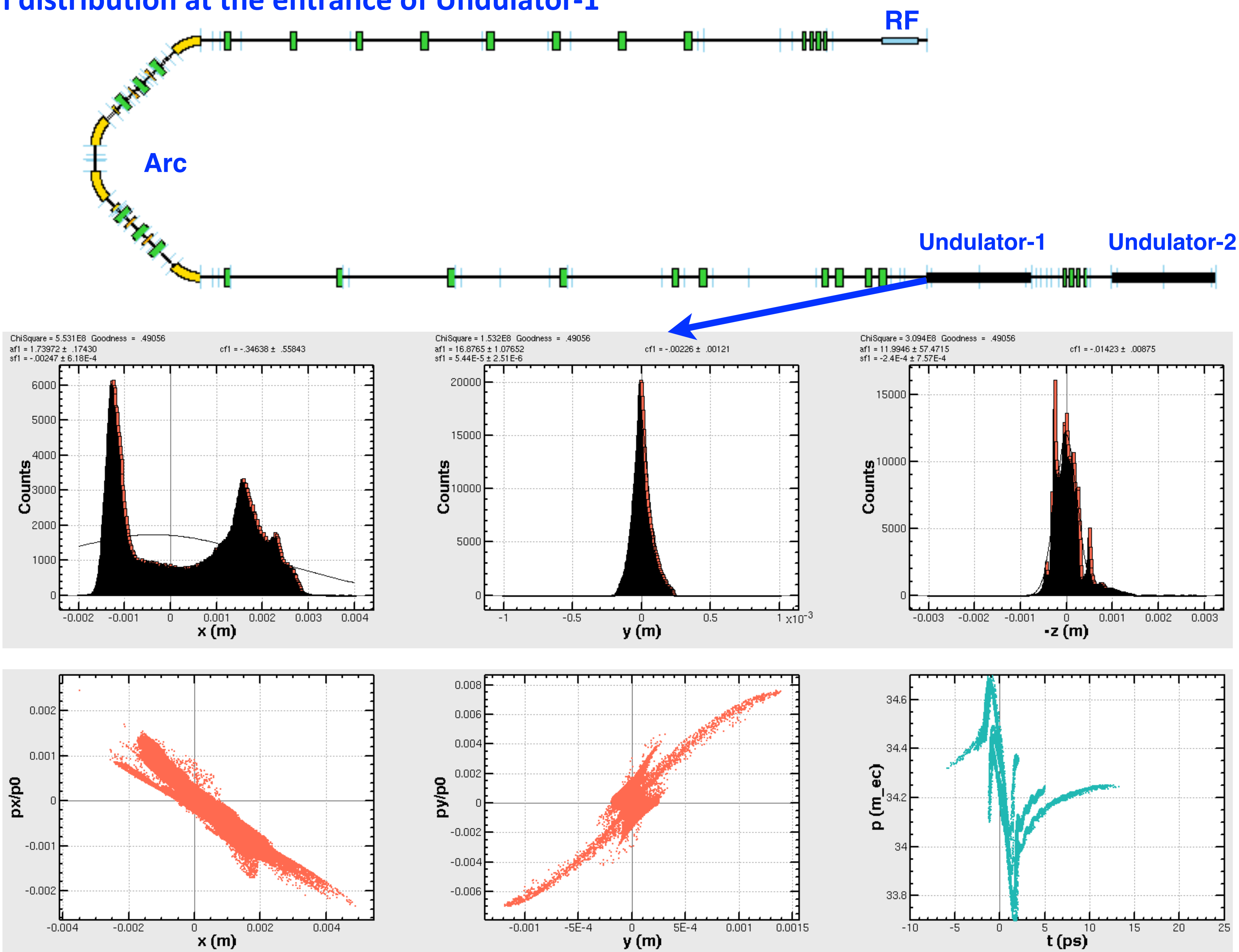




3. Simulation of SC and CSR effects for cERL-FEL

