Linear fringe map for a quadrupole

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1. Treatment of fringe effects in SAD

➤ **Nonlinear Maxwellian hard-edge fringe: included with default value DISFRIN=0**

➤ **Soft-edge fringe: F1, F2 with flag FRINGE**

	DISFRIN=0		DISFRIN<>0	
	Nonlinear	Linear	Nonlinear	Linear
FRINGE=0	entr & exit	none	none	none
FRINGE=1	entr	entr	none	entr
FRINGE=2	exit	exit	none	exit
FRINGE=3	entr & exit	entr & exit	none	entr & exit

SAD manual (online)

1. Treatment of fringe effects in SAD

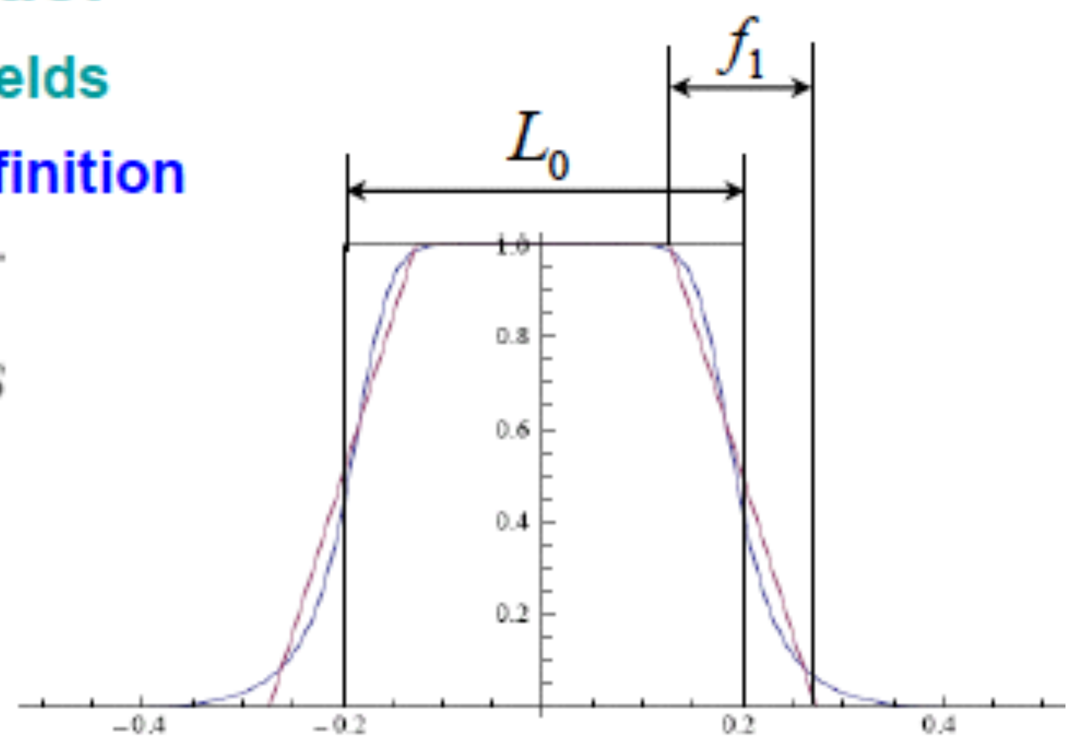
- **Hard edge approximation: simple but unphysical**

$$L_0 = \frac{1}{G_0}\int_{-\infty}^{\infty} G(s)ds$$

- **Trapezoidal fringe model**
  - Approximation of fringe fields
  - Fringe extension: **SAD definition**

$$f_1 = \sqrt{24\left|\int_0^{\infty} \frac{\widetilde{G}(s)}{G_0}(s - s_0)ds\right|}$$

