Theories on beam-beam effects - I

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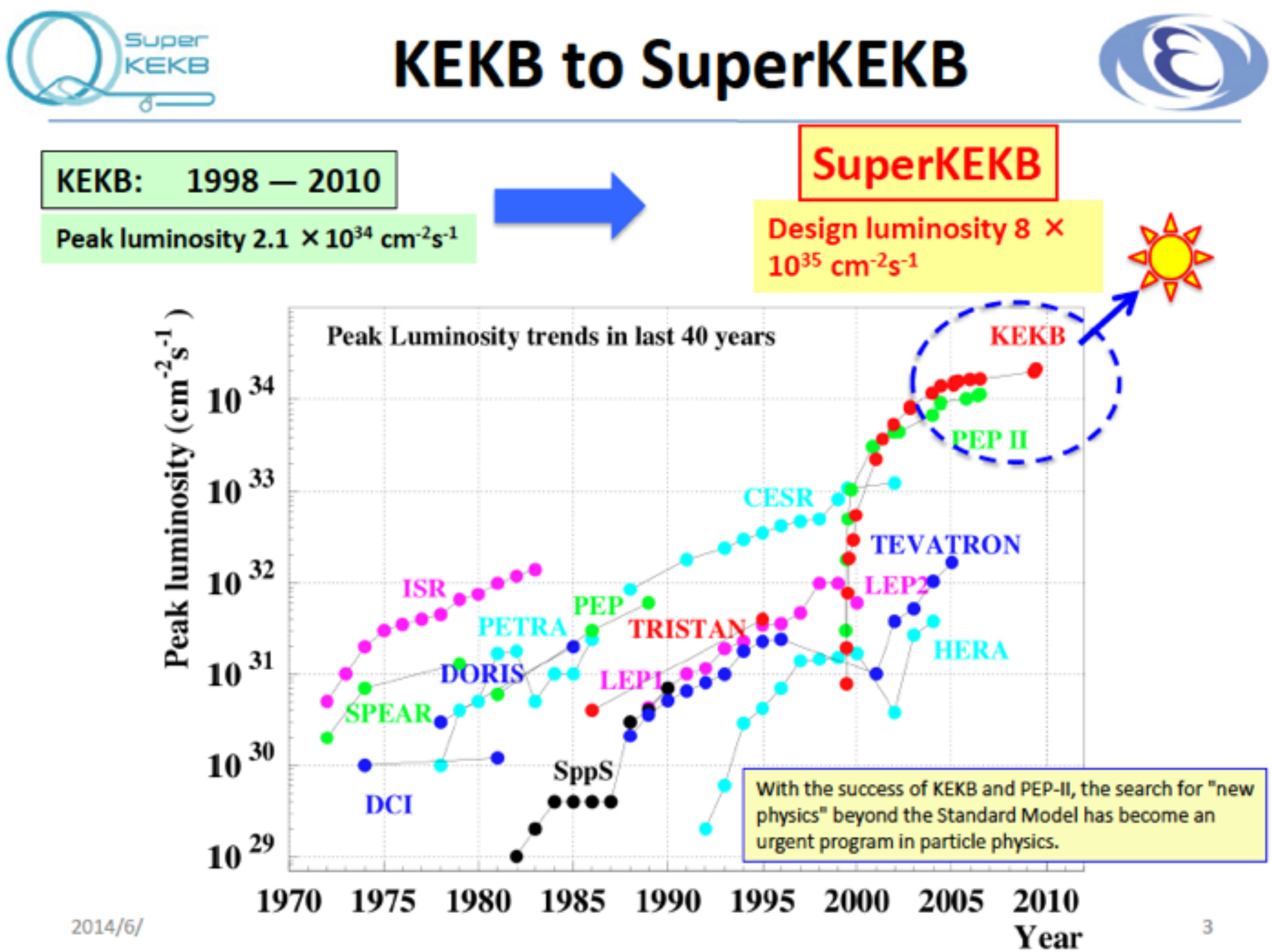
Outline

- ➤ **Introduction**
- ➤ **Physics of beam-beam collision (This talk)**
  - Electric potential and fields of a bunched beam
  - Beam-beam kick and tune shifts
  - Hamiltonian for beam-beam interaction
  - Amplitude-dependent beam-beam tune shifts
  - Hourglass effects (if possible)
- ➤ **Nonlinear beam-beam resonances (Next talk)**
- ➤ **Beam-beam simulations (Next next talk)**

1. Introduction

➤ **Beam-beam interaction is always important in a collider**



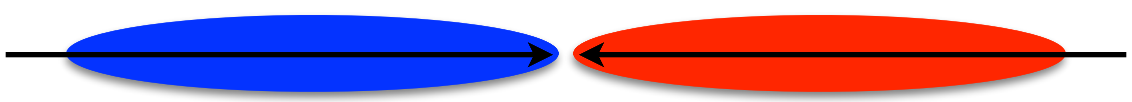
From N. Ohuchi, IPAC14, WEOCA01

1. Introduction

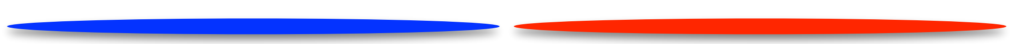
➤ Collision schemes

Head-on

X-Z plane

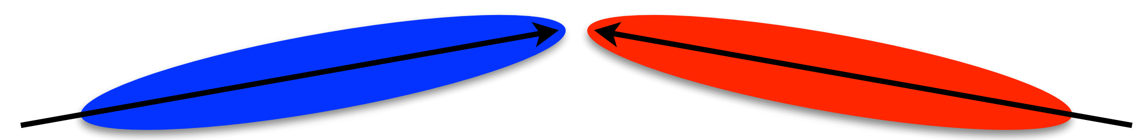


Y-Z plane



Crossing angle

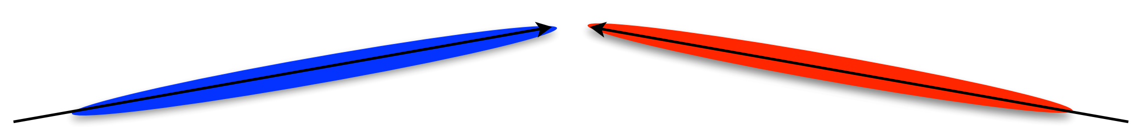
X-Z plane



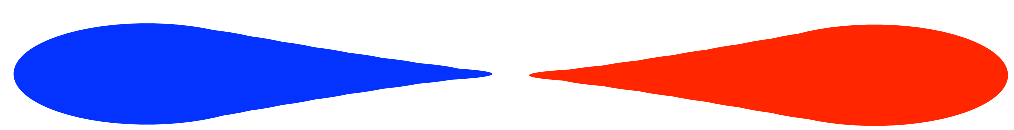
Y-Z plane

“Nano-beam”

X-Z plane



Y-Z plane



Hourglass effect

2. Electric potential (beam frame)

➤ Fundamental equations

- **Beam frame**
- **Poisson's equation**
- **Solution: Green's function or Fourier transform**

$$\nabla^2 \Phi_+(\vec{r}_+) = -\frac{1}{\epsilon_0}\rho_+(\vec{r}_+)$$

General solution based on Green's function:

$$\Phi_+(\vec{r}_+) = \frac{1}{4\pi^{3/2}\epsilon_0}\int_0^\infty dt \frac{1}{t^{3/2}} \int\int\int_{-\infty}^{\infty} d\vec{r}'\rho(\vec{r}')e^{-\frac{|\vec{r}_+ - \vec{r}'|^2}{t}}$$