Tracking simulations using SAD

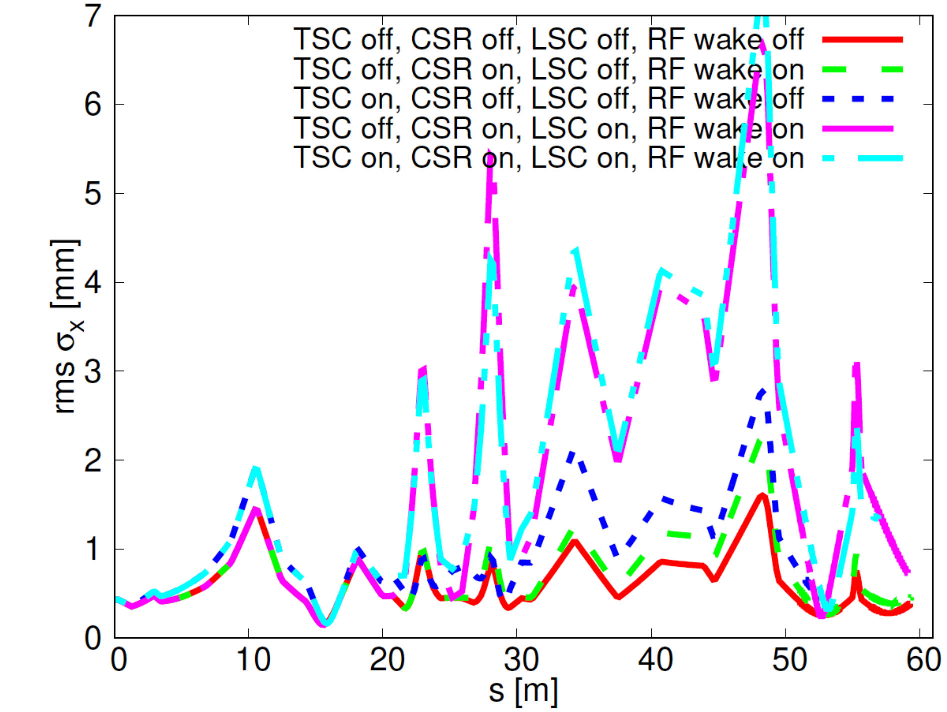
* Beam distribution at the entrance of Undulator-1

