Fractional Instantons and Bions

KEK theory workshop
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See also
Field Theory on Compactified Space $\mathbb{R}^{D-1} \times S^1$ with a Twisted Boundary Condition (TBC) along $S^1$

* Resurgence of quantum field theory (long history ...., Dunne & Unsal ‘12--)
* Gauge-Higgs unification (Hosotani mechanism)
* Large extra dimension

Topological solitons, instantons have *fractional topological charge*

*Bions = composite of fractional instantons with zero instanton charge*
O(3) model = CP¹ model on $\mathbb{R}^1 \times S^1$

Sigma model instanton (lump) (skyrmion in cond-mat)

$$\pi_2(S^2) \cong \mathbb{Z} \quad \text{for } \mathbb{R}^2$$

$$L = \partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n} \quad \mathbf{n} = (n_1, n_2, n_3) \quad \mathbf{n}^2 = 1$$

$$u \equiv \frac{n_1 + in_2}{1 - n_3} = \lambda(z - z_0) \quad \lambda \in \mathbb{C}^*$$

Size, phase moduli

$$z_0 \in \mathbb{C}$$

Position moduli
Sigma model instanton (lump) \( \pi_2(S^2) \cong \mathbb{Z} \) for \( \mathbb{R}^2 \)

(skyrmion in cond-mat)

\[
R^2 \quad \xrightarrow{\quad} \quad S^2
\]

\[
u = \frac{n_1 + in_2}{1-n_3} = \lambda(z-z_0)
\]

\( (n_1, n_2) \)

\( N: n_3 = +1 \)

\( S^2 \)

\( S^1 \)

\( n_3 = 0 \)

\( S: n_3 = -1 \)
Sigma model instanton (lump) \( \pi_2(S^2) \cong \mathbb{Z} \) for \( \mathbb{R}^2 \)

(skyrmion in cond-mat)

\[ \mathbb{R}^2 \rightarrow S^2 \]

*In the presence of a potential*

\[ V = m^2 (1 - n_3^2) \]

A domain wall ring

\[ N: n_3 = +1 \]

\[ S: n_3 = -1 \]
Sigma model instanton
(skyrmion in cond-mat)

\[ \mathbb{R}^1 \times S^1 \rightarrow S^2 \]

Twisted boundary condition (tbc)

\[
(n_1, n_2, n_3)(x^1, x^2 + R) = (-n_1, -n_2, n_3)(x^1, x^2)
\]

\[
u(x^1, x^2 + R) = -\nu(x^1, x^2)
\]

\[
(H^1, H^2) = \frac{1}{\sqrt{1 + |\nu|^2}}(1, \nu)
\]

Sigma model instanton (skyrmion in cond-mat)

\[ \mathbb{R}^1 \times S^1 \longrightarrow S^2 \]

Twisted boundary condition (tbc)

\[(n_1, n_2, n_3)(x^1, x^2 + R) = (-n_1, -n_2, n_3)(x^1, x^2)\]

Scherk-Schwarz dimensional reduction

\[(n_1, n_2) = \left( \hat{n}_1(x^1) \cos \frac{\pi}{R} x^2, \hat{n}_2(x^1) \sin \frac{\pi}{R} x^2 \right)\]

Effective potential (twisted mass)

\[ V = \int_0^R dx^2 \left[ (\partial_2 n_1)^2 + (\partial_2 n_2)^2 \right] \]
\[ = m^2 (\hat{n}_1^2 + \hat{n}_2^2) = m^2 (1 - \hat{n}_3^2) \]

Sigma model instanton (skyrmion in cond-mat)

$R^1 \times S^1 \rightarrow S^2$

Identified because of TBC

Sigma model instanton (skyrmion in cond-mat)

$R^1 \times S^1 \rightarrow S^2$

Reconnection occurs!!