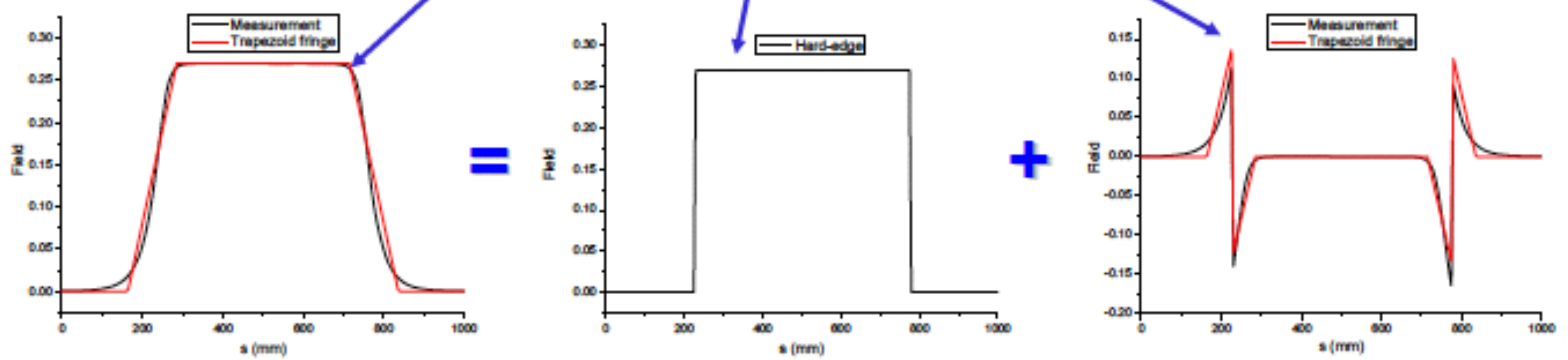
$$\widetilde{G}(s) = \begin{cases} G(s) - G_0 & 0 < s < s_0 \\ G(s) & s \geq s_0 \end{cases}$$



SAD: http://acc-physics.kek.jp/sad

1. Treatment of fringe effects in SAD

$$K(s) = K_0(s) + k(s)$$



Real magnet **Ideal magnet** **Gradient errors**

2. Derive fringe map for a quadrupole

Lie Algebra technique

- Hamiltonian system
- Solve the problem analytically
- Perturbation treatment if necessary
- Preserve the semplecticity of the solution

$$\vec{r}\,'' = f(\vec{r}, \vec{r}\,') \rightarrow X_i' = [H, X_i]$$

$$X^{(f)} = e^{-:\int_0^t H(X,t')dt':} X^{(i)}$$

Generating function: $F(t) = \int_0^t H(X,t')dt'$

2. Derive fringe map for a quadrupole

Step 1: s-dependent Hamiltonian in the field of a normal quad

- Frenet-Serret coordinate system
- On-momentum particle
- Expand H(s) in polynomials

$$H(q,p,t) = e\phi + c\sqrt{(\vec{P} - c\vec{A})^2 + m_0^2 c^2}$$

ϕ : scalar potential

$\vec{A}$: vector potential

$$H(s) = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}K(s)(x^2 - y^2) - \frac{1}{4}K'(s)(xp_x + yp_y)(x^2 - y^2)$$
$$- \frac{1}{12}K''(s)(x^4 - y^4) + \frac{1}{32}K'^2(s)(x^4 - y^4)(x^2 - y^2)$$
$$+ \frac{1}{48}K'''(s)(xp_x + yp_y)(x^4 - y^4) + \frac{1}{256}K^{(4)}(s)(x^4 - y^4)(x^2 + y^2) + O(X^8)$$