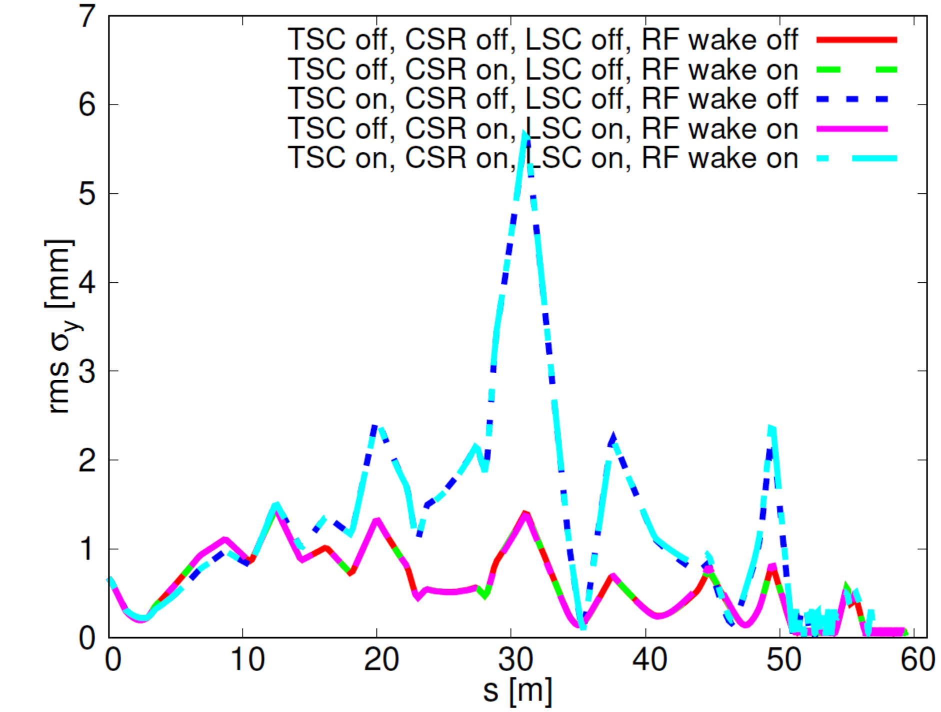


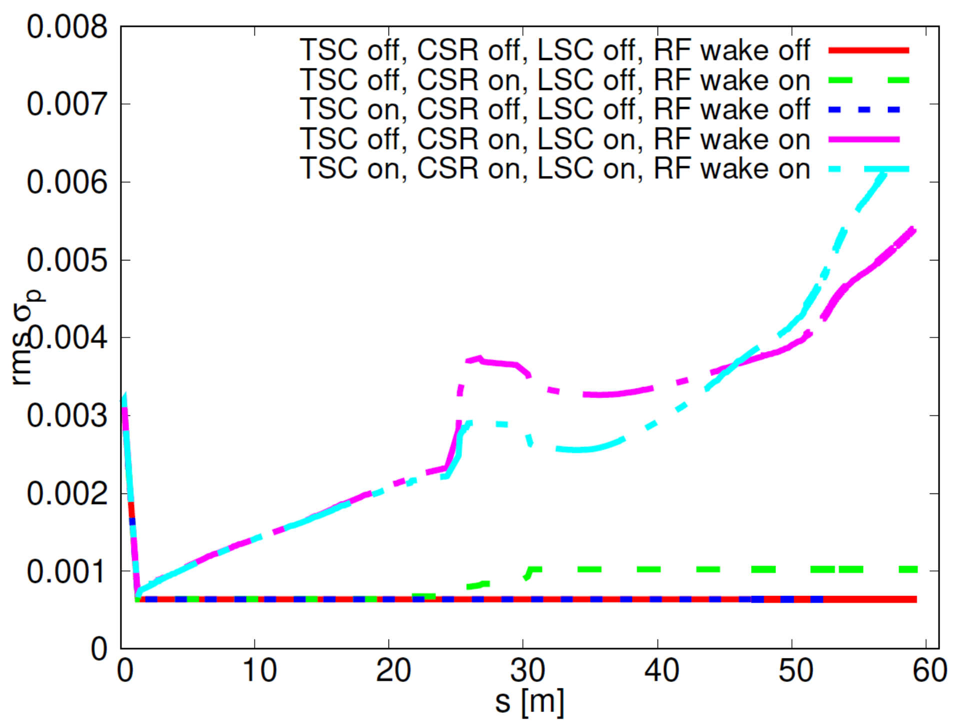
3. Simulation of SC and CSR effects for cERL-FEL

