Principles derived from Haissinski equation for electron storage rings

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Connecting Hassinski equation and Zotter's equation

- Haissinski equation [1]
  - Stationary solution of Vlasov-Fokker-Planck equation.
  - Bottom-up predictions of potential-well lengthening.

$$\lambda_0(z) = Ae^{-\frac{z^2}{2\sigma_{z0}^2} - \frac{I}{\sigma_{z0}}\int_z^\infty dz' \int_{-\infty}^\infty W_{||}(z'-z'')\lambda_0(z'')dz''} \qquad I = -Ne^2/(2\pi\nu_s E\sigma_\delta)$$

- Zotter's equation [2]
  - Extraction of effective inductance from bunch length measurements.

$$x^3 - x - D = 0 \qquad D = \frac{cI_bL}{\boxed{4\sqrt{\pi}}\eta\sigma_{z0}\sigma_\delta^2(E/e)}$$

$\sqrt{2\pi}$ was used, but it is incorrect [3].

[1] J. Haissinski, Il NuovoCimento B (1971-1996) 18, 72 (1973).

[2] B. W. Zotter, Potential-well bunch lengthening, CERN-SPS-81-14-DI (1981).

[3] V. Smaluk, NIMA 888, 22 (2018).

Principles derived from Hassinski equation

- Some principles can be trivially derived from Haissinski equation
  - Starting from the differential equation instead

$$\frac{d\lambda_0(z)}{dz} + \left[\frac{z}{\sigma_{z0}^2} - \frac{1}{\eta\sigma_\delta^2}F_0(z)\right]\lambda_0(z) = 0 \longrightarrow \lambda_0(z) = Ae^{-\frac{z^2}{2\sigma_{z0}^2} - \frac{I}{\sigma_{z0}}\int_z^\infty dz'\int_{-\infty}^\infty W_\parallel(z'-z'')\lambda_0(z'')dz''}$$

  - Center of mass: Sensitive to real part of impedance

$$z_c = \int_{-\infty}^{\infty} z\lambda_0(z)dz \longrightarrow z_c(I) = I\sigma_{z0}\kappa_\parallel \qquad \kappa_\parallel = \int_{-\infty}^{\infty} dz\lambda_0(z)\mathbb{W}_\parallel(z)$$

$$\mathbb{W}_\parallel(z) = \int_{-\infty}^{\infty} W_\parallel(z-z')\lambda_0(z')dz' \qquad \kappa_\parallel = \frac{c}{\pi}\int_0^\infty \mathbf{Re}[Z_\parallel(k)]h(k)dk$$

Principles derived from Hassinski equation

- Some principles can be trivially derived from Haissinski equation
  - Peak position of bunch profile: Sensitive to real part of impedance

$$\frac{d\lambda_0(z)}{dz} = 0 \longrightarrow z_m = I\sigma_{z0}\mathbb{W}_{\|}(z_m)$$

  - rms bunch length: Sensitive to imaginary part of impedance

$$\sigma_z^2 = \int_{-\infty}^{\infty} (z - z_c)^2 \lambda_0(z) dz = \int_{-\infty}^{\infty} z^2 \lambda_0(z) dz - z_c^2$$

$$x^2 - 1 - \frac{cI}{2\pi\sigma_{z0}} Z_{\|}^{\mathbf{eff}}(x) = 0$$

Exact given Haissinski equation

$$Z_{\|}^{\mathbf{eff}} = \frac{2\pi}{c}\int_{-\infty}^{\infty} dz (z - z_c)\lambda_0(z)\mathbb{W}_{\|}(z)$$

$$Z_{\|}^{\mathbf{eff}} = -\int_{-\infty}^{\infty} dk Z_{\|}(k)\tilde{\lambda}_0(k)\left[i\frac{d}{dk}\tilde{\lambda}_0^*(k) + z_c\tilde{\lambda}_0^*(k)\right]$$

"Effective impedance" for bunch lengthening

Principles derived from Hassinski equation

- Some principles can be trivially derived from Haissinski equation
  - Zotter's equation can be easily obtained with conditions of pure inductance $Z_{\parallel}(k) = -ikcL$ and Gaussian bunch profile. Exactly, the numerical coefficient should be $4\sqrt{\pi}$.

$$x^2 - 1 - \frac{cI}{2\pi\sigma_{z0}} Z_{\parallel}^{\textbf{eff}}(x) = 0$$

$$\downarrow$$

$$x^3 - x - D = 0 \qquad D = \frac{cI_bL}{\boxed{4\sqrt{\pi}}\,\eta\sigma_{z0}\sigma_\delta^2(E/e)}$$

  - Zotter's equation is one special case of self-consistent quadratic equation. But it is only a good approximation for electron storage rings where the inductance is the dominant impedance.

Principles derived from Hassinski equation

- Bunch lengthening for some impedance models with Gaussian bunch approximation
  - Bunch shortening for positive momentum compaction: Pure capacitance, Free-space steady-state CSR and CWR

TABLE I. Effective impedances for a Gaussian bunch with specific impedance forms.

Description	Impedances $Z_\parallel(k)$	Effective impedances $Z_\parallel^{\mathrm{eff}}(x)$
Pure inductance	$-ikcL$	$\frac{\sqrt{\pi}cL}{2\sigma_{z0}x}$
Pure resistance	R	$-\frac{IcR^2}{2x^2}$
Pure capacitance	$\frac{i}{kcC}$	$-\frac{\sqrt{\pi}\sigma_{z0}x}{cC}$
Resistive wall	$\frac{L}{2\pi b}\left[1-i\,\mathrm{sgn}[k]\right]\sqrt{\frac{\lvert k\rvert Z_0}{2\sigma_c}}$	$\frac{L\Gamma\left(\frac{5}{4}\right)}{2\pi b\sqrt{x\sigma_{z0}}}\sqrt{\frac{Z_0}{2\sigma_c}}$
	L: chamber length; b: chamber radius; σ_c: Conductivity.	
Free-space steady-state CSR	$\frac{Z_0}{3^{1/3}}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\Gamma\left(\frac{2}{3}\right)(k\rho)^{1/3}$	$-\frac{Z_0\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{7}{6}\right)}{2\cdot 3^{1/3}}\left(\frac{\rho}{x\sigma_{z0}}\right)^{1/3}$
	ρ: Bending radius; total length of dipoles is $2\pi\rho$.	
Free-space steady-state CWR	$\frac{1}{16}Z_0\theta_0^2Lk\left(1-\frac{2i}{\pi}\ln\frac{k}{k_c}\right)$	$-\frac{Z_0\theta_0^2L}{32\sqrt{\pi}x\sigma_{z0}}\left[Y+2\ln\left(k_cx\sigma_{z0}\right)\right]$
	L: wiggler length; θ_0: wiggler deflection angle; k_c: fundamental frequency of wiggler radiation; $Y=-2+\gamma_E+\ln 4\approx -0.0365$.	

Principles derived from Hassinski equation

- Inverse problem of Haissinski equation
  - Wake potential extracted from simulated or measured bunch profile

$$\lambda_0(z) = Ae^{-\frac{z^2}{2\sigma_{z0}^2}-\frac{I}{\sigma_{z0}}\int_z^\infty dz' \int_{-\infty}^\infty W_\parallel(z'-z'')\lambda_0(z'')dz''}$$

$$\downarrow$$

$$\mathbb{W}_\parallel(z) = \frac{\sigma_{z0}}{I}\left[\frac{d\ln\lambda_0(z)}{dz} + \frac{z}{\sigma_{z0}^2}\right]$$