Sigma model instanton
(skyrmion in cond-mat)

\[ \mathbb{R}^1 \times S^1 \]

\[ \pi_2 = 1/2 \]

\[ \pi_0 = 1 \]

\[ \pi_2 = 1/2 \]

\[ \pi_0 = -1 \]

\( S^2 \)

Positions & phases

Domain wall

Fractional instantons = (anti-)domain walls with half SG kink

\[ \pi_2 = 1 \]

\[ \pi_2 = -1 \]

instanton

anti-instanton

wall

KK wall

anti-wall

anti-KK wall
Dunne & Unsal (‘12)

**bion**

\[ \pi_2 = 0 \]

---

**bion**

\[ \pi_2 = 0 \]
Interaction energy $E_{\text{int}} \sim -\exp(-\xi d)$ asymptotically

Dunne & Unsal (‘12) resurgence
Explicit configuration of a bion at \textit{arbitrary} distance

Misumi, MN & Sakai (‘14)

JHEP 1406 (2014)164

[arXiv:1404.7225]
Explicit configuration of a bion at *arbitrary* distance

*Misumi, MN & Sakai ('14)*

JHEP 1406 (2014)164

[arXiv:1404.7225]
Action density

Topological charge density

\[ n = \frac{\omega^\dagger(x) \overline{\sigma} \omega(x)}{\omega^\dagger(x) \omega(x)} \]

\[ \omega = \left( 1 + \lambda_2 e^{i\theta_2} e^{\pi(z + \bar{z})}, \lambda_1 e^{i\theta_1} e^{\pi z} \right)^T \]
O(3) model [= CP$^1$ model] on $\mathbb{R}^1 \times S^1$

Target space $M = S^2 \cong O(3) / O(2) \cong CP^1 \cong SU(2) / U(1)$

Generalizations of fractional instantons

(1) $CP^{N-1}$ model on $\mathbb{R}^1 \times S^1$

$CP^{N-1} \cong SU(N) / [SU(N-1) \times U(1)]$

(2) Grassmann model on $\mathbb{R}^1 \times S^1$

$Gr_{N,M} \cong \frac{SU(N)}{SU(N-M) \times SU(M) \times U(1)} \quad \pi_2 = \mathbb{Z}$


later by F.Bruckman (‘08)

O(3) model [= $CP^1$ model] on $R^1 \times S^1$

Target space $M = S^2 \cong O(3)/O(2) \cong CP^1 \cong SU(2)/U(1)$

Generalizations of bions

(1) $CP^{N-1}$ model on $R^1 \times S^1$

$CP^{N-1} \cong SU(N)/[SU(N-1) \times U(1)]$

Dunne & Unsal, JHEP1211(2012)170

PRD87(2013)025015

(2) Grassmann model on $R^1 \times S^1$

$Gr_{N,M} \cong \frac{SU(N)}{SU(N-M) \times SU(M) \times U(1)}$


Dunne & Unsal, arXiv:1505.07803

(3) $O(N)$ model on $R^{N-2} \times S^1$

$O(N)/O(N-1) \cong S^{N-1}$

$\pi_{N-1} = Z$


(4) $SU(N)$ principal chiral model on $R^2 \times S^1$

$M = SU(N)$ or $G$ \hspace{1cm} $\pi_3(M) = Z$

§ $\mathbb{C}P^N$ and Grassmann models
(1) \(CP^{N-1}\) model on \(R^1 \times S^1\)

\[CP^{N-1} \cong SU(N) /[SU(N-1) \times U(1)]\]

(2) Grassmann model on \(R^1 \times S^1\)

\[Gr_{N,M} \cong \frac{SU(N)}{SU(N - M) \times SU(M) \times U(1)}\]

\(M = 1\)

\[\pi_2 = \mathbb{Z}\]

\(U(M)\) gauge theory with complex \(M \times N\) matrix \(H\)

\[H \rightarrow g_C H g_F, \quad g_C \in U(N_C = M), \quad g_F \in SU(N_F = N)_F\]

\[\mathcal{L}_{\text{gauge}} = \text{Tr} \left[ \frac{1}{2g^2} F_{\mu \nu} F_{\mu \nu} + \mathcal{D}_\mu H (\mathcal{D}_\mu H)^\dagger \right] + \text{Tr} \left[ \frac{g^2}{4} \left( v^2 \mathbf{1}_{N_C} - HH^\dagger \right)^2 \right]\]

Twisted b.c. (\(Z_{N_f}\) symmetric)

\[H(x^1, x^2 + R) = H(x^1, x^2) \exp \left[ \frac{2\pi i}{N_f} \text{diag}(1,2,\ldots, N_f - 1) \right]\]

Promote this to supersymmetric theory (8 SUSY)

\(\left( H, \tilde{H} (= 0) \right)\) \(N_f\) hypermulets \(\left( A_\mu, \Sigma \right)\) \(U(N_C)\) gauge multiplets