2. Electric potential (beam frame)

3D Gaussian distribution (Green's function):

$$\rho_+(\vec{r}_+) = \frac{N_+ e_+}{(2\pi)^{3/2}\sigma_{x+}\sigma_{y+}\sigma_{z+}} e^{-\frac{x_+^2}{2\sigma_{x+}^2}-\frac{y_+^2}{2\sigma_{y+}^2}-\frac{z_+^2}{2\sigma_{z+}^2}}$$

$$\Phi_+(\vec{r}_+) = \frac{N_+ e_+}{4\pi^{3/2}\epsilon_0}\int_0^\infty dt \frac{e^{-\frac{x_+^2}{2\sigma_x^2+t}-\frac{y_+^2}{2\sigma_y^2+t}-\frac{z_+^2}{2\sigma_z^2+t}}}{\sqrt{(2\sigma_{x+}^2+t)(2\sigma_{y+}^2+t)(2\sigma_{z+}^2+t)}}$$

-1 term added to remove the singularity:

$$\Phi_+(\vec{r}_+) = \frac{N_+ e_+}{4\pi^{3/2}\epsilon_0}\int_0^\infty dt \frac{-1+e^{-\frac{x_+^2}{2\sigma_x^2+t}-\frac{y_+^2}{2\sigma_y^2+t}-\frac{z_+^2}{2\sigma_z^2+t}}}{\sqrt{(2\sigma_{x+}^2+t)(2\sigma_{y+}^2+t)(2\sigma_{z+}^2+t)}}$$

2. Electric potential (beam frame)

3D Gaussian distribution (Fourier transform):

$$\rho_+(\vec{r}_+) = \frac{N_+ e_+}{(2\pi)^{3/2}\sigma_{x+}\sigma_{y+}\sigma_{z+}} e^{-\frac{x_+^2}{2\sigma_{x+}^2}-\frac{y_+^2}{2\sigma_{y+}^2}-\frac{z_+^2}{2\sigma_{z+}^2}}$$

$$\tilde{\rho}_+(\vec{k}_+) = \int_{-\infty}^{\infty} dx_+ \int_{-\infty}^{\infty} dy_+ \int_{-\infty}^{\infty} dz_+ \rho_+(\vec{r}_+) e^{ik_{x+}x_+ + ik_{y+}y_+ + ik_z z_+}$$
$$= N_+ e_+ e^{-\frac{1}{2}k_{x+}^2\sigma_{x+}^2 - \frac{1}{2}k_{y+}^2\sigma_{y+}^2 - \frac{1}{2}k_{z+}^2\sigma_{z+}^2}.$$

$$\tilde{\Phi}_+(\vec{k}_+) = \frac{N_+ e_+}{\epsilon_0(k_{x+}^2 + k_{y+}^2 + k_{z+}^2)} e^{-\frac{1}{2}k_{x+}^2\sigma_{x+}^2 - \frac{1}{2}k_{y+}^2\sigma_{y+}^2 - \frac{1}{2}k_{z+}^2\sigma_{z+}^2}$$

$$\Phi_+(\vec{r}_+) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} dk_{x+} \int_{-\infty}^{\infty} dk_{y+} \int_{-\infty}^{\infty} dk_{z+} \tilde{\Phi}_+(\vec{k}_+) e^{-ik_{x+}x_+ - ik_{y+}y_+ - ik_z z_+}$$
$$= \frac{N_+ e_+}{(2\pi)^3 \epsilon_0} \int_{-\infty}^{\infty} d\vec{k}_+ \frac{e^{-ik_{x+}x_+ - ik_{y+}y_+ - ik_z z_+}}{k_{x+}^2 + k_{y+}^2 + k_{z+}^2} e^{-\frac{1}{2}k_{x+}^2\sigma_{x+}^2 - \frac{1}{2}k_{y+}^2\sigma_{y+}^2 - \frac{1}{2}k_{z+}^2\sigma_{z+}^2}$$

2. Electric potential (beam frame)

3D Gaussian distribution (Fourier transform):

$$\Phi_+(\vec{r}_+) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} dk_{x+}\int_{-\infty}^{\infty} dk_{y+}\int_{-\infty}^{\infty} dk_{z+}\tilde{\Phi}_+(\vec{k}_+)e^{-ik_{x+}x_+ - ik_{y+}y_+ - ik_z z_+}$$

$$= \frac{N_+ e_+}{(2\pi)^3\epsilon_0}\int_{-\infty}^{\infty} d\vec{k}_+ \frac{e^{-ik_{x+}x_+ - ik_{y+}y_+ - ik_z z_+}}{k_{x+}^2 + k_{y+}^2 + k_{z+}^2} e^{-\frac{1}{2}k_{x+}^2\sigma_{x+}^2 - \frac{1}{2}k_{y+}^2\sigma_{y+}^2 - \frac{1}{2}k_{z+}^2\sigma_{z+}^2}$$

Identity:

$$\frac{1}{k_{x+}^2 + k_{y+}^2 + k_{z+}^2} = \frac{1}{4}\int_0^{\infty} dt e^{-\frac{1}{4}t(k_{x+}^2 + k_{y+}^2 + k_{z+}^2)}$$

$$\Phi_+(\vec{r}_+) = \frac{N_+ e_+}{4\pi^{3/2}\epsilon_0}\int_0^{\infty} dt \frac{e^{-\frac{x_+^2}{2\sigma_x^2+t} - \frac{y_+^2}{2\sigma_y^2+t} - \frac{z_+^2}{2\sigma_z^2+t}}}{\sqrt{(2\sigma_{x+}^2 + t)(2\sigma_{y+}^2 + t)(2\sigma_{z+}^2 + t)}}$$

2. Electric potential (beam frame)

2D Gaussian distribution:

$$\rho_+(x_+, y_+) = \frac{N_+ e_+}{2\pi\sigma_{x+}\sigma_{y+}} e^{-\frac{x_+^2}{2\sigma_{x+}^2} - \frac{y_+^2}{2\sigma_{y+}^2}}$$

$$\Phi_+(x_+, y_+) = \frac{N_+ e_+}{4\pi\epsilon_0} \int_0^\infty dt \frac{e^{-\frac{x_+^2}{2\sigma_x^2 + t} - \frac{y_+^2}{2\sigma_y^2 + t}}}{\sqrt{(2\sigma_{x+}^2 + t)(2\sigma_{y+}^2 + t)}}$$

Round beam:

$$\Phi_+(x_+, y_+) = \frac{N_+ e_+}{4\pi\epsilon_0} \int_0^\infty dt \frac{e^{-\frac{x_+^2}{2\sigma_+^2 + t} - \frac{y_+^2}{2\sigma_+^2 + t}}}{2\sigma_+^2 + t}$$

2. Electric potential (lab frame)

1D Gaussian distribution (flat beam):

$$\rho_+(x_+) = \frac{N_+ e_+}{\sqrt{2\pi}\sigma_{x+}} e^{-\frac{x_+^2}{2\sigma_{x+}^2}}$$

$$\Phi_+(x_+) = \frac{N_+ e_+}{4\sqrt{\pi}\epsilon_0} \int_0^\infty dt \frac{e^{-\frac{x_+^2}{2\sigma_{x+}^2+t}}}{\sqrt{2\sigma_{x+}^2 + t}}$$

2. Electric potential (lab frame)

➤ From beam frame to lab frame

- **Lorentz transform**

$$x_+ \rightarrow x_+,\ y_+ \rightarrow y_+,\ z_+ \rightarrow \gamma_+(z_+ - c\tau)$$

$$\sigma_{x+} \rightarrow \sigma_{x+},\ \ \sigma_{y+} \rightarrow \sigma_{y+},\ \sigma_{z+} \rightarrow \gamma_+\sigma_{z+}$$

$$\Phi_+(\vec{r}_+) = \frac{N_+ e_+}{4\pi^{3/2}\epsilon_0}\int_0^\infty dt \frac{e^{-\frac{x_+^2}{2\sigma_{x+}^2+t}-\frac{y_+^2}{2\sigma_{y+}^2+t}-\frac{\gamma_+^2(z_+-c\tau)^2}{2\gamma_+^2\sigma_{z+}^2+t}}}{\sqrt{(2\sigma_{x+}^2+t)(2\sigma_{y+}^2+t)(2\gamma_+^2\sigma_{z+}^2+t)}}$$