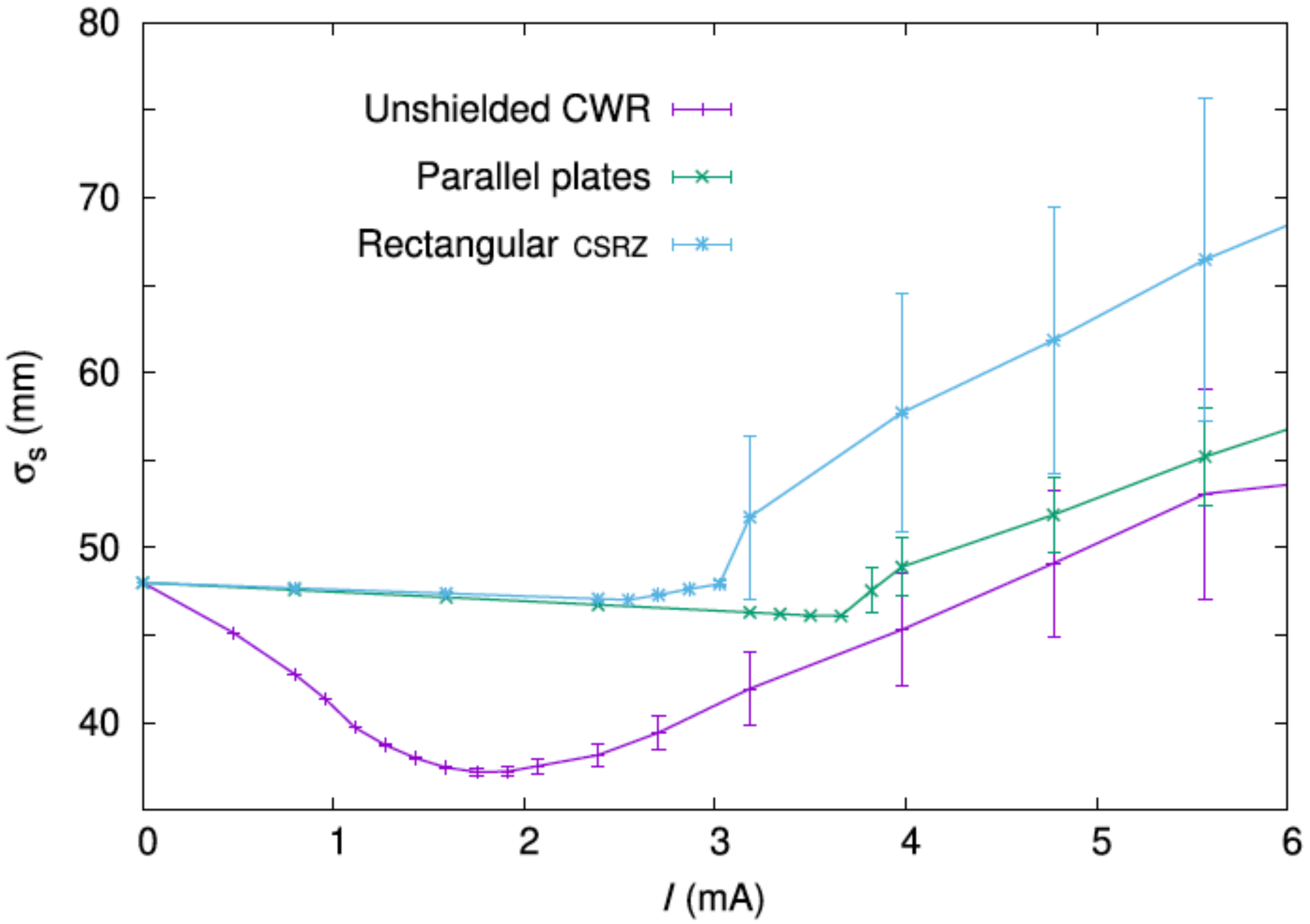
  - Impedance extracted from wake potential [1]

$$Z_\parallel(k) = \frac{\sigma_{z0}}{Ic^2\tilde{\lambda}_0(k)}\int_{-\infty}^{\infty}\left[\frac{d\ln\lambda_0(z)}{dz} + \frac{z}{\sigma_{z0}^2}\right]e^{-ikz}dz$$

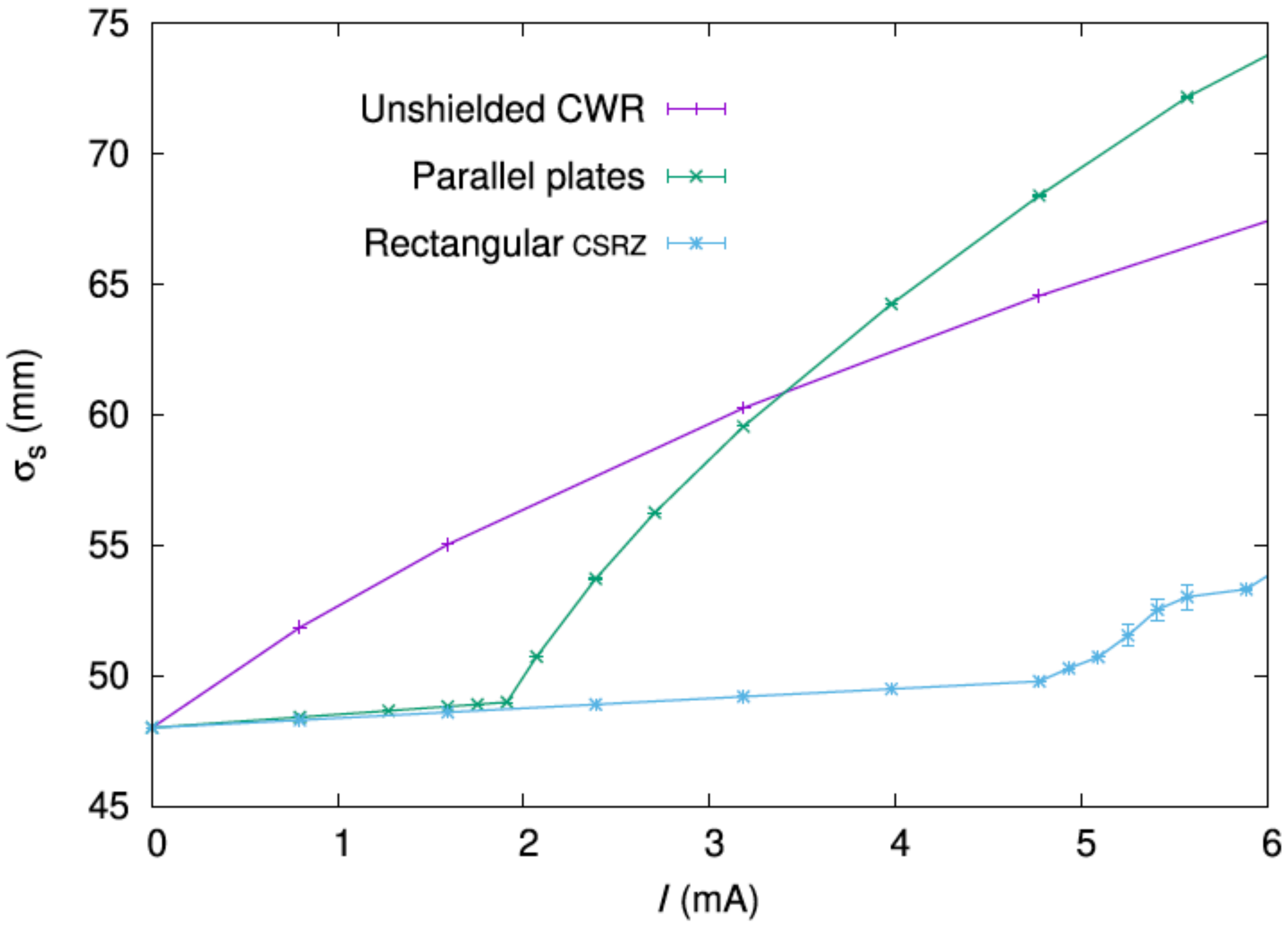
[1] A. Chao, Lectures on Accelerator Physics (World Scientific, 2020)

Some practical examples

- Bunch shortening by free-space CSR/CWR
  - Significant bunch shortening/lengthening for positive/negative momentum compaction in simulations for EIC ring electron cooler [1]
  - Practically, chamber shielding suppresses low-frequency CSR/CWR, such effects have not be observed in measurements



Positive α_c



Negative α_c

[1] A. Blednykh et al., PRAB 26, 051002 (2023).