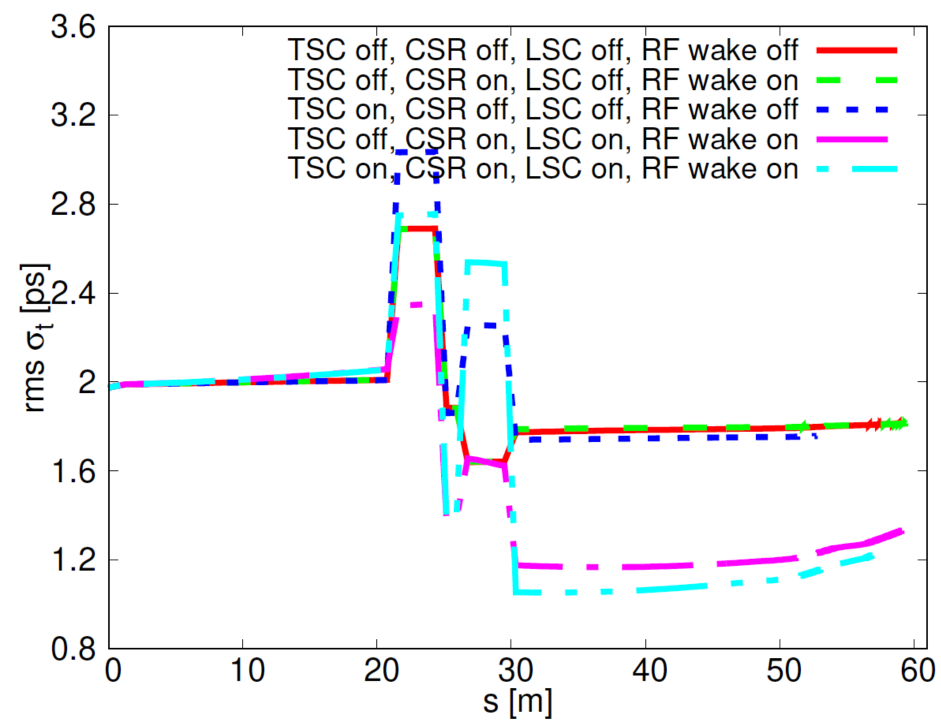
Tracking simulations using SAD

* CSR effects from the arc dipoles are not important in cERL-FEL
* LSC dominates the collective effects; TSC causes vertical beam-size blowup
* There are coupled effects of CSR, TSC and LSC









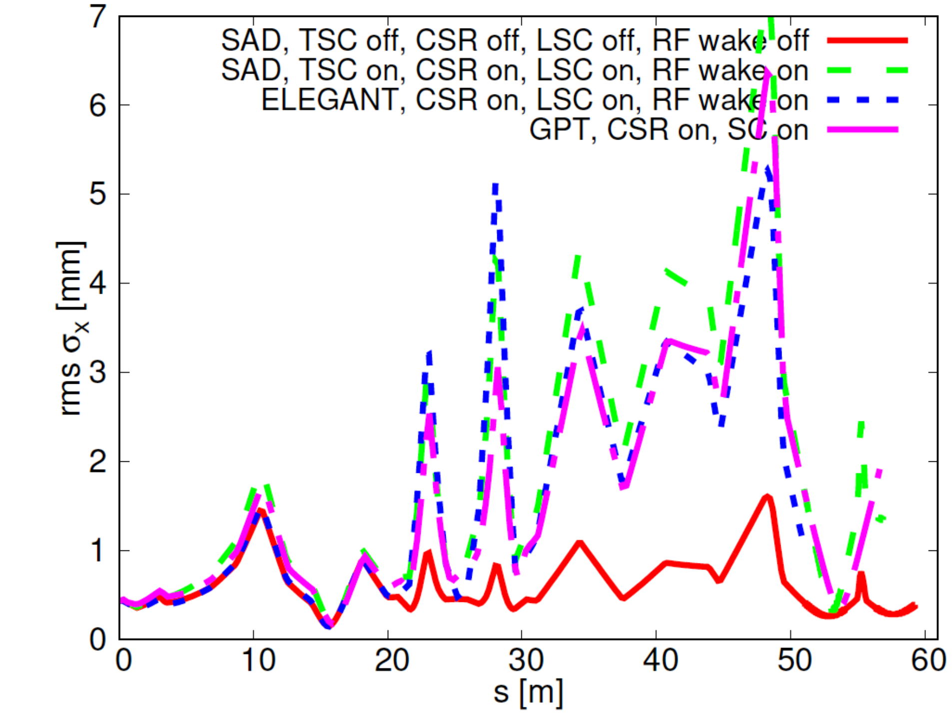
3. Simulation of SC and CSR effects for cERL-FEL