2. Electric potential (lab frame)

Electromagnetic fields:

$$E_{x+}(\vec{r}_+) = -\gamma_+ \frac{\partial \Phi_+}{\partial x_+}$$

$$= \frac{\gamma_+ N_+ e_+ x_+}{2\pi^{3/2}\epsilon_0} \int_0^\infty dt \frac{e^{-\frac{x_+^2}{2\sigma_{x+}^2+t}-\frac{y_+^2}{2\sigma_{y+}^2+t}-\frac{\gamma_+^2(z_+-c\tau)^2}{2\gamma_+^2\sigma_{z+}^2+t}}}{(2\sigma_{x+}^2+t)\sqrt{(2\sigma_{x+}^2+t)(2\sigma_{y+}^2+t)(2\gamma_+^2\sigma_{z+}^2+t)}}$$

$$E_{y+}(\vec{r}_+) = -\gamma_+ \frac{\partial \Phi_+}{\partial y_+}$$

$$= \frac{\gamma_+ N_+ e_+ y_+}{2\pi^{3/2}\epsilon_0} \int_0^\infty dt \frac{e^{-\frac{x_+^2}{2\sigma_{x+}^2+t}-\frac{y_+^2}{2\sigma_{y+}^2+t}-\frac{\gamma_+^2(z_+-c\tau)^2}{2\gamma_+^2\sigma_{z+}^2+t}}}{(2\sigma_{y+}^2+t)\sqrt{(2\sigma_{x+}^2+t)(2\sigma_{y+}^2+t)(2\gamma_+^2\sigma_{z+}^2+t)}}$$

$$B_{x+}(\vec{r}_+) = -\frac{1}{c} E_{y+}(\vec{r}_+),$$

$$B_{y+}(\vec{r}_+) = \frac{1}{c} E_{x+}(\vec{r}_+).$$

2. Electric potential (lab frame)

Take the limit of $\gamma_+ \to \infty$

$$E_{x+}(\vec{r}_+) = \frac{N_+ e_+ x_+}{(2\pi)^{3/2}\epsilon_0 \sigma_{z+}} e^{-\frac{(z_+ - c\tau)^2}{2\sigma_{z+}^2}} \int_0^\infty dt \frac{e^{-\frac{x_+^2}{2\sigma_{x+}^2+t} - \frac{y_+^2}{2\sigma_{y+}^2+t}}}{(2\sigma_{x+}^2 + t)\sqrt{(2\sigma_{x+}^2 + t)(2\sigma_{y+}^2 + t)}}$$

$$E_{y+}(\vec{r}_+) = \frac{N_+ e_+ y_+}{(2\pi)^{3/2}\epsilon_0 \sigma_{z+}} e^{-\frac{(z_+ - c\tau)^2}{2\sigma_{z+}^2}} \int_0^\infty dt \frac{e^{-\frac{x_+^2}{2\sigma_{x+}^2+t} - \frac{y_+^2}{2\sigma_{y+}^2+t}}}{(2\sigma_{y+}^2 + t)\sqrt{(2\sigma_{x+}^2 + t)(2\sigma_{y+}^2 + t)}}$$

Bassetti-Erskine equation ($\sigma_{x+} > \sigma_{y+}$ **):**

$$E_{x+}(\vec{r}_+) = \frac{Q_+}{2\epsilon_0\sqrt{2\pi(\sigma_{x+}^2 - \sigma_{y+}^2)}} \mathrm{Im}\left[w(X) - e^{-\frac{x_+^2}{2\sigma_{x+}^2} - \frac{y_+^2}{2\sigma_{y+}^2}} w(Y)\right]$$

$$E_{y+}(\vec{r}_+) = \frac{Q_+}{2\epsilon_0\sqrt{2\pi(\sigma_{x+}^2 - \sigma_{y+}^2)}} \mathrm{Re}\left[w(X) - e^{-\frac{x_+^2}{2\sigma_{x+}^2} - \frac{y_+^2}{2\sigma_{y+}^2}} w(Y)\right]$$

$$Q_+ = \frac{N_+ e_+}{\sqrt{2\pi}\sigma_{z+}} e^{-\frac{(z_+ - c\tau)^2}{2\sigma_{z+}^2}} \quad X = \frac{x + iy}{\sqrt{2(\sigma_{x+}^2 - \sigma_{y+}^2)}} \quad Y = \frac{x\frac{\sigma_{y+}}{\sigma_{x+}} + iy\frac{\sigma_{x+}}{\sigma_{y+}}}{\sqrt{2(\sigma_{x+}^2 - \sigma_{y+}^2)}} \quad w(z) = e^{-z^2}\left[1 + \frac{2i}{\pi}\int_0^z e^{t^2 dt}\right]$$