Some practical examples

- Using BPM to measure beam phase
  - It's not trivial to detect the center of mass using BPM signals
  - Detecting the peak position of the bunch profile using BPM signals (zero-cross point) was developed at SuperKEKB

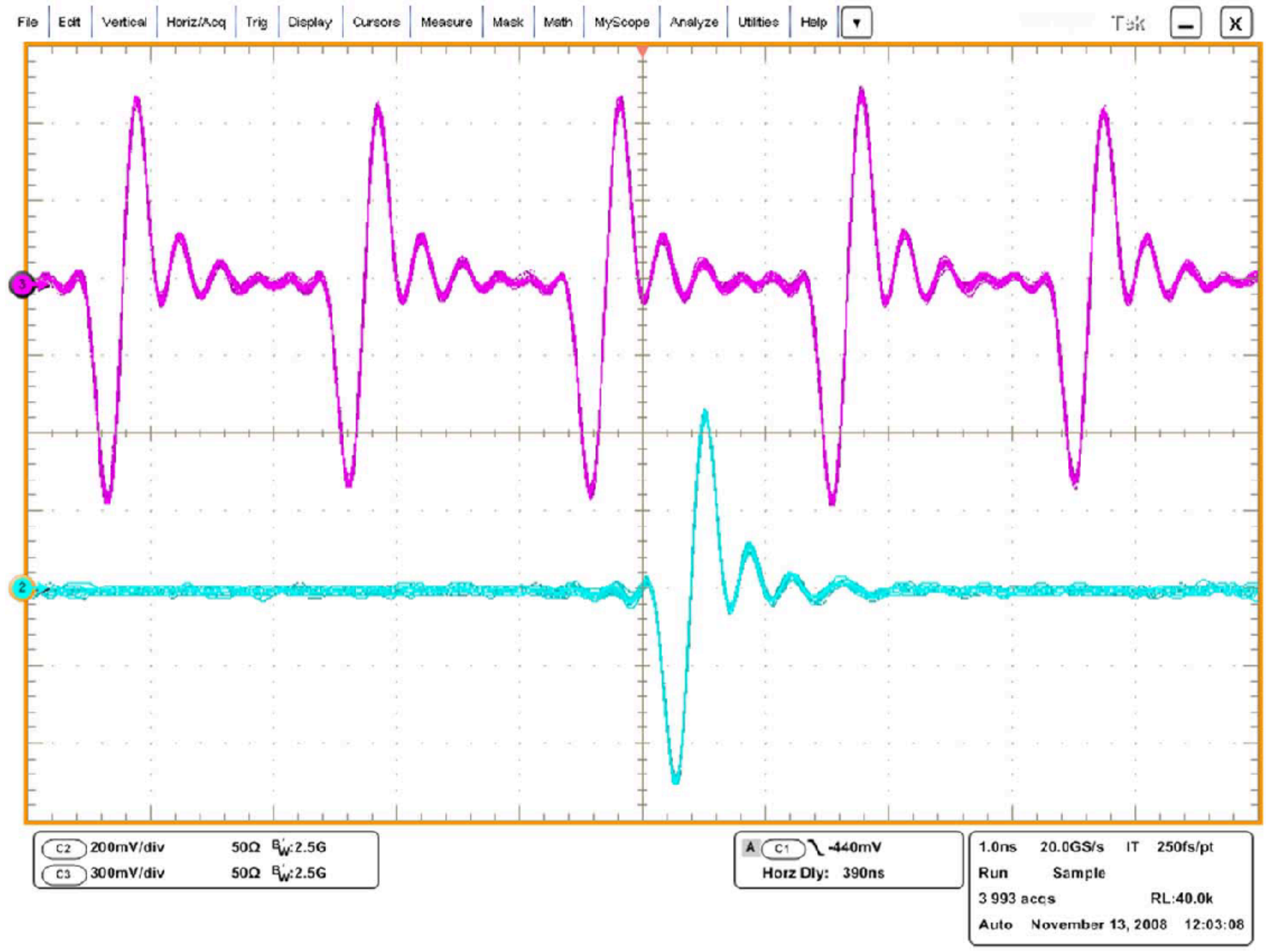
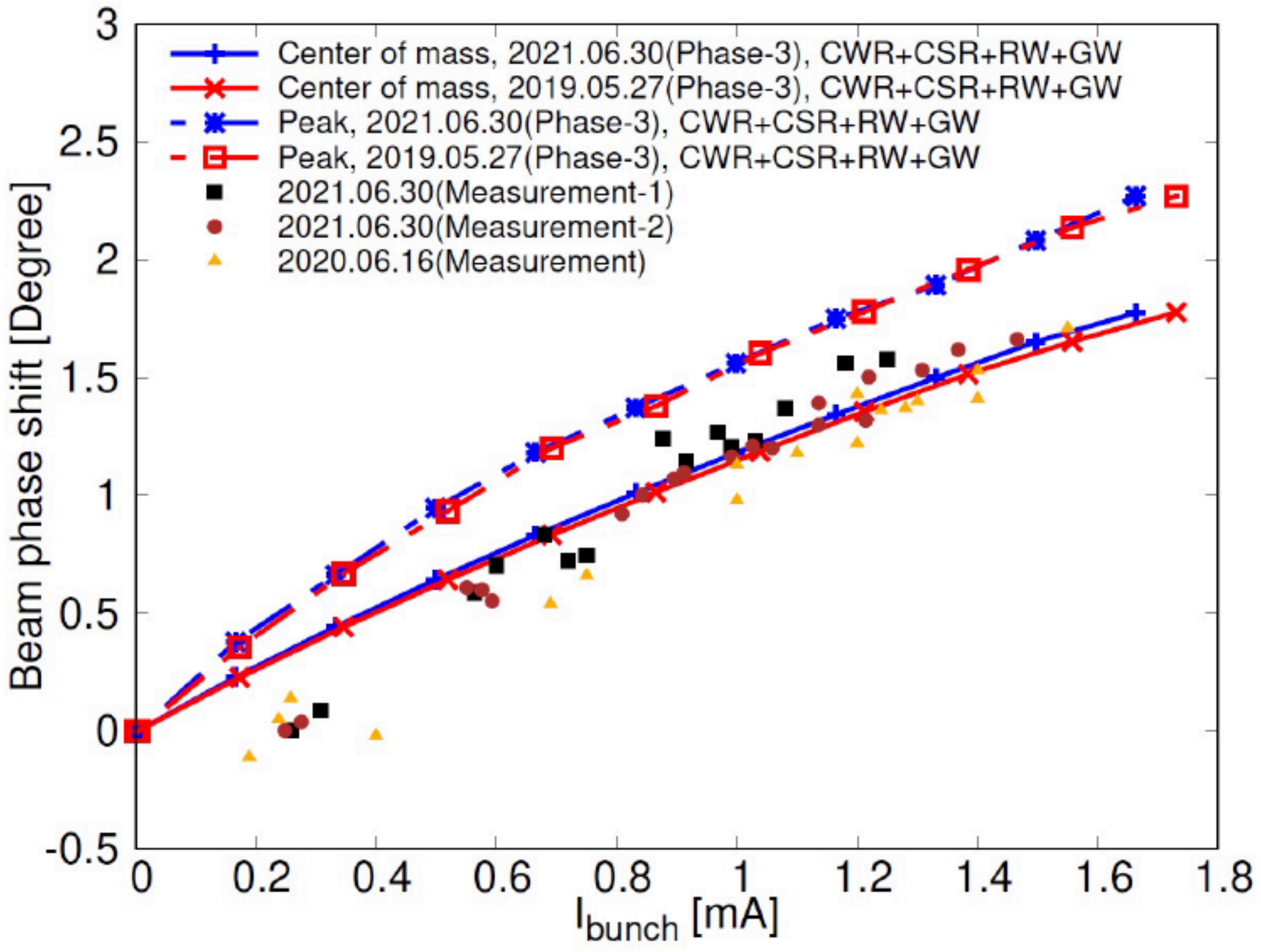
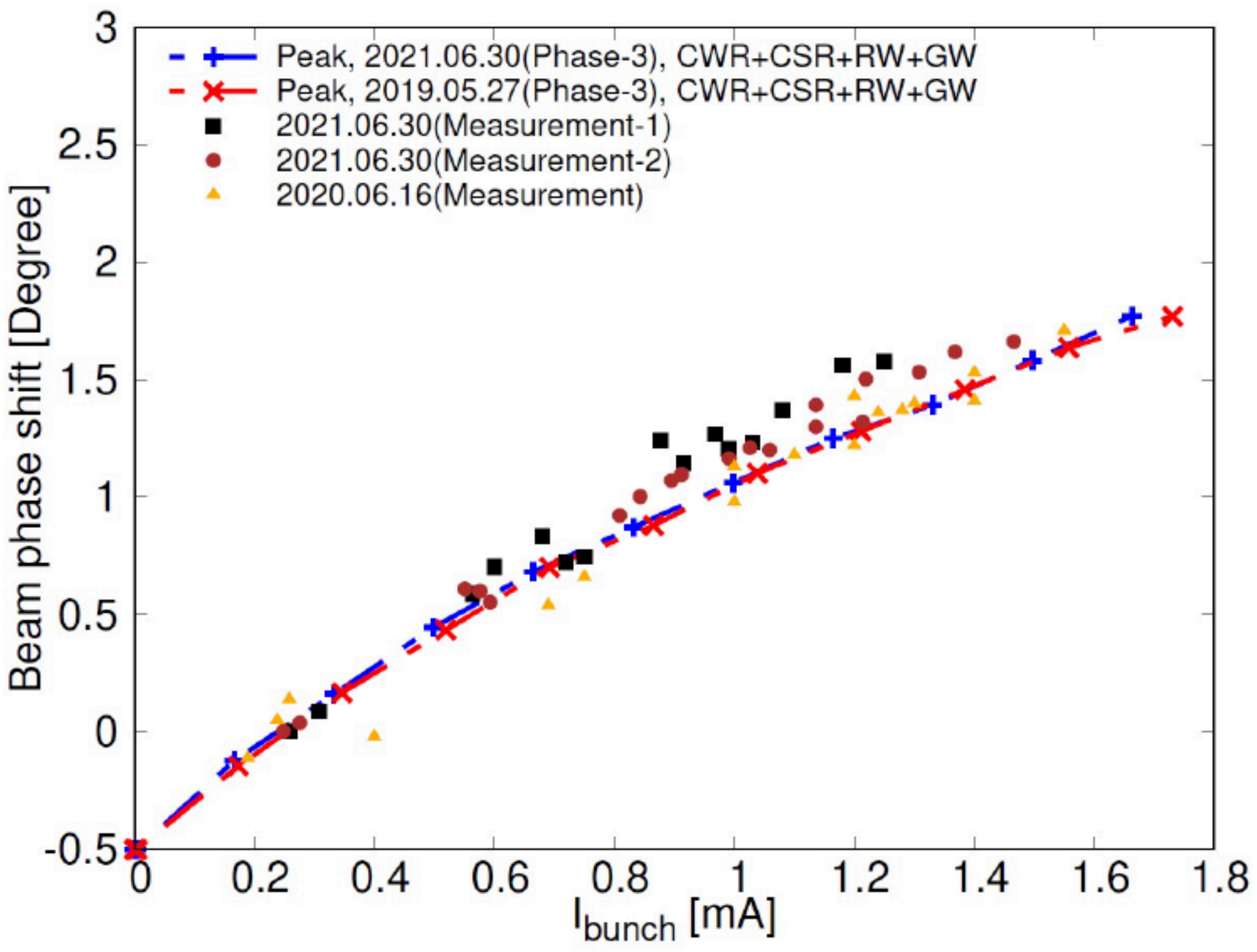