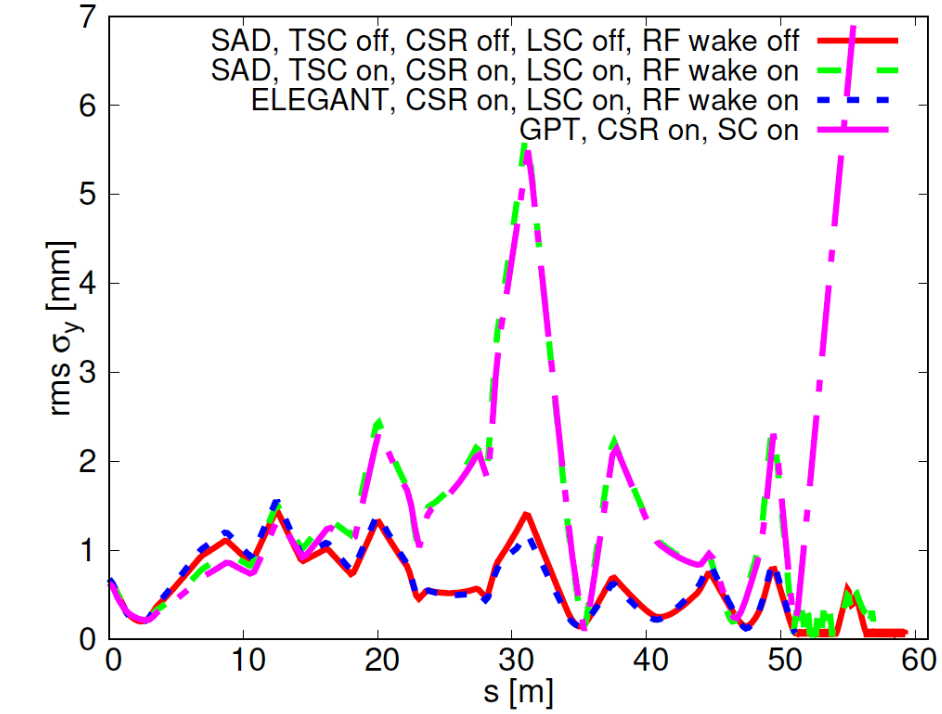
Tracking simulations: Compare SAD, ELEGANT, and GPT codes [11]

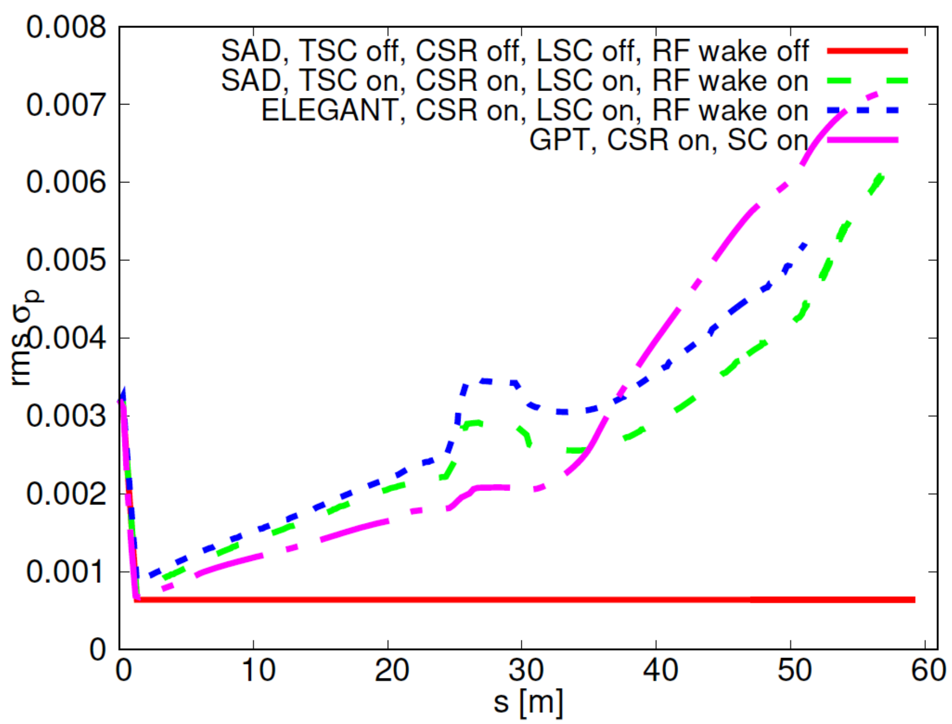
* Evolutions of beam sizes and energy spread along the beam line