Ref.[1] J. Irwin and C.X. Wang

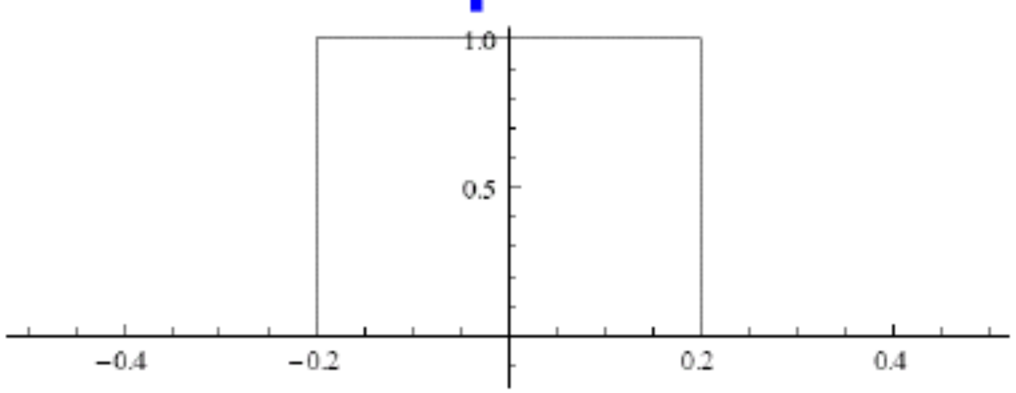
2. Derive fringe map for a quadrupole

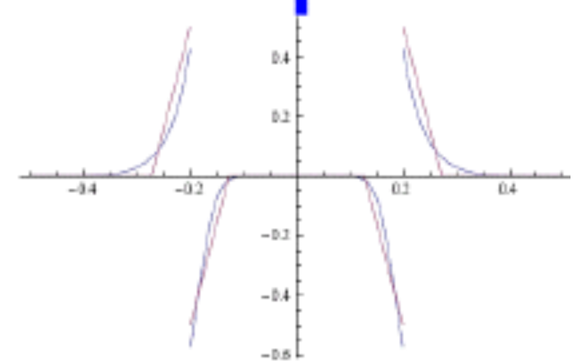
Step 2: Perturbation treatment

- Solutions for s-dependent Hamiltonian system are hard to be found, even for linear system
- Offer clear physical picture of perturbations
- Evaluate the significancy of fringe field effect

$$H(s) \cong \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}K(s)(x^2 - y^2) = H_0(s) + \widetilde{H}(s)$$

$$H_0(s) = \begin{cases} \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}K_0(x^2 - y^2) & s_1 \le s \le s_0 \\ \frac{1}{2}(p_x^2 + p_y^2) & s_0 < s \le s_2 \end{cases} \qquad \widetilde{H}(s) = \frac{1}{2}\widetilde{K}(s)(x^2 - y^2) = \begin{cases} \frac{1}{2}[K(s) - K_0](x^2 - y^2) & s_1 \le s \le s_0 \\ \frac{1}{2}K(s)(x^2 - y^2) & s_0 < s \le s_2 \end{cases}$$



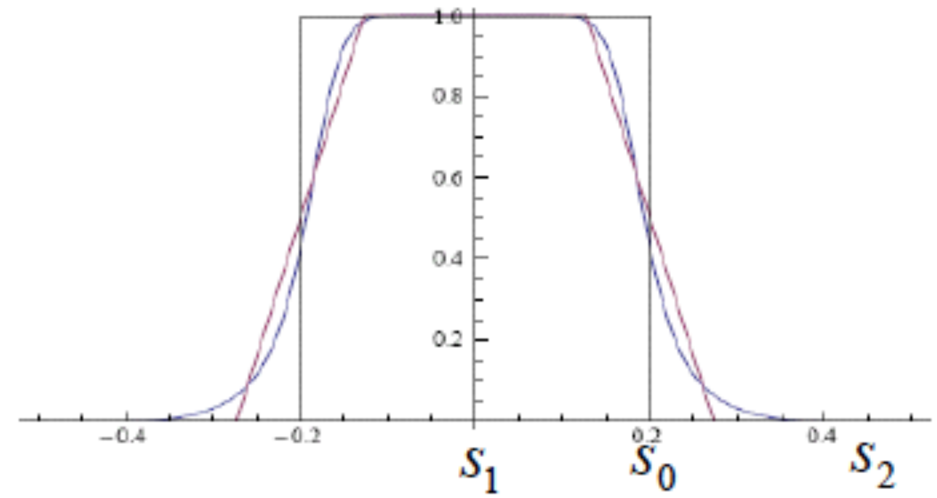


Ref.[1] J. Irwin and C.X. Wang

2. Derive fringe map for a quadrupole

Step 3: Linear map (from quad center to far right side)

- Map of ideal quad
- Map of fringe
- Map of drift



$$M(s_1 \to s_2) = R_-(s_1 \to s_0) R_+(s_0 \to s_2)$$

$$R_-(s_1 \to s_0) = M_Q(s_1 \to s_0) e^{:f_2^-:}$$

$$R_+(s_0 \to s_2) = e^{:f_2^+:} M_{drift}(s_0 \to s_2)$$

$$R_f = e^{:f_2^-:} e^{:f_2^+:} = e^{:f_2:}$$

$$M_Q(s_1 \to s) \leftrightarrow \begin{bmatrix} \cos\sqrt{K_0}s & \frac{\sin\sqrt{K_0}s}{\sqrt{K_0}} & 0 & 0 \\ -\sqrt{K_0}\sin\sqrt{K_0}s & \cos\sqrt{K_0}s & 0 & 0 \\ 0 & 0 & \cosh\sqrt{K_0}s & \frac{\sinh\sqrt{K_0}s}{\sqrt{K_0}} \\ 0 & 0 & \sqrt{K_0}\sinh\sqrt{K_0}s & \cosh\sqrt{K_0}s \end{bmatrix}$$

$$M_{drift}(s_0 \to s) \leftrightarrow \begin{bmatrix} 1 & s-s_0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s-s_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Ref.[1] J. Irwin and C.X. Wang