3. Beam-beam kick and tune shifts

➤ The first look of beam-beam effects

- Coordinates system (assume v=c)
- Beam-beam force
- Tune shifts

$$x_+(\tau) = x_-(\tau)\cos\theta + (z_- - c\tau)\sin\theta,$$

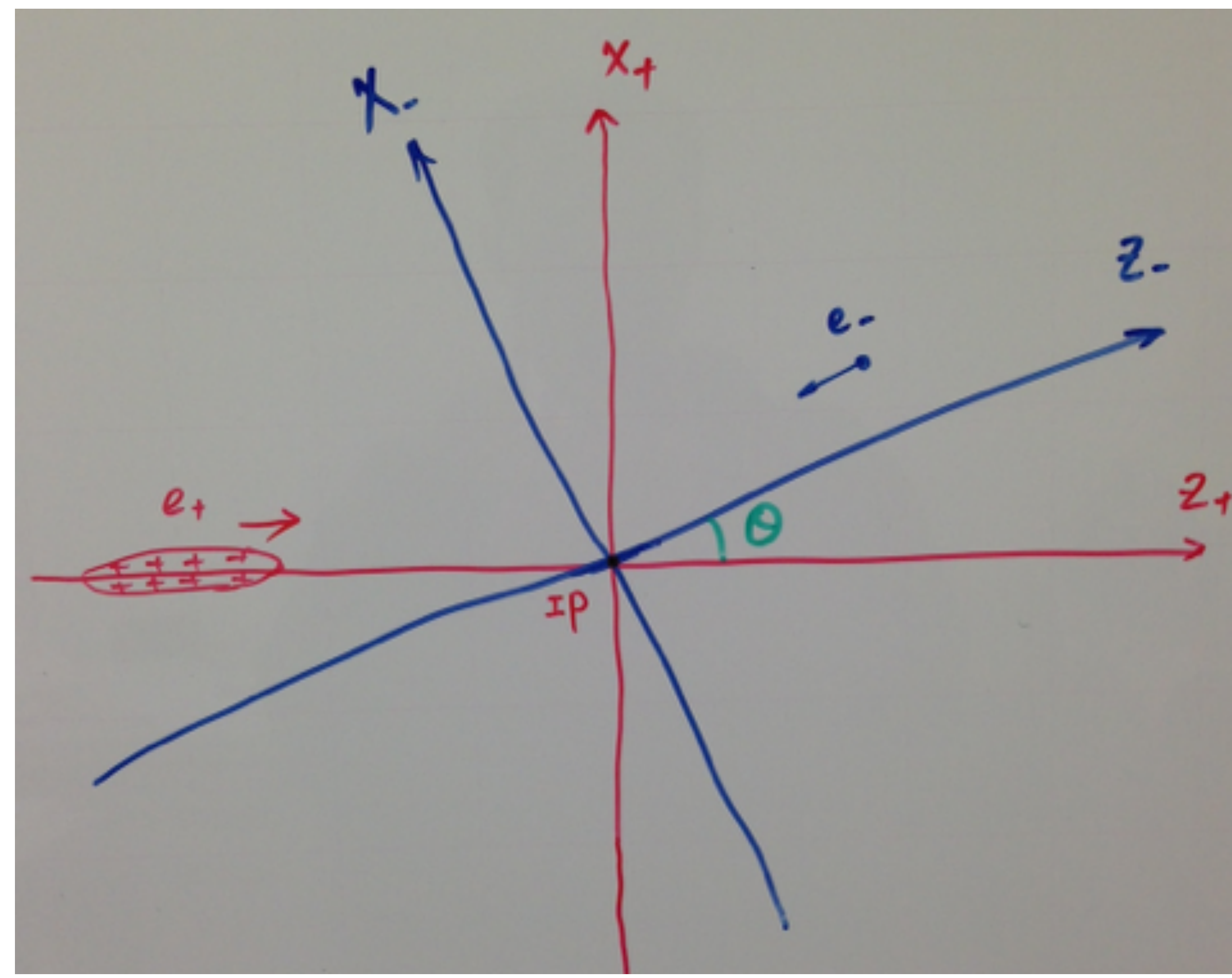
$$y_+(\tau) = y_-(\tau),$$

$$z_+(\tau) = (z_- - c\tau)\cos\theta - x_-(\tau)\sin\theta,$$

$$v_{x+} = -c\sin\theta,$$

$$v_{y+} = 0,$$

$$v_{z+} = -c\cos\theta.$$



3. Beam-beam kick and tune shifts

Lorentz force (in e+ beam's frame):

$$\vec{F}_+ = e_-(\vec{E}_+ + \vec{v}_+ \times \vec{B}_+)$$

$$\vec{v}_+ \times \vec{B}_+ = -v_{z+}B_{y+}\vec{i} + v_{z+}B_{x+}\vec{j} + v_{x+}B_{y+}\vec{k}$$

$$F_{x+} = e_-(E_{x+} - v_{z+}B_{y+}) = e_-E_{x+}(1+\cos\theta)$$

$$F_{y+} = e_-(E_{y+} + v_{z+}B_{x+}) = e_-E_{y+}(1+\cos\theta)$$

$$F_{z+} = e_-v_{x+}B_{y+} = -e_-E_{x+}\sin\theta$$

Lorentz force (in e- beam's frame):

$$F_{x-} = F_{x+}\cos\theta - F_{z+}\sin\theta = F_{x+}$$

$$F_{y-} = F_{y+}$$

$$F_{z-} = F_{x+}\sin\theta + F_{z+}\cos\theta = F_{x+}\tan\frac{\theta}{2}$$

3. Beam-beam kick and tune shifts

Momentum kick (in e- beam's frame, perturbation theory):

$$\Delta p_{x-} = \frac{\Delta P_{x-}}{P_{x-}} = \frac{1}{p_0}\int_{-\infty}^{\infty} F_{x-} d\tau$$

$$\Delta p_{y-} = \frac{\Delta P_{y-}}{P_{y-}} = \frac{1}{p_0}\int_{-\infty}^{\infty} F_{y-} d\tau$$

Tune shifts:

$$\xi_{x-}(x_-, y_-, z_-) = -\frac{1}{4\pi}\beta_{x-}\frac{\partial}{\partial x_-}\Delta p_{x-}$$

$$\xi_{y-}(x_-, y_-, z_-) = -\frac{1}{4\pi}\beta_{y-}\frac{\partial}{\partial y_-}\Delta p_{y-}$$

Or:

$$\xi_{x-}(x_-, y_-, z_-) = -\frac{1}{4\pi p_0}\int_{-\infty}^{\infty} d\tau \beta_{x-}\frac{\partial F_{x-}}{\partial x_-}$$

$$\xi_{y-}(x_-, y_-, z_-) = -\frac{1}{4\pi p_0}\int_{-\infty}^{\infty} d\tau \beta_{y-}\frac{\partial F_{y-}}{\partial y_-}$$

3. Beam-beam kick and tune shifts

Examples from Handbook (Sec. 2.5.4.2, Second edition):

Momentum kick

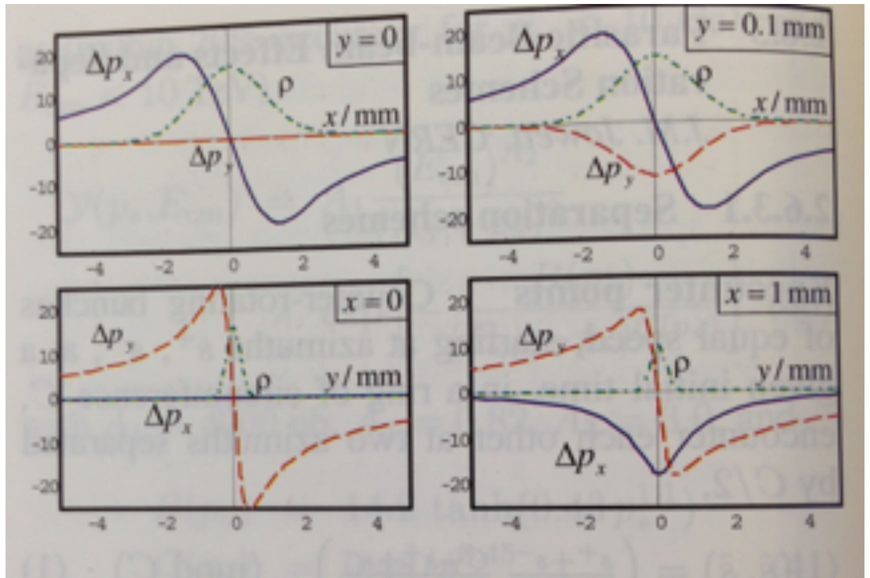


Figure 4: Beam-beam kicks in μrad felt by a 1 GeV, $Z = -1$, particle (e.g., electron) as a function of horizontal (left) and vertical (right) separations around a bunch with $N = 10^{10}$, $\sigma_x = 1$ mm, $\sigma_y = 0.2$ mm (e.g., positron bunch). In the plots on the left, the other separation is zero, while on the right a non-zero separation is included. Each plot has the same scales and the (scaled) Gaussian charge density is also shown.

Vector field

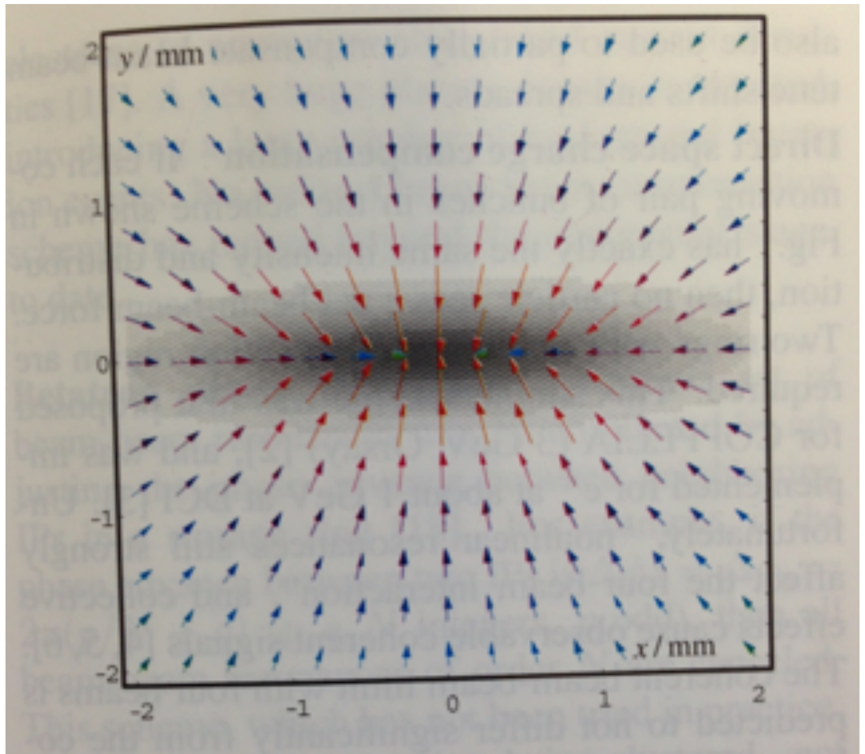


Figure 5: Kick vector field where the length (and color) of the arrows indicates the kick strength. Gray shading indicates the charge density. Same parameters as Fig.4.