Fig. 10. Bunch signal of the electron beam picked-up by a button electrode. Upper trace corresponds to a part of a bunch train with a spacing of 2 ns, the bunch current is 0.5 mA. Lower trace corresponds to a gated bunch signal. The horizontal scale is 10 ns in full range or 1 ns/div. and the vertical scale is 300 mV/div. for the upper trace and 200 mV/div. for the lower one.

BPM signal measured at KEKB in 2008 [1]



Simulated center of mass did not fit the measured data well

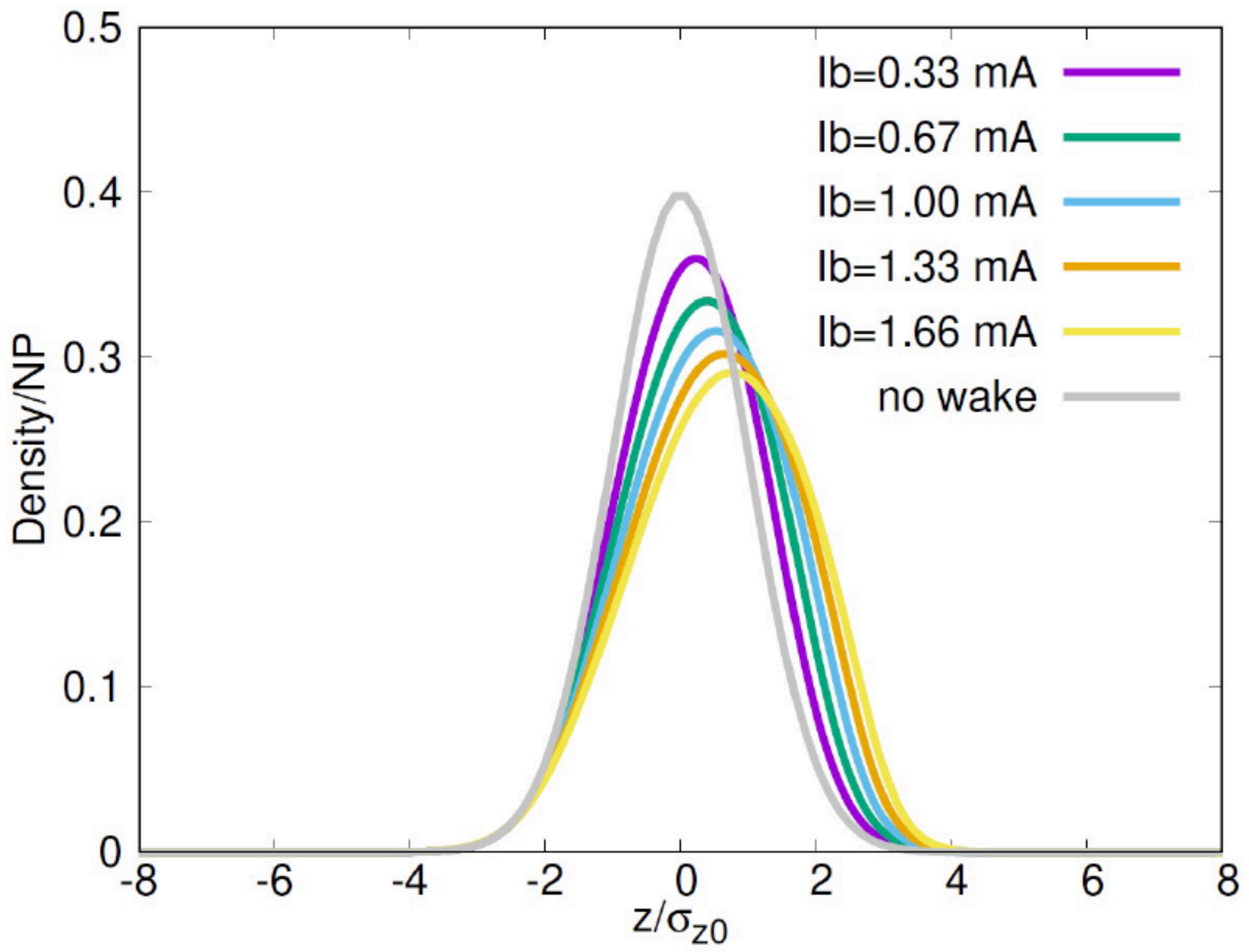


Simulated peak position did fit the measured data well, especially at low bunch currents

[1] T. Ieiri et al., NIMA 606 (2009) 248–256.