2. Derive fringe map for a quadrupole

Step 4: Generating functions

- s-dependent dynamical variables: Taylor expansion
- Assumption: **fringe region is short**
- 2rd BCH formula is enough

$$X = [x, p_x, y, p_y]^T$$

$$f_2^- = -\int_{s_1}^{s_0} \overline{H}(s)ds + \frac{1}{2}\int_{s_1}^{s_0} ds \int_{s}^{s_0} ds' [\overline{H}(s), \overline{H}(s')]$$

$$f_2^- = -\int_{s_0}^{s_2} \overline{H}(s)ds + \frac{1}{2}\int_{s_0}^{s_2} ds \int_{s}^{s_2} ds' [\overline{H}(s), \overline{H}(s')]$$

$$M_Q(s_1 \to s) \leftrightarrow \begin{bmatrix} \cos\sqrt{K_0}s & \frac{\sin\sqrt{K_0}s}{\sqrt{K_0}} & 0 & 0 \\ -\sqrt{K_0}\sin\sqrt{K_0}s & \cos\sqrt{K_0}s & 0 & 0 \\ 0 & 0 & \cosh\sqrt{K_0}s & \frac{\sinh\sqrt{K_0}s}{\sqrt{K_0}} \\ 0 & 0 & \sqrt{K_0}\sinh\sqrt{K_0}s & \cosh\sqrt{K_0}s \end{bmatrix}$$

$$\downarrow$$

$$M_Q(s_0 \to s) \leftrightarrow \begin{bmatrix} 1-\frac{1}{2}K_0(s)\Delta s^2 & \Delta s-\frac{1}{6}K_0(s)\Delta s^3 & 0 & 0 \\ -K_0\Delta s & 1-\frac{1}{2}K_0(s)\Delta s^2 & 0 & 0 \\ 0 & 0 & 1+\frac{1}{2}K_0(s)\Delta s^2 & \Delta s+\frac{1}{6}K_0(s)\Delta s^3 \\ 0 & 0 & K_0\Delta s & 1+\frac{1}{2}K_0(s)\Delta s^2 \end{bmatrix}$$

$$\overline{H}(s) = \begin{cases} \widetilde{H}(s, M_Q(s_0 \to s)X) & s_1 \le s \le s_0 \\ \widetilde{H}(s, M_{drift}(s_0 \to s)X) & s_0 \le s \le s_2 \end{cases}$$

$$\widetilde{H}(s) = \frac{1}{2}\widetilde{K}(s)(x^2 - y^2) = \begin{cases} \frac{1}{2}[K(s) - K_0](x^2 - y^2) & s_1 \le s \le s_0 \\ \frac{1}{2}K(s)(x^2 - y^2) & s_0 < s \le s_2 \end{cases}$$

2. Derive fringe map for a quadrupole

Step 4: Generating functions (cont)

- Represented by fringe field integrals (FFI)

$$f_2^- \cong -\frac{1}{2}I_0^-(x^2-y^2) - I_1^-(xp_x - yp_y) - \frac{1}{2}I_2^-(p_x^2 - p_y^2)$$
$$+\frac{1}{2}K_0 I_2^-(x^2+y^2) + \frac{2}{3}K_0 I_3^-(xp_x - yp_y) + \frac{1}{2}\Lambda_2^-(x^2+y^2)$$

$$f_2^+ \cong -\frac{1}{2}I_0^+(x^2-y^2) - I_1^+(xp_x - yp_y) - \frac{1}{2}I_2^+(p_x^2 - p_y^2) + \frac{1}{2}\Lambda_2^+(x^2+y^2)$$

$$f_2 \equiv f_2^- + f_2^+ + \frac{1}{2}[f_2^-, f_2^+]$$

$$I_0^- + I_0^+ \equiv 0$$

$$\approx -(I_1^- + I_1^+)(xp_x - yp_y) - \frac{I_2^- + I_2^+}{2}(p_x^2 - p_y^2)$$
$$+\frac{K_0 I_2^-}{2}(x^2+y^2) + \frac{2K_0 I_3^-}{3}(xp_x + yp_y) + \frac{\Lambda_2^- + \Lambda_2^+}{2}(x^2+y^2)$$
$$-\frac{1}{2}I_0^+(I_1^- + I_1^+)(x^2+y^2) - \frac{1}{2}I_0^+(I_2^- + I_2^+)(xp_x + yp_y)$$