Tune shift

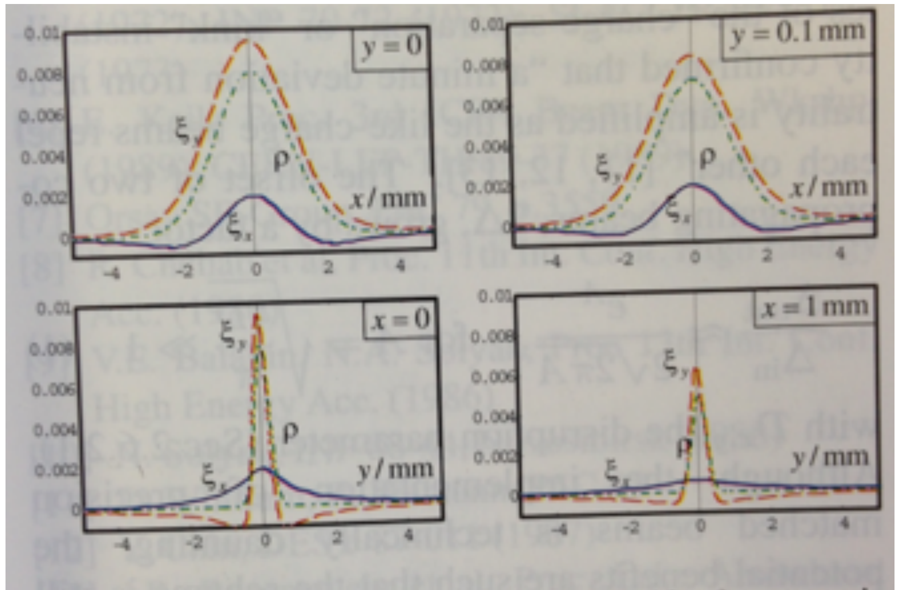


Figure 6: Parasitic beam-beam tune-shifts for a particle on the closed orbit, calculated perturbatively, for the same conditions as Fig.4 with $\beta_x = \beta_y = 1$ m. The charge density is also shown.

Conclusions:

1) Different from normal magnetic kicks, beam-beam kicks are very nonlinear.

2) Beam-beam forces are similar as that of space-charge and electron cloud.

4. Hamiltonian for beam-beam interaction

➤ Hamiltonian

- Linear one-turn map in the 6D phase space
- Beam-beam effects lumped at the IP
- Analysis of beam-beam dynamics

General form of Hamiltonian:

$$H(x, p_x, y, p_y, z, p_z; s) = H_0 + U(x, y, z) \sum_n \delta_p(s - nC)$$

Q: Is this definition correct?

$$H_0 = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}\left[K_x(s)x^2 + K_y(s)y^2\right] - \frac{1}{2}\alpha_p \left[p_z^2 + \left(\frac{\mu_z}{\alpha_p}\right)^2 z^2\right]$$

4. Hamiltonian for beam-beam interaction

General form of Hamiltonian:

$$U(x_-, y_-, z_-) = \frac{N_+ r_e \gamma_+ (1+\cos\theta) c}{\sqrt{\pi}\gamma_-} \int_{-\infty}^{\infty} d\tau \int_0^{\infty} dt$$

$$\times \frac{1}{\sqrt{(2\sigma_{x+}^2 + t)(2\sigma_{y+}^2 + t)(2\gamma_+^2 \sigma_{z+}^2 + t)}}$$

$$\times e^{-\frac{[x_- \cos\theta + (z_- - c\tau)\sin\theta]^2}{2\sigma_{x+}^2 + t} - \frac{y_-^2}{2\sigma_{y+}^2 + t} - \frac{\gamma_+^2 [(z_- - c\tau)\cos\theta - x_- \sin\theta - c\tau]^2}{2\gamma_+^2 \sigma_{z+}^2 + t}}$$

Qs: Is this equation correct?

How to get it from electric potential?

Integration over τ:

$$U(x_-, y_-, z_-) = \frac{N_+ r_e \gamma_+ (1+\cos\theta) c}{\sqrt{\pi}\gamma_-} \int_0^{\infty} dt$$

$$\times \frac{1}{\sqrt{(2\sigma_{x+}^2 + t)(2\sigma_{y+}^2 + t)(2\gamma_+^2 \sigma_{z+}^2 + t)}}$$

$$\times \frac{1}{c} \frac{\sqrt{\pi}}{\sqrt{\frac{\sin^2\theta}{2\sigma_{x+}^2 + t} + \frac{\gamma_+^2 (1+\cos\theta)^2}{2\gamma_+^2 \sigma_{z+}^2 + t}}}$$

$$\times e^{-\frac{y_-^2}{2\sigma_{y+}^2 + t} - \frac{\gamma_+^2 [x(1+\cos\theta) + z_- \sin\theta]^2}{(2\gamma_+^2 \sigma_{z+}^2 + t)\sin^2\theta + \gamma_+^2 (2\sigma_{x+}^2 + t)(1+\cos\theta)^2}}$$

4. Hamiltonian for beam-beam interaction

Simplified case 1: $\gamma_+ \to \infty$, $\theta \neq 0$, **and finite** σ_{z+}

$$U(x_-, y_-, z_-) = \frac{N_+ r_e (1+\cos\theta) c}{\sqrt{2\pi}\gamma_- \sigma_{z+}} \int_{-\infty}^{\infty} d\tau \int_0^{\infty} dt$$

$$\times \frac{1}{\sqrt{(2\sigma_{x+}^2 + t)(2\sigma_{y+}^2 + t)}}$$

$$\times e^{-\frac{[x_- \cos\theta + (z_- - c\tau)\sin\theta]^2}{2\sigma_{x+}^2 + t} - \frac{y_-^2}{2\sigma_{y+}^2 + t} - \frac{[(z_- - c\tau)\cos\theta - x_- \sin\theta - c\tau]^2}{2\sigma_{z+}^2}}$$

Integration over τ:

$$U(x_-, y_-, z_-) = \frac{N_+ r_e (1+\cos\theta)}{\sqrt{2\pi}\gamma_- \sigma_{z+}} \int_0^{\infty} dt$$

$$\times \frac{1}{\sqrt{(2\sigma_{x+}^2 + t)(2\sigma_{y+}^2 + t)}}$$

$$\times \frac{\sqrt{\pi}}{\sqrt{\frac{\sin^2\theta}{2\sigma_{x+}^2 + t} + \frac{(1+\cos\theta)^2}{2\sigma_{z+}^2}}}$$

$$\times e^{-\frac{y_-^2}{2\sigma_{y+}^2 + t} - \frac{[x_-(1+\cos\theta) + z_- \sin\theta]^2}{2\sigma_{z+}^2 \sin^2\theta + (2\sigma_{x+}^2 + t)(1+\cos\theta)^2}}$$

4. Hamiltonian for beam-beam interaction

Simplified case 2: $\theta \neq 0$, **and super bunch with large** σ_{z+}

$$U(x_-, y_-, z_-) = \frac{N_+ r_e (1+\cos\theta) c}{\sqrt{2\pi}\sigma_{z+}\gamma_-} \int_{-L/c}^{L/c} d\tau \int_0^\infty dt \times \frac{1}{\sqrt{(2\sigma_{x+}^2 + t)(2\sigma_{y+}^2 + t)}} \times e^{-\frac{\left[x_- \cos\theta + (z_- - c\tau)\sin\theta\right]^2}{2\sigma_{x+}^2 + t} - \frac{y_-^2}{2\sigma_{y+}^2 + t}},$$

Comment: This case is not interesting at this moment.