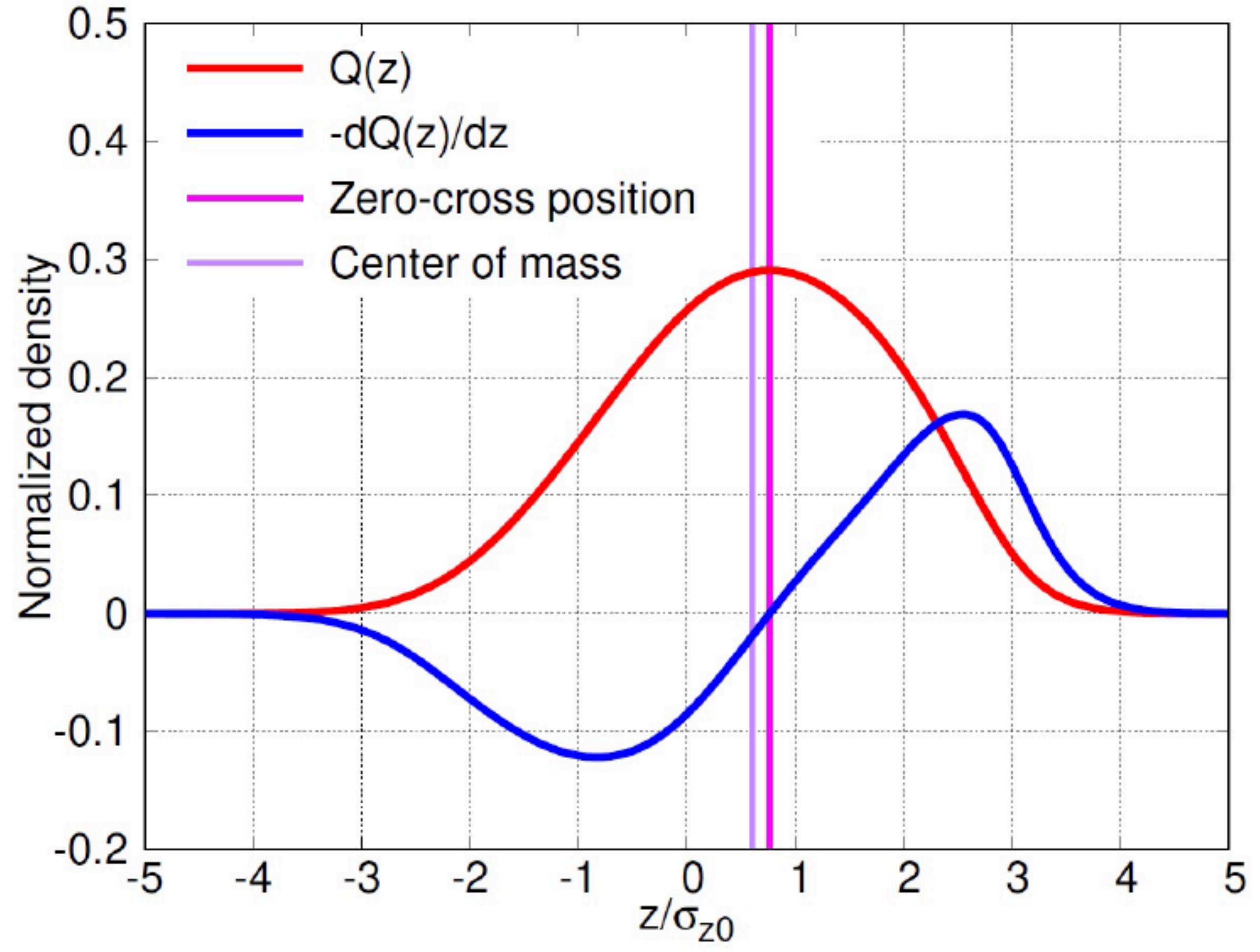
Some practical examples

- Using BPM to measure beam phase
  - It's not trivial to detect the center of mass using BPM signals
  - Detecting the peak position of the bunch profile using BPM signals (zero-cross point) was developed at SuperKEKB



Simulated bunch profiles

BPM signal



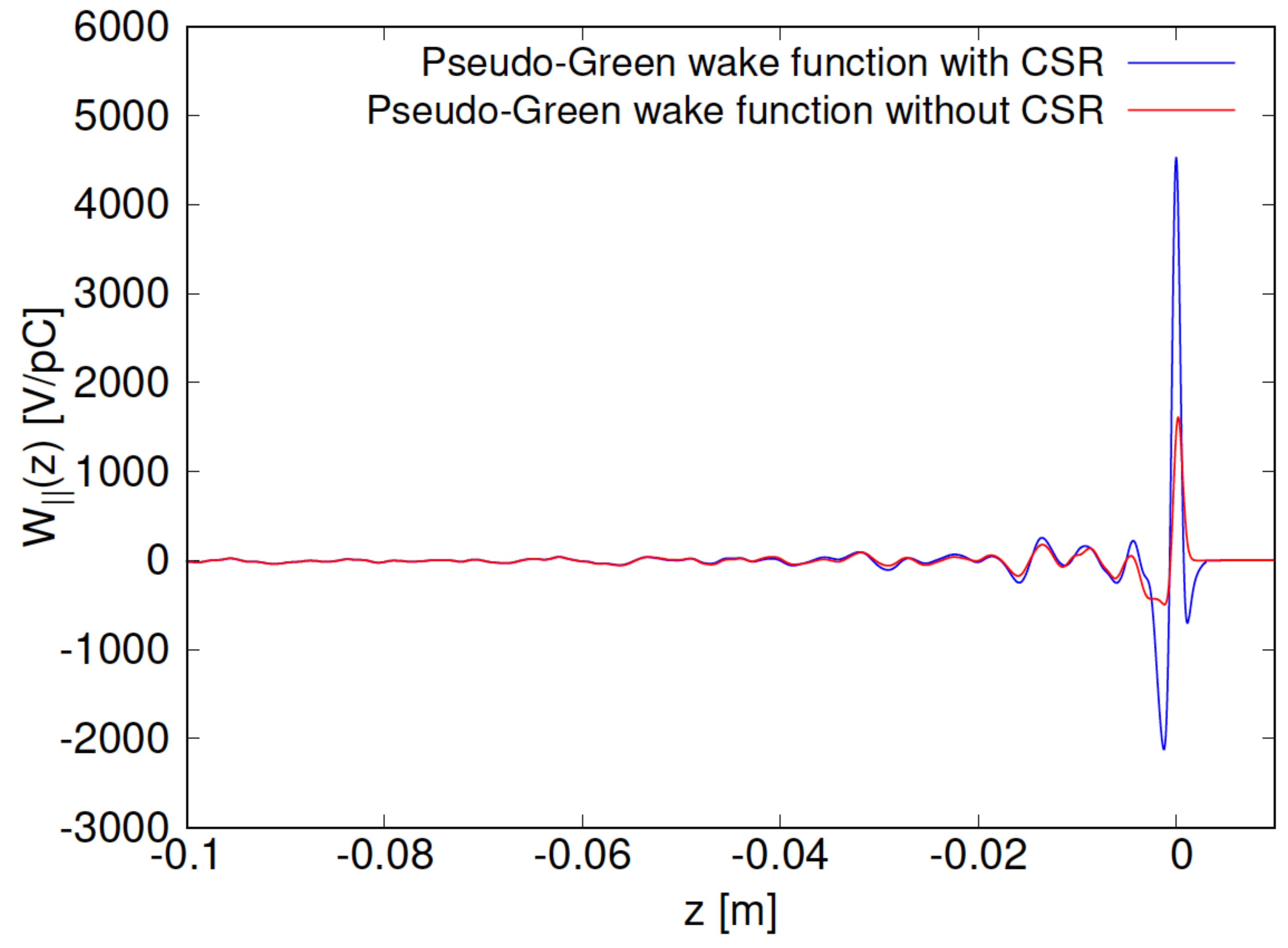
Simulated bunch profiles have tilt and shift in center-of-mass.
BPM signal: $\boldsymbol{i(t) = -dQ(t)/dt}$
-> Zero-cross point of BPM signal is more relevant to peak of bunch profile

Indeed the transport line of BPM system filters the signal from buttons. It such filtering is well understood, details of bunch profile could be extracted (Discussion with B. Pedobedov)