2. Derive fringe map for a quadrupole

Fringe field integrals

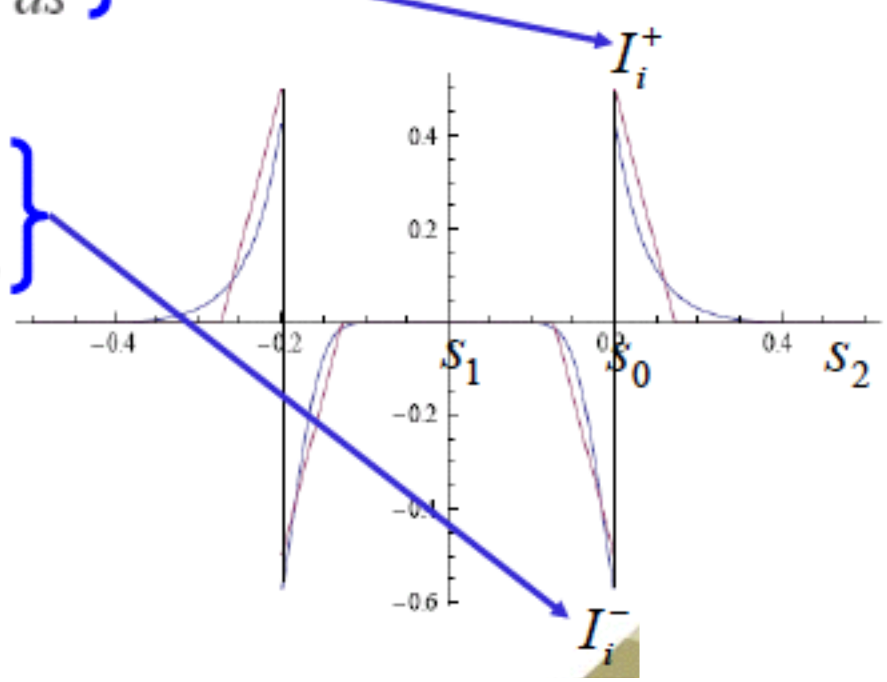
- Anti-symmetric assumption not necessary

$$I_0^- = \int_{s_1}^{s_0} \widetilde{K}(s)ds \qquad I_1^- = \int_{s_1}^{s_0} \widetilde{K}(s)(s-s_0)ds$$

$$I_2^- = \int_{s_1}^{s_0} \widetilde{K}(s)(s-s_0)^2 ds \qquad I_3^- = \int_{s_1}^{s_0} \widetilde{K}(s)(s-s_0)^3 ds$$

$$I_0^+ = \int_{s_0}^{s_2} \widetilde{K}(s)ds \qquad I_1^+ = \int_{s_0}^{s_2} \widetilde{K}(s)(s-s_0)ds$$

$$I_2^+ = \int_{s_0}^{s_2} \widetilde{K}(s)(s-s_0)^2 ds \qquad I_3^+ = \int_{s_0}^{s_2} \widetilde{K}(s)(s-s_0)^3 ds$$



$$\Lambda_2^- = \int_{s_1}^{s_0} ds \int_{s}^{s_0} ds' K(s)K(s')(s'-s)$$

$$\Lambda_2^+ = \int_{s_0}^{s_2} ds \int_{s}^{s_2} ds' K(s)K(s')(s'-s)$$

2. Derive fringe map for a quadrupole

Step 5: Correction matrix of fringe field

- **Linear fringe effects**
  - Scale change
  - Drift
  - Quadrupole

$$f_2 \equiv f_2^- + f_2^+ + \frac{1}{2}[f_2^-, f_2^+]$$

$$\approx -(I_1^- + I_1^+)(xp_x - yp_y) - \frac{I_2^- + I_2^+}{2}(p_x^2 - p_y^2)$$

$$+\frac{K_0 I_2^-}{2}(x^2+y^2) + \frac{2K_0 I_3^-}{3}(xp_x + yp_y) + \frac{\Lambda_2^- + \Lambda_2^+}{2}(x^2+y^2)$$

$$-\frac{1}{2}I_0^+(I_1^- + I_1^+)(x^2+y^2) - \frac{1}{2}I_0^+(I_2^- + I_2^+)(xp_x + yp_y)$$

$$MR_x = \begin{bmatrix} 1 & 0 \\ J_3 & 1 \end{bmatrix} \begin{bmatrix} 1 & J_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{J_1} & 0 \\ 0 & e^{-J_1} \end{bmatrix}$$