4. Hamiltonian for beam-beam interaction

Simplified case 3: **Head-on collision with $\theta = 0$**

$$U(x_-, y_-, z_-) = \frac{2N_+ r_e \gamma_+ c}{\sqrt{\pi}\gamma_-} \int_{-\infty}^{\infty} d\tau \int_0^{\infty} dt$$
$$\times \frac{1}{\sqrt{(2\sigma_{x+}^2 + t)(2\sigma_{y+}^2 + t)(2\gamma_+^2 \sigma_{z+}^2 + t)}}$$
$$\times e^{-\frac{x_-^2}{2\sigma_{x+}^2 + t} - \frac{y_-^2}{2\sigma_{y+}^2 + t} - \frac{\gamma_+^2 (z_- - 2c\tau)^2}{2\gamma_+^2 \sigma_{z+}^2 + t}}.$$

Integration over τ:

$$U(x_-, y_-, z_-) = \frac{N_+ r_e}{\gamma_-} \int_0^{\infty} dt \frac{e^{-\frac{x_-^2}{2\sigma_{x+}^2 + t} - \frac{y_-^2}{2\sigma_{y+}^2 + t}}}{\sqrt{(2\sigma_{x+}^2 + t)(2\sigma_{y+}^2 + t)}}$$

Comments:

1) This is the popular form of beam-beam potential.

2) The form is the same as that of 2D electric potential.

5. Amplitude-dependent tune shifts

Change to action-angle variables:

$$u = \sqrt{2\beta_u(s)J_u}\cos\phi_u$$

$$p_u = -\sqrt{\frac{2J_u}{\beta_u}}\left(\sin\phi_u + \alpha_u\cos\phi_u\right) \qquad u = x, y, z$$

$$\phi_u = \psi_u + \int_0^s \frac{1}{\beta_u(s)}ds - \nu_u\frac{s}{R}$$

Effective Hamiltonian:

$$H(J_x, \psi_x, J_y, \psi_y, J_z, \psi_z) = \mu_x J_x + \mu_y J_y + \mu_z J_z + U\sum_n \delta_p(s - nC)$$

5. Amplitude-dependent tune shifts

Amplitude-dependent tune shifts (2D):

$$\Delta\nu_x(J_x, J_y) = \frac{1}{(2\pi)^3}\int_0^{2\pi} d\phi_x \int_0^{2\pi} d\phi_y \frac{\partial U(J_x, \phi_x, J_y, \phi_y)}{\partial J_x}$$

$$\Delta\nu_y(J_x, J_y) = \frac{1}{(2\pi)^3}\int_0^{2\pi} d\phi_x \int_0^{2\pi} d\phi_y \frac{\partial U(J_x, \phi_x, J_y, \phi_y)}{\partial J_y}$$

Or (3D):

$$\Delta\nu_x(J_x, J_y, J_z) = \frac{1}{(2\pi)^4}\int_0^{2\pi} d\phi_x \int_0^{2\pi} d\phi_y \int_0^{2\pi} d\phi_z \frac{\partial U(J_x, \phi_x, J_y, \phi_y, J_z, \phi_z)}{\partial J_x}$$

$$\Delta\nu_y(J_x, J_y, J_z) = \frac{1}{(2\pi)^4}\int_0^{2\pi} d\phi_x \int_0^{2\pi} d\phi_y \int_0^{2\pi} d\phi_z \frac{\partial U(J_x, \phi_x, J_y, \phi_y, J_z, \phi_z)}{\partial J_y}$$

5. Amplitude-dependent tune shifts

Case 1: **Head-on collision with $\theta = 0$**

$$U(J_x, \phi_x, J_y, \phi_y) = \frac{N_+ r_e}{\gamma_-} \int_0^\infty dt \frac{1 - e^{-\frac{2\beta_{x-} J_x \cos^2\phi_x}{2\sigma_{x+}^2 + t} - \frac{2\beta_{y-} J_y \cos^2\phi_y}{2\sigma_{y+}^2 + t}}}{\sqrt{(2\sigma_{x+}^2 + t)(2\sigma_{y+}^2 + t)}}$$

Q: Why should I change the sign here?

Solution:

$$\Delta\nu_x = \frac{\beta_{x-} N_+ r_e}{2\pi\gamma_-} \int_0^\infty dt \frac{e^{-\frac{\beta_{x-} J_x}{2\sigma_{x+}^2 + t}} e^{-\frac{\beta_{y-} J_y}{2\sigma_{y+}^2 + t}}}{(2\sigma_{x+}^2 + t)^{3/2} (2\sigma_{y+}^2 + t)^{1/2}} \times \left[I_0\left(\frac{\beta_{x-} J_x}{2\sigma_{x+}^2 + t}\right) - I_1\left(\frac{\beta_{x-} J_x}{2\sigma_{x+}^2 + t}\right)\right] I_0\left(\frac{\beta_{y-} J_y}{2\sigma_{y+}^2 + t}\right)$$

$$\Delta\nu_y = \frac{\beta_{y-} N_+ r_e}{2\pi\gamma_-} \int_0^\infty dt \frac{e^{-\frac{\beta_{x-} J_x}{2\sigma_{x+}^2 + t}} e^{-\frac{\beta_{y-} J_y}{2\sigma_{y+}^2 + t}}}{(2\sigma_{y+}^2 + t)^{3/2} (2\sigma_{x+}^2 + t)^{1/2}} \times \left[I_0\left(\frac{\beta_{y-} J_y}{2\sigma_{y+}^2 + t}\right) - I_1\left(\frac{\beta_{y-} J_y}{2\sigma_{y+}^2 + t}\right)\right] I_0\left(\frac{\beta_{x-} J_x}{2\sigma_{x+}^2 + t}\right)$$

5. Amplitude-dependent tune shifts

Equivalent form of the previous solution:

$$\Delta\nu_x(A_x, A_y) = \frac{\beta_{x-}N_+r_e}{4\pi\gamma_-\sigma_{x+}\sigma_{y+}}\int_0^1 d\eta \frac{e^{-\frac{A_y^2\sigma_{y-}^2\eta}{4\sigma_{y+}^2[\eta+(1-\eta)/R^2]}}e^{-\frac{A_x^2\sigma_{x-}^2\eta}{4\sigma_{x+}^2}}}{\sqrt{\eta+(1-\eta)/R^2}} \qquad \eta = \frac{2\sigma_{x+}^2}{2\sigma_{x+}^2+t}$$

$$\times I_0\left(\frac{A_y^2\sigma_{y-}^2\eta}{4\sigma_{y+}^2\left[\eta+(1-\eta)/R^2\right]}\right)\left[I_0\left(\frac{A_x^2\sigma_{x-}^2\eta}{4\sigma_{x+}^2}\right) - I_1\left(\frac{A_x^2\sigma_{x-}^2\eta}{4\sigma_{x+}^2}\right)\right]$$

$$\Delta\nu_y(A_x, A_y) = \frac{\beta_{y-}N_+r_e}{4\pi\gamma_-\sigma_{x+}\sigma_{y+}}\int_0^1 d\eta \frac{e^{-\frac{A_x^2\sigma_{x-}^2\eta}{4\sigma_{x+}^2[\eta+(1-\eta)R^2]}}e^{-\frac{A_y^2\sigma_{y-}^2\eta}{4\sigma_{y+}^2}}}{\sqrt{\eta+(1-\eta)R^2}} \qquad \eta = \frac{2\sigma_{y+}^2}{2\sigma_{y+}^2+t}$$

$$\times I_0\left(\frac{A_x^2\sigma_{x-}^2\eta}{4\sigma_{x+}^2\left[\eta+(1-\eta)R^2\right]}\right)\left[I_0\left(\frac{A_y^2\sigma_{y-}^2\eta}{4\sigma_{y+}^2}\right) - I_1\left(\frac{A_y^2\sigma_{y-}^2\eta}{4\sigma_{y+}^2}\right)\right]$$