Vacuum moduli = \(T^* \left[ \frac{SU(N_F)}{SU(N_F - N_C) \times SU(N_C) \times U(1)} \right]\)
Embed it to a D-brane configuration for usefulness
Hanany-Witten setup + Hanany-Tong’s vortices

D3 world-volume = 2+1d $U(N_c)$ gauge + $N_f$ fund matter

$k$ Non-Abelian vortices (lump)

Hanany & Tong ('04)
Compactification along $x^2$

Nontrivial Wilson line on D5

Each D3 ends on one of D5s (s-rule)

For $Gr_{N_f, N_c}$, $N_f!/N_c!(N_f-N_c)!$ vacua
Taking a **T-dual** along $x^2$ (without vortices)

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Taking a **T-dual** along $x^2$ *(with vortices)*

kinky D-brane

Eto, Fujimori, Isozumi, MN, Ohashi, Ohta & Sakai
$CP^1$ fractional instanton and bion

\[ \pi_2 = \frac{1}{2} \]
\[ \pi_0 = 1 \]

wall

\[ \pi_2 = \frac{1}{2} \]
\[ \pi_0 = -1 \]

KK wall

\[ \pi_2 = 1 \]
\[ \pi_0 = 1 \]

instanton

\[ \pi_2 = 1 \]

anti-instanton

\[ \pi_2 = -1 \]

anti-KK wall

\[ \pi_2 = -\frac{1}{2} \]
\[ \pi_0 = 1 \]

anti-wall

\[ \pi_2 = -\frac{1}{2} \]
\[ \pi_0 = -1 \]
$CP^1$ fractional instanton and bion

\[ \pi_2 = 0 \]

\[ \pi_2 = 1/2 \]
\[ \pi_0 = 1 \]

wall

\[ \pi_2 = -1/2 \]
\[ \pi_0 = -1 \]

anti-wall

bion

anti-KK wall

\[ \pi_2 = 0 \]

\[ \pi_2 = -1/2 \]
\[ \pi_0 = 1 \]

KK wall

\[ \pi_2 = 1/2 \]
\[ \pi_0 = -1 \]
$CP^2$ fractional instanton and bion

$CP^2$ instanton $\pi_2 = 1$

\[
\begin{aligned}
\text{fractional instanton} \\
\pi_2 = 1/3
\end{aligned}
\]

$CP^2$ bion $\pi_2 = 0$

$CP^{N_f-1}$ fractional instanton $\pi_2 = 1/N_f$
Gr$_{4,2}$ fractional instanton and bion

Gr$_{Nf,Nc}$ fractional instanton $\pi_2 = 1/N_f$

Gr$_{4,2}$ instanton

$\pi_2 = 1$

fractional instanton

$\pi_2 = 1/4$

Gr$_{4,2}$ bion (the maximum)

For any given config, we gave ansatz with arbitrary positions
§ O(N) model and Principal chiral model
O(4) model = SU(2) principal chiral model on $\mathbb{R}^2 \times S^1$
= Skyrme model (if 4 deriv term added)

$\pi_3[SU(2)] \cong \mathbb{Z}$

$U = \begin{pmatrix} \phi_1 & -\phi_2^* \\ \phi_2 & \phi_1^* \end{pmatrix} \in SU(2) \, |\phi_1|^2 + |\phi_2|^2 = 1$

Skyrmion

A vortex ring of $\phi_1$
with phase of $\phi_2$
twisted once (vorton)

First found in two-component BECs
Ruostekoski & Anglin
PRL ('01)

085007 [arXiv:1407.7210]
$O(4) \text{ model} = \text{SU}(2) \text{ principal chiral model on } \mathbb{R}^2 \times S^1$

$= \text{Skyrme model (if 4 deriv term added)}$

twisted b.c

$U(x + R) = WU(x)W^\dagger \quad W = \sigma_3 = \text{diag}(1, -1)$

vacua $[U, W] = 0$

$vortex$

$\phi_1 \in U(1)$

$\pi_1[U(1)] \cong \mathbb{Z}$

A vortex ring of $\phi_1$

with phase of $\phi_2$
twisted once (vorton)
O(4) model = SU(2) principal chiral model on $\mathbb{R}^2 \times S^1$