SAD linear fringe:

$$MR_x(\mathrm{SAD}) = \begin{bmatrix} 1 & J_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{I_1} & 0 \\ 0 & e^{-I_1} \end{bmatrix}$$

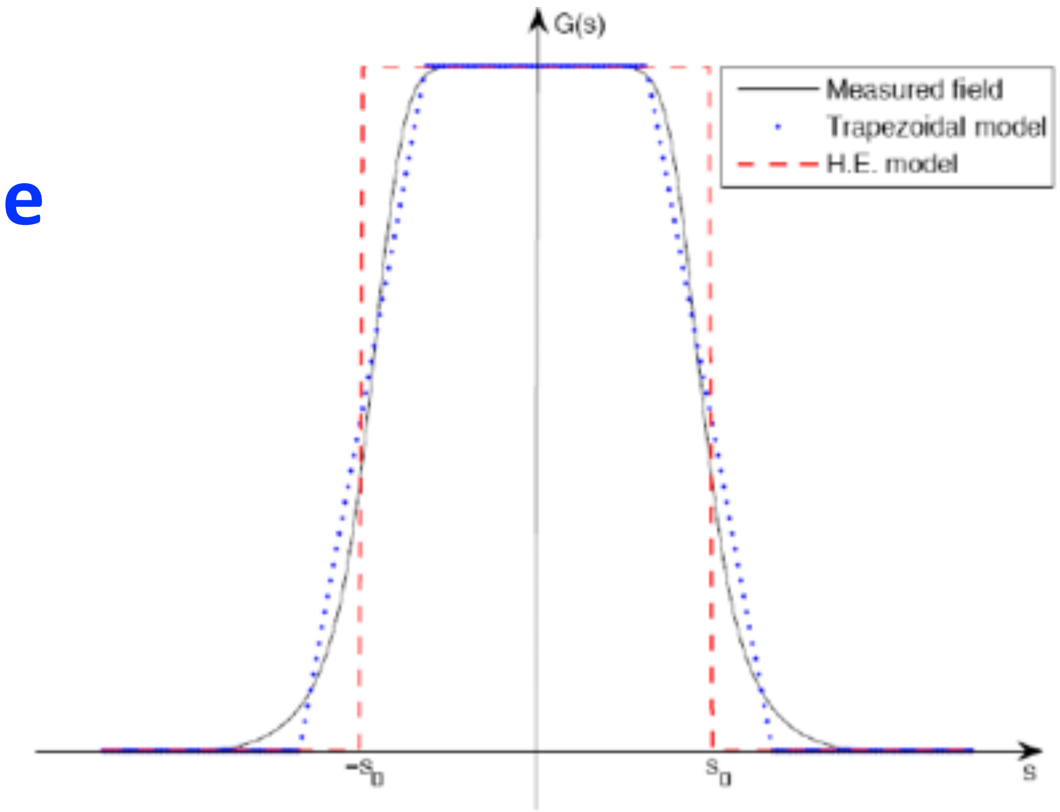
$$J_1 = (I_1^- + I_1^+) - \frac{2K_0 I_3^-}{3} + \frac{1}{2} I_0^+ (I_2^- + I_2^+)$$

$$J_2 = I_2^- + I_2^+ \quad J_3 = K_0 I_2^- + (\Lambda_2^- + \Lambda_2^+) - I_0^+ (I_1^- + I_1^+)$$

3. Summary

➤ **Findings for quad fringe:**

- Theory accepts asymmetric field profile
- Anti-symmetry fringe is not necessary
- SAD implicitly assumes $I_0^+ + I_0^- = 0$
- F1 in SAD corresponds to $I_1^+ + I_1^-$
- F2 in SAD corresponds to $I_2^+ + I_2^-$



➤ **Analogy to dipole magnet:**

- Is it natural to choose:
  $K_{max}=Max[K(s)]$?
- Does SAD assume:
  $I_0^+ + I_0^- = 0$?
- How to determine the magnet center?
  Personal opinion:
  1st: Choose $K_{max}=Max[K(s)]$
  2nd: Determine boundary
  3rd: Determine magnet center for hard-edge model