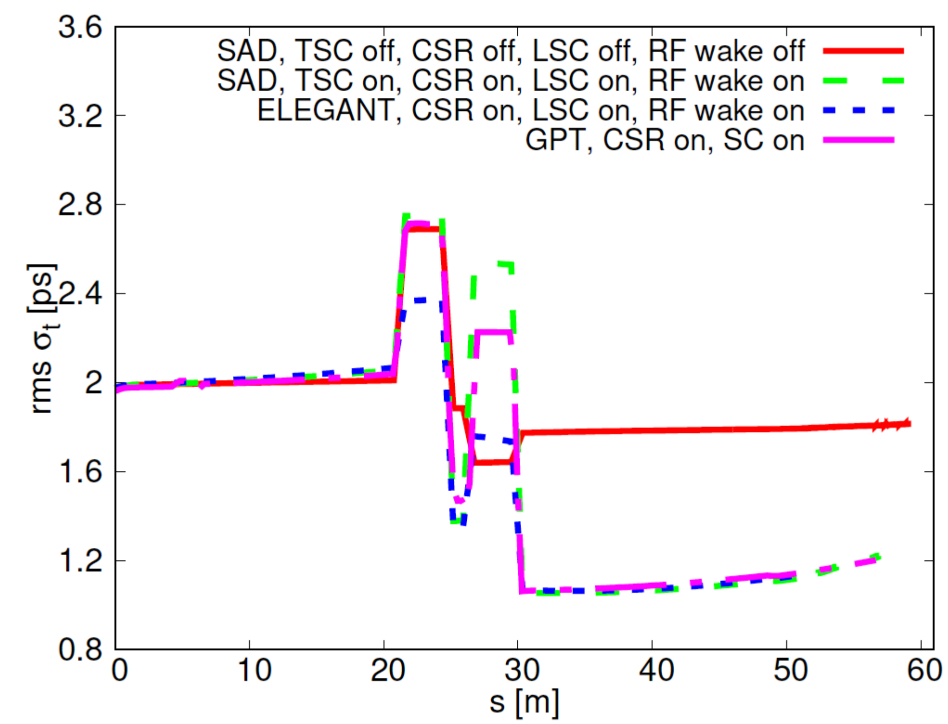
* Fairly good agreements were seen

* The discrepancies can be attributed to differences in models of SC and CSR of the codes









[11] Collaborate with N. Nakamura(ELEGANT) and M. Shimada(GPT)

4. Summary

* Theories of SC impedance and wake functions

** 3D models of SC impedance and wake functions are formulated

** Consistent with the existing theories/models of SC

** Conditions of use for various simple SC models are understood

* Theories and calculation of CSR impedance

** Interference of CSR and SC is a possible issue in low energy machines

** When chamber shielding is important (depend on beam parameters), numerically calculated impedance can be an alternative choice of simulating CSR effects [12]

* Simulation of SC and CSR effects for cERL-FEL

** Impedance models are useful in evaluating the SC and CSR effects

** Fairly good agreements are found between SAD, ELEGANT, and GPT codes

[12] D. Zhou, IPAC2012, MOOBB03