= Skyrme model (if 4 deriv term added)

twisted b.c $U(x + R) = WU(x)W^\dagger \quad W = \sigma_3 = \text{diag}(1, -1)$

Fractional instantons

= vortices of $\phi_1$

$\phi_2$ confined inside

phase of $\phi_2$

twisted half

$\pi_3 = 1/2$

$\pi_1 = -1$ vortex

$\pi_3 = 1/2$

$\pi_1 = 1$ KK vortex
vortex $\pi_3$

SG $\pi_1$

kink

$\pi_1$

Instanton

$\pi_1$

Anti-Instanton

$\pi_3$

+1/2

+1/2

-1/2

-1/2

-1

+1

-1/2

+1/2

+1/2

-1/2
vortex $\pi_3$ + 1/2

SG $\pi_1$ + 1 + 1/2

kink $\pi_1$ + 1/2

Interaction energy $E_{\text{int}} \sim \log(d / \xi)$ asymptotically

They are “confined”
SU($N$) principal chiral model on $\mathbb{R}^2 \times S^1$

$U(x + R) = WU(x)W^\dagger$ \quad $W = \text{diag}(1, \omega, \omega^2, \ldots, \omega^{N-1})$

**Vacua:** $[U, W] = 0$

$\omega = \exp(2\pi i / N)$

$U(1)^{N-1}$ Cartan subalgebra of SU($N$)

\[ \pi_1[U(1)^{N-1}] \cong \mathbb{Z}^{N-1} \] \quad \left\{ \begin{array}{c}
\pi_2 \left[ \frac{SU(1)}{U(1)^{N-1}} \right] \cong \mathbb{Z}^{N-1} \\
\text{Monopole charge}
\end{array} \right. \]

**Fractional instantons = global vortices**

$N$ constituents with \[ \pi_3 = 1 / N \]
§ Relations among them
What relations with Yang-Mills instantons, bions?

SU($N$) Yang-Mills instanton
What relations with Yang-Mills instantons, bions?

SU($N$) Yang-Mills instanton

Compactification with TBC (nontrivial holonomy)
What relations with Yang-Mills instantons, bions?

Compactification with TBC (nontrivial holonomy)

monopole  Kalzua-Klein(KK) monopole
What relations with Yang-Mills instantons, bions?

SU($N$) Yang-Mills instanton
What relations with Yang-Mills instantons, bions?

Hanany & Tong ('04)

$\mathbb{CP}^{N-1}$ sigma model instanton

$U(N)$ Non-Abelian vortex

Eff theory = $\mathbb{CP}^{N-1}$ model
What relations with Yang-Mills instantons, bions?

Monopole, KK monopole

\( \mathbb{C}P^{N-1} \) sigma model kinks

U(\( N \)) Non-Abelian vortex

Eff theory = \( \mathbb{C}P^{N-1} \) model

Higgsing

Eto, Isozumi, MN, Ohashi & Sakai ('05)
What relations with Yang-Mills instantons, bions?

Eto, MN, Ohashi & Tong
[hep-th/0508130]

(another)

**Higgsing**

SU($N$) Yang-Mills instanton
(Josephson instanton)

SU($N$) instanton (Skyrmion) in SU($N$) PCM

$U(N)$ Non-Abelian domain wall
Eff theory = SU($N$) PCM
What relations with Yang-Mills instantons, bions?

Monopole, KK monopole

Vortices in SU(N) PCM

U(N) Non-Abelian domain wall Eff theory = SU(N) PCM

(another) Higgsing

**Summary**

SU(\(N\)) Yang-Mills in 4dim

SU(\(N\)) YM instanton
Monopole
KK monopole
bion

\(C^P^{N-1}\) model in 2dim

\(C^P^{N-1}\) instanton
\(C^P^{N-1}\) domain wall
\(C^P^{N-1}\) KK wall
\(C^P^{N-1}\) bion

Non-Abelian vortex

\((d=2+0)\)

1/\(N\) instanton
charge

Non-Abelian domain wall

\((d=3+0)\)

SU(\(N\)) PCM in 3dim

SU(\(N\)) PCM instanton
SU(\(N\)) PCM vortex
SU(\(N\)) PCM KK vortex
SU(\(N\)) PCM bion

**Discussion:** Relations among *resurgence* of these theories
Thank you for your attention!!


See also

Keio U. has started Topological Science Project. We will look for postdocs starting in April.
contact: nitta (at) phys-h.keio.ac.jp