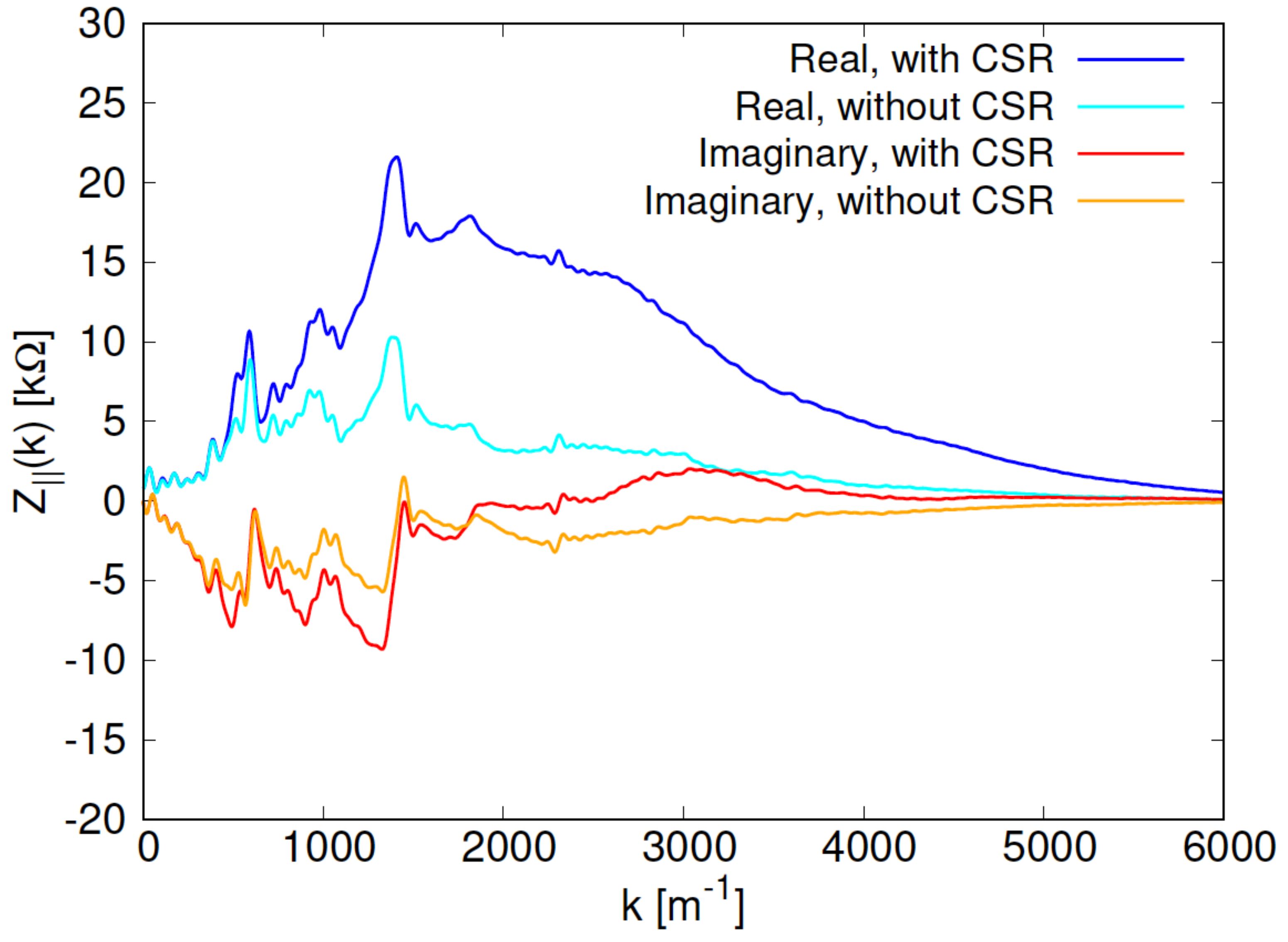
→ A challenge to monitor experts

Some practical examples

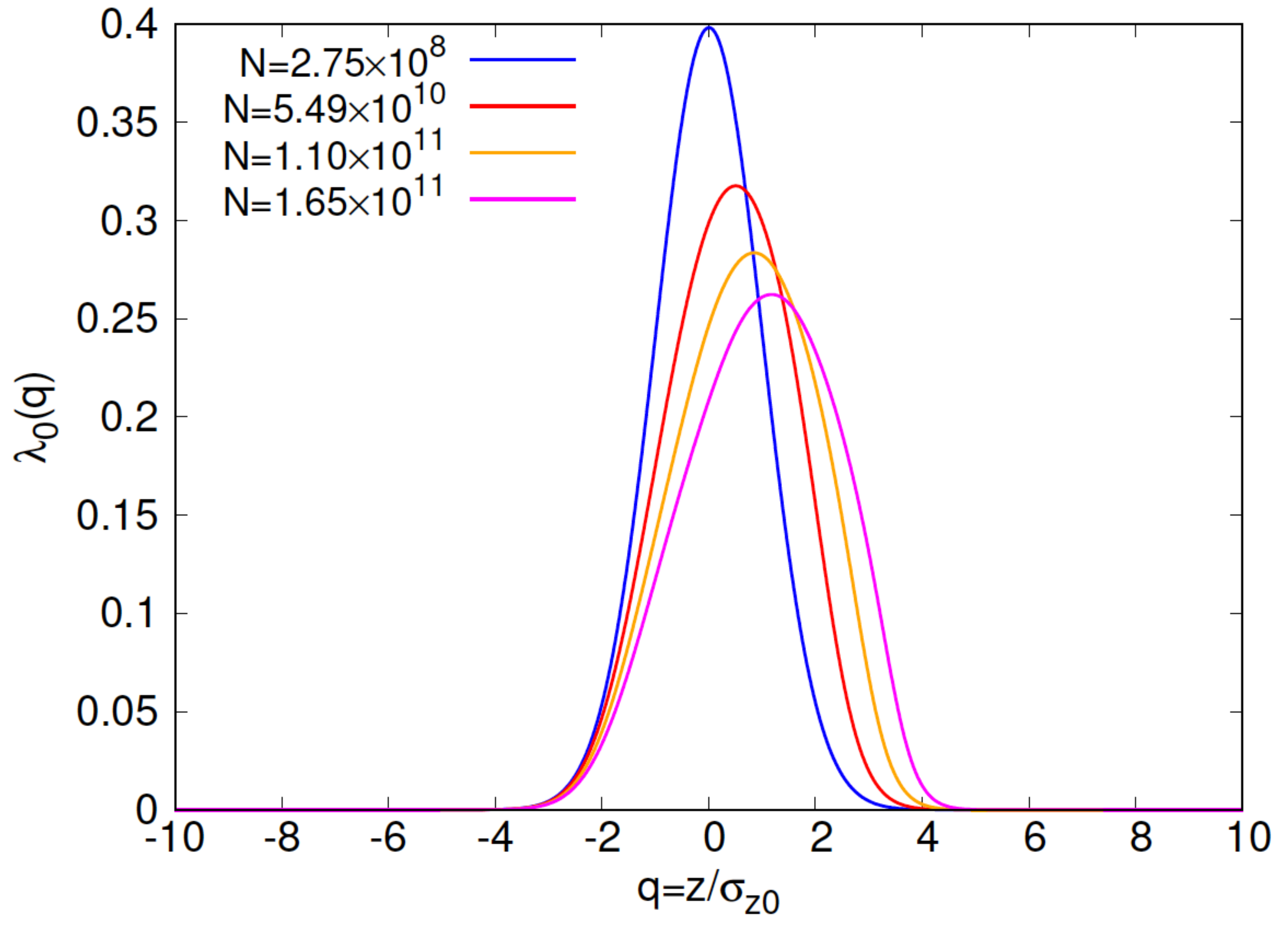
- SuperKEKB LER
  - Pseudo-Green function wakes constructed and used inputs of simulations