$$A_u = \frac{\sqrt{2\beta_{u-}J_u}}{\sigma_{u-}} \qquad u = x, y$$

5. Amplitude-dependent tune shifts

Case 2: **Crossing angle with finite σ_{z+}**

$$U(x_-, y_-, z_-) = \frac{N_+ r_e (1+\cos\theta)}{\sqrt{2}\gamma_- \sigma_{z+}} \int_0^\infty dt \times \frac{1}{\sqrt{(2\sigma_{x+}^2 + t)(2\sigma_{y+}^2 + t)}} \times \frac{1}{\sqrt{\frac{\sin^2\theta}{2\sigma_{x+}^2 + t} + \frac{(1+\cos\theta)^2}{2\sigma_{z+}^2}}} \times \left[1 - e^{-\frac{y_-^2}{2\sigma_{y+}^2 + t} - \frac{[x_-(1+\cos\theta) + z_- \sin\theta]^2}{2\sigma_{z+}^2 \sin^2\theta + (2\sigma_{x+}^2 + t)(1+\cos\theta)^2}}\right]$$

$$U(J_x, \phi_x, J_y, \phi_y, J_z, \phi_z) = \frac{N_+ r_e (1+\cos\theta)}{\gamma_-} \int_0^\infty dt \frac{1}{\sqrt{2\sigma_{y+}^2 + t}} \times \frac{1}{\sqrt{2\sigma_{z+}^2 \sin^2\theta + (2\sigma_{x+}^2 + t)(1+\cos\theta)^2}} \times \left[1 - e^{-\frac{2\beta_{y-} J_y \cos^2\phi_y}{2\sigma_{y+}^2 + t} - \frac{[\sqrt{2\beta_{x-} J_x} \cos\phi_x (1+\cos\theta) + \sqrt{2\beta_{z-} J_z} \cos\phi_z \sin\theta]^2}{2\sigma_{z+}^2 \sin^2\theta + (2\sigma_{x+}^2 + t)(1+\cos\theta)^2}}\right]$$

$$\beta_{z-} = \frac{\sigma_{z-}}{\sigma_{\delta-}} = \frac{\alpha_{p-} C_-}{2\pi\nu_{s-}}$$

5. Amplitude-dependent tune shifts

Case 2: **Crossing angle with finite σ_{z+}**

In the case of $J_z = 0$:

$$U(J_x, \phi_x, J_y, \phi_y, 0, \phi_z) = \frac{N_+ r_e}{\gamma_-} \int_0^\infty dt \frac{1}{\sqrt{2\sigma_{y+}^2 + t}} \times \frac{1}{\sqrt{2\sigma_{z+}^2 \tan^2\left(\frac{\theta}{2}\right) + 2\sigma_{x+}^2 + t}} \times \left[1 - e^{-\frac{2\beta_{y-} J_y \cos^2 \phi_y}{2\sigma_{y+}^2 + t} - \frac{2\beta_{x-} J_x \cos^2 \phi_x}{2\sigma_{z+}^2 \tan^2\left(\frac{\theta}{2}\right) + 2\sigma_{x+}^2 + t}}\right]$$

Effective beam width:

$$\bar{\sigma}_{x+} = \sqrt{\sigma_{x+}^2 + \sigma_{z+}^2 \tan^2\left(\frac{\theta}{2}\right)}$$

Piwinski angle (Important parameter for e+e- colliders) :

$$\Phi_p = \frac{\sigma_z}{\sigma_x} \tan\left(\frac{\theta}{2}\right)$$

5. Amplitude-dependent tune shifts

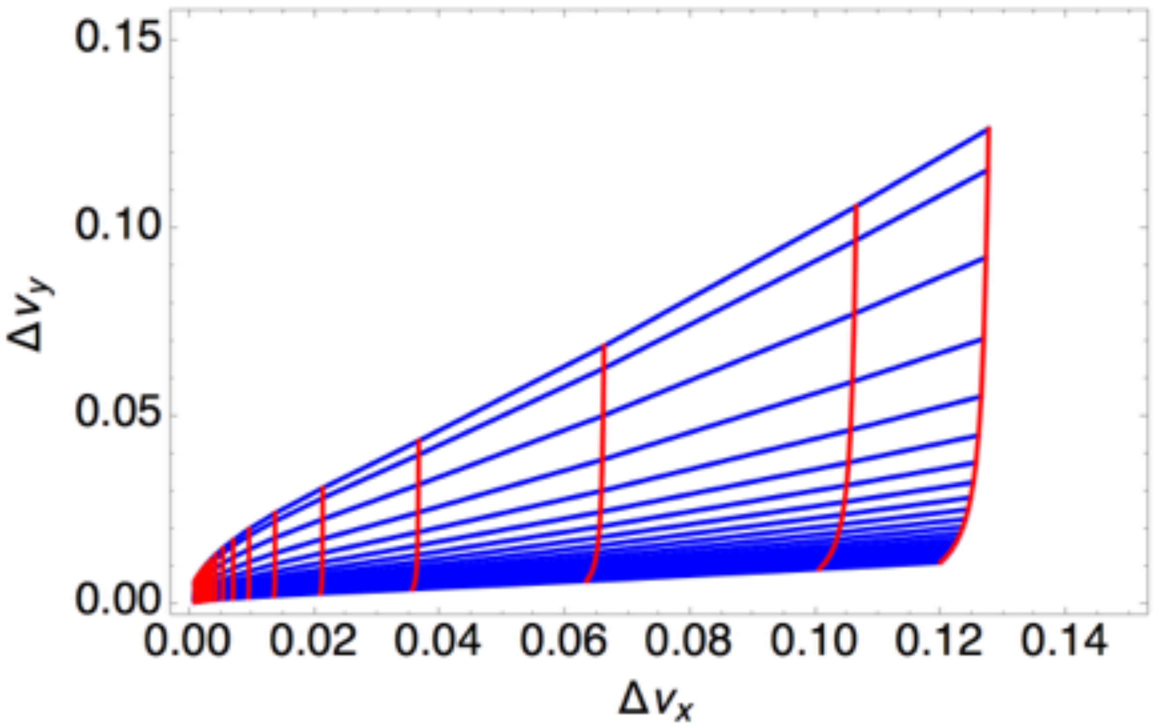
Apply equations in p.26 to KEKB:

	KEKB		SuperKEKB	
	HER	LER	HER	LER
E(GeV)	8	3.5	7.007	4
$N_p(10^{10})$	4.7	6.47	6.53	9.04
ε_x(nm)	24	18	4.6	3.2
$\varepsilon_y/\varepsilon_x$(%)	0.5	0.5	0.28	0.27
β_x^*(m)	1.2	1.2	0.025	0.032
β_y^*(mm)	5.9	5.9	0.3	0.27
σ_z(mm)	6	7	5	6
Θ(rad)	0.022		0.083	