Pseudo-Green function wakes with 0.5 mm Gaussian bunch



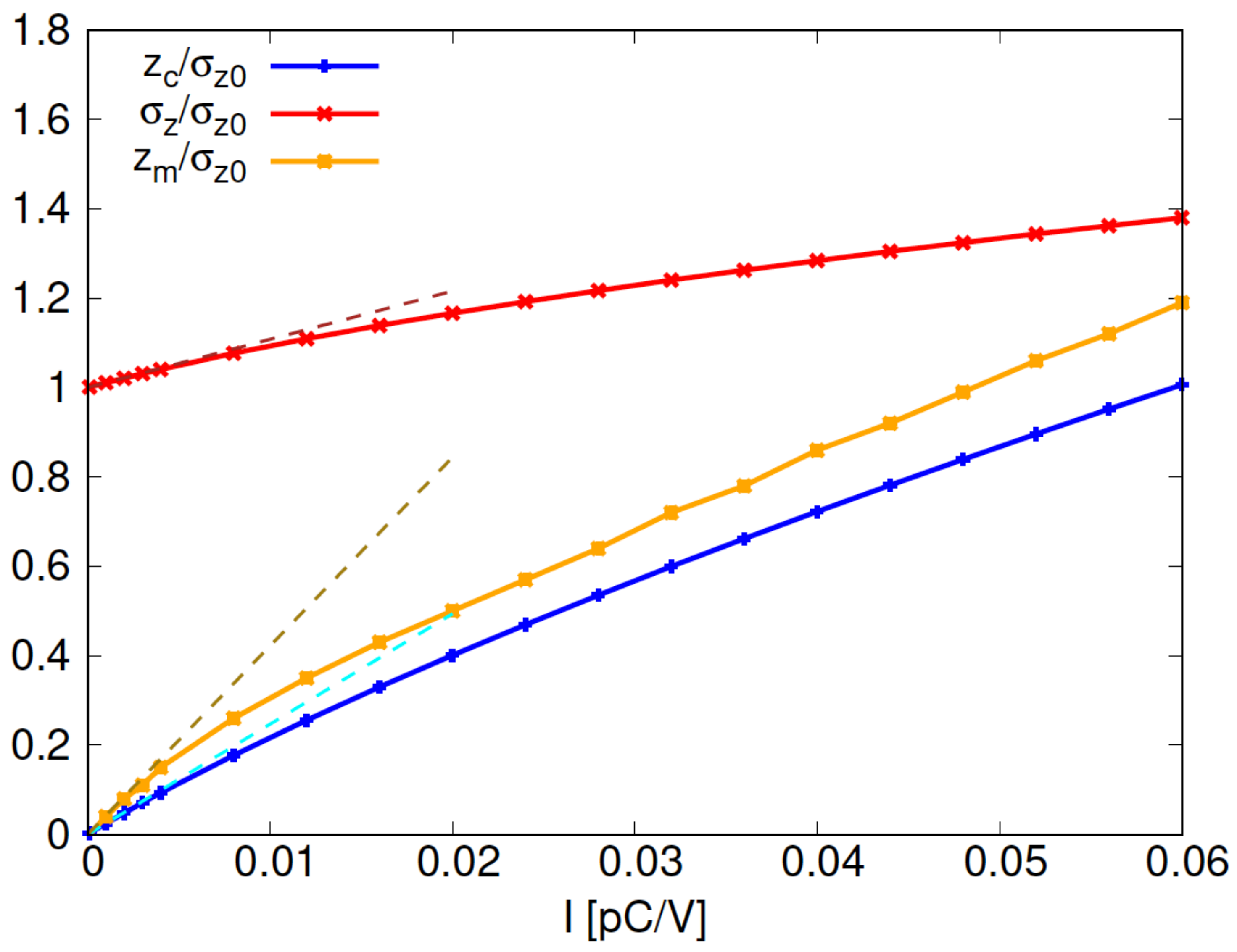
Fourier transform of short-bunch wakes



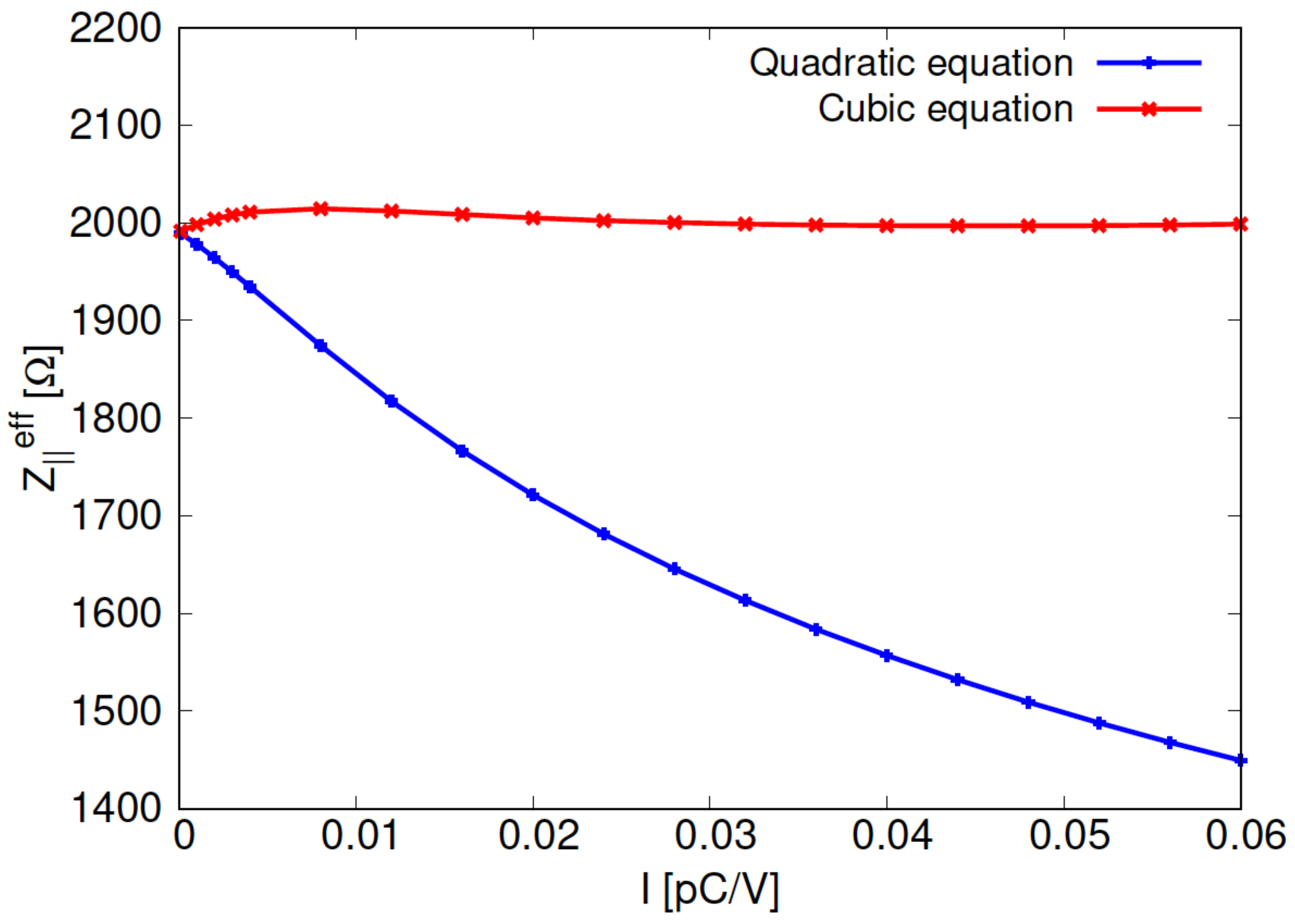
Haissinski solutions

Some practical examples

- SuperKEKB LER
  - Pseudo-Green function wakes constructed and used inputs of simulations



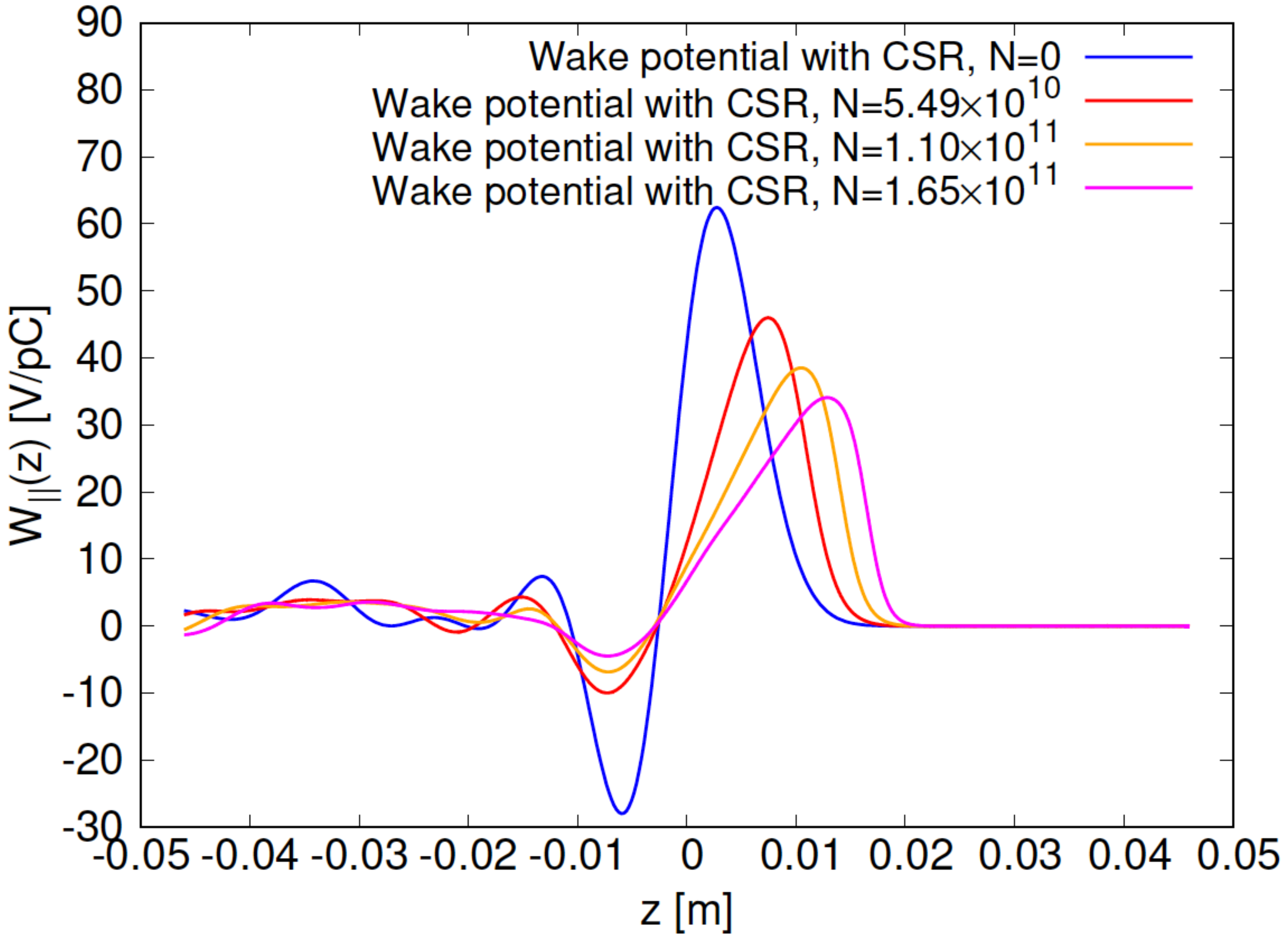
The slopes at zero current have clear meanings with given impedance and nominal bunch



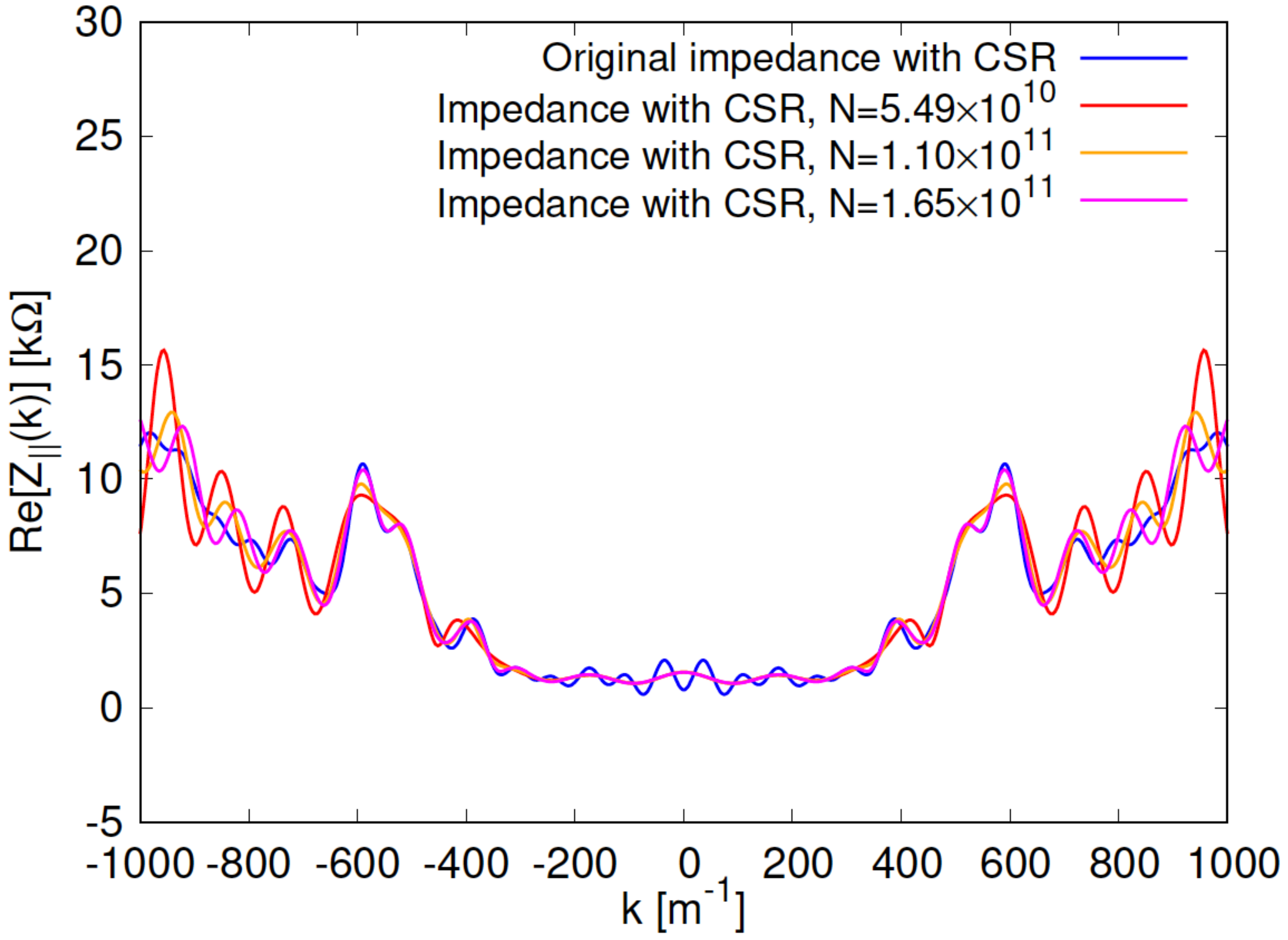
Effective impedance shows machine properties

Some practical examples

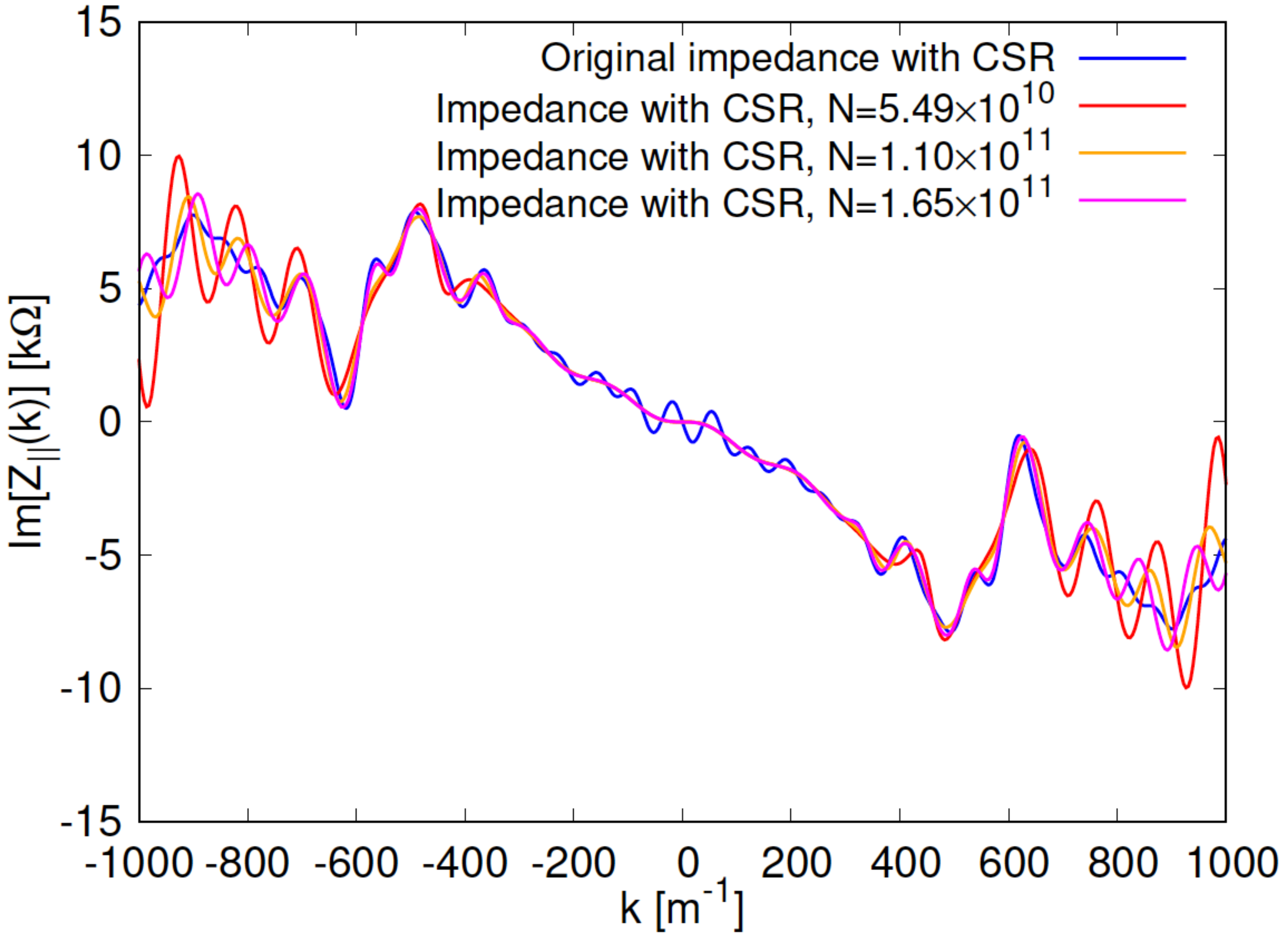
- SuperKEKB LER
  - Pseudo-Green function wakes constructed and used inputs of simulations



Wake potential with different bunch profiles



Real part of impedance extracted from Haissinski solutions



Imaginary part of impedance extracted from Haissinski solutions

Summary

- Simple scaling laws are derived from Haissinski equation, useful for correlating impedance computations with beam-based measurements
- It is possible to extract impedance information from BPM-based measurements
- In addition to bunch lengthening, streak camera measurements can provide details of ring impedances