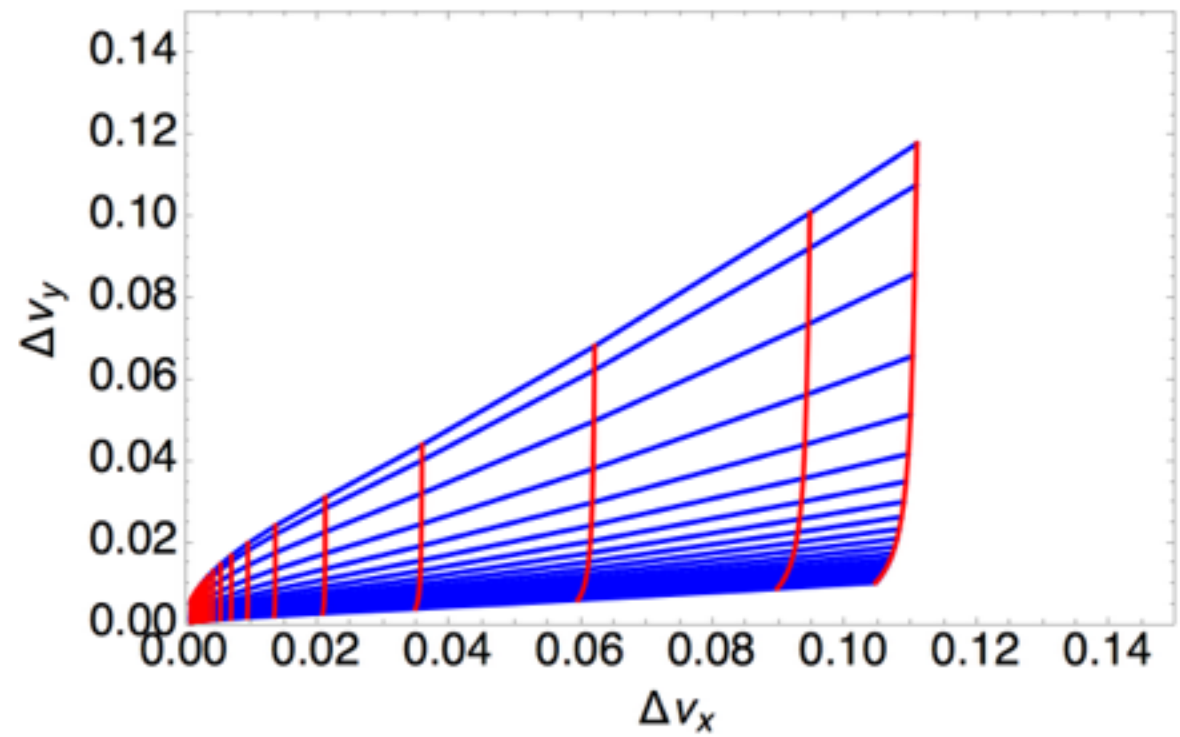
5. Amplitude-dependent tune shifts

Apply equations in p.26 to KEKB LER:
PlotRange->{$20\sigma_x$,$20\sigma_y$}

Head-on



Crossing angle

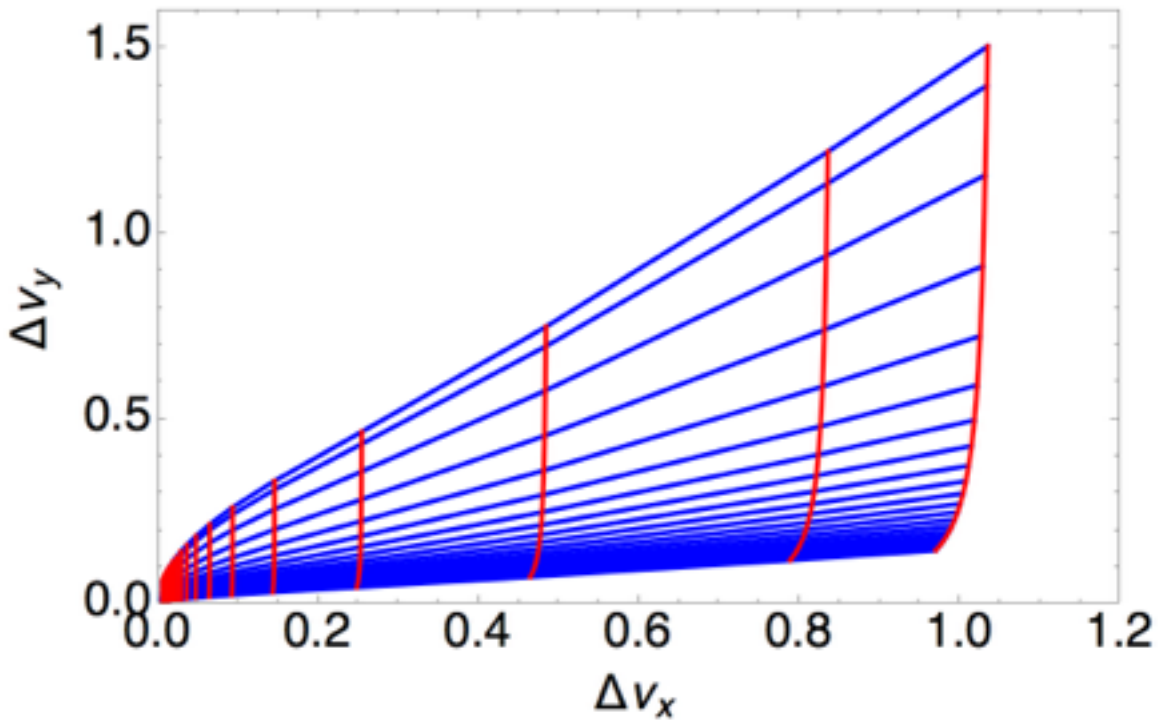


$$\Phi_{pe} \approx 0.4$$

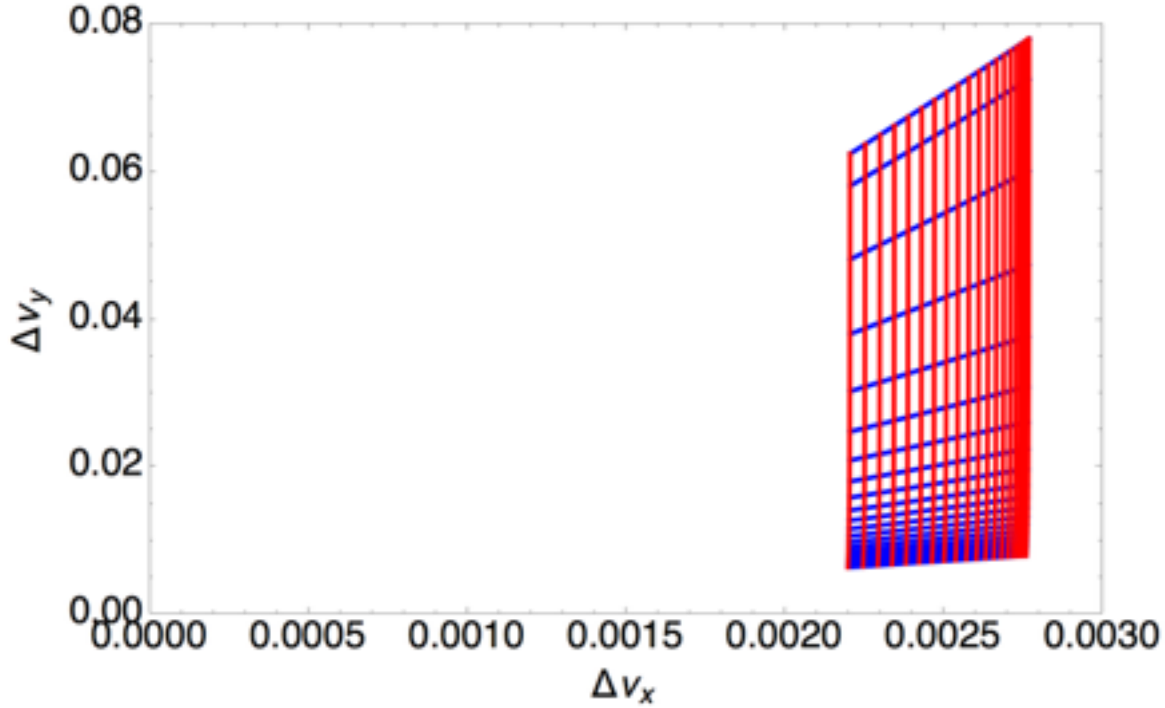
5. Amplitude-dependent tune shifts

Apply equations in p.26 to SuperKEKB LER:
PlotRange->{$20\sigma_x$,$20\sigma_y$}

Head-on



Crossing angle



$$\Phi_{pe} \approx 20$$