Resurgence and Non-Perturbative Physics: Decoding the Path Integral

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GD, M. Ünsal, G. Başar, 1210.2423, 1210.3646, 1306.4405, 1401.5202, 1501.05671, 1505.07803, 1509.05046, 1511.05977

GD, introductory lectures at CERN 2014 Winter School
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Some Physical Motivation

- improved asymptotics in QFT
- infrared renormalon puzzle in asymptotically free QFT
- non-perturbative physics without instantons: physical meaning of non-BPS saddle configurations

Bigger Picture

- non-perturbative definition of nontrivial QFT in continuum
- analytic continuation of path integrals (Lefschetz thimbles)
- dynamical and non-equilibrium physics from path integrals
Some Mathematical Motivation

Resurgence: ‘new’ idea in mathematics (Écalle, 1980; Stokes, 1850)

resurgence = unification of perturbation theory and non-perturbative physics

- perturbation theory $\rightarrow$ divergent (asymptotic) series
- formal series expansion $\rightarrow$ trans-series expansion
- trans-series ‘well-defined under analytic continuation’
- perturbative and non-perturbative physics entwined
- applications: ODEs, PDEs, fluids, QM, Matrix Models, QFT, String Theory, ...
- philosophical shift: view asymptotics/semiclassics as potentially exact
trans-series expansion in QM and QFT applications:

\[ f(g^2) = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{k-1} c_{k,l,p} g^{2p} \left( \exp \left[ -\frac{c}{g^2} \right] \right)^k \left( \ln \left[ \pm \frac{1}{g^2} \right] \right)^l \]

perturbative fluctuations

k–instantons

quasi-zero-modes

Écalle (1980): functions ‘closed under all operations’:

(Borel transform) + (analytic continuation) + (Laplace transform)

trans-monomial elements: \( g^2, e^{-\frac{1}{g^2}}, \ln(g^2) \), are familiar

“multi-instanton calculus” in QFT

new: analytic continuation encoded in trans-series

new: trans-series coefficients \( c_{k,l,p} \) highly correlated

new: exponential asymptotics (Olver, Kruskal, Segur, Costin, …)
Resurgence

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle, 1980

local analysis encodes more global information than one might naively think
recap: rough basics of Borel summation

(i) divergent, alternating:
\[ \sum_{n=0}^{\infty} (-1)^n \frac{n! g^{2n}}{n!} = \int_0^{\infty} dt \, e^{-t} \frac{1}{1+g^2 t} \]

(ii) divergent, non-alternating:
\[ \sum_{n=0}^{\infty} \frac{n! g^{2n}}{n!} = \int_0^{\infty} dt \, e^{-t} \frac{1}{1-g^2 t} \]

⇒ ambiguous imaginary non-pert. term: \( \pm \frac{i \pi}{g^2} e^{-1/g^2} \)

avoid singularities on \( \mathbb{R}^+ \): **directional** Borel sums:

\[ \theta = 0^\pm \longrightarrow \text{non-perturbative ambiguity: } \pm \text{Im}[Bf(g^2)] \]

challenge: use physical input to resolve ambiguity
Resurgent relations within Trans-series

- trans-series (neglecting logs for now for simplicity)

\[ F(g^2) \sim \left( c_0^{(0)} + c_1^{(0)} g^2 + c_2^{(0)} g^4 + \ldots \right) \]
\[ + \sigma e^{-S/g^2} \left( c_0^{(1)} + c_1^{(1)} g^2 + c_2^{(1)} g^4 + \ldots \right) \]
\[ + \sigma^2 e^{-2S/g^2} \left( c_0^{(2)} + c_1^{(2)} g^2 + c_2^{(2)} g^4 + \ldots \right) \]
\[ + \ldots \]

- **basic idea:** ambiguous imaginary non-perturbative contributions from Borel summation of non-alternating divergent series in one sector must cancel against terms in some other non-perturbative sector

- implies very strong relations between trans-series expansion coefficients in different non-perturbative sectors
Hint of Resurgence in QM Spectral Problems

- QM analog of IR renormalon problem in QFT

- degenerate vacua: double-well, Sine-Gordon, ...

splitting of levels: a real one-instanton effect: \( \Delta E \sim e^{-S/g^2} \)

surprise: pert. theory non-Borel-summable: \( c_n \sim \frac{n!}{(2S)^n} \)

- stable systems
- ambiguous imaginary part
- \( \pm i e^{-2S/g^2} \), a 2-instanton effect
• degenerate vacua: double-well, Sine-Gordon, ...

1. perturbation theory non-Borel-summable:
   ill-defined/incomplete

2. instanton gas picture ill-defined/incomplete:
   $\mathcal{I}$ and $\bar{\mathcal{I}}$ attract

• regularize both by analytic continuation of coupling

⇒ ambiguous, imaginary non-perturbative terms cancel !

“resurgence” ⇒ cancellation to all orders
Mariño, Schiappa, Weiss: *Nonperturbative Effects and the Large-Order Behavior of Matrix Models and Topological Strings* 0711.1954; Mariño, *Nonperturbative effects and nonperturbative definitions in matrix models and topological strings* 0805.3033

- resurgent Borel-Écalle analysis of matrix models

\[ Z(g_s, N) = \int dU \exp \left[ \frac{2}{g_s} \text{tr} V(U) \right] \]

- two variables: \( g_s \) and \( N \) (’t Hooft coupling: \( \lambda = g_s N \))

- e.g. Gross-Witten-Wadia: \( V = U + U^{-1} \)

- 3rd order phase transition at \( \lambda = 1 \), associated with condensation of instantons (Neuberger)

- double-scaling limit: Painlevé II
QFT: new source of divergence in perturbation theory

IR renormalons (pert theory): \[ \rightarrow \pm i e^{-\frac{2S}{\beta_0 g^2}} \]

non-pert instantons (\( \mathbb{R}^2 \) or \( \mathbb{R}^4 \)): \[ \rightarrow \pm i e^{-\frac{2S}{g^2}} \]

appears that BZJ cancellation cannot occur

asymptotically free theories remain inconsistent

't Hooft, 1980; David, 1981
resolution: there is another problem with the non-perturbative instanton gas analysis: **scale modulus of instantons**

- spatial compactification and principle of continuity
- 2 dim. $\mathbb{CP}^{N-1}$ model: $S_{\text{inst}} \rightarrow \frac{S_{\text{inst}}}{N} = \frac{S_{\text{inst}}}{\beta_0}$

\[ \text{inst} \quad \text{anti-inst} \quad \beta_0 \]

\[ \text{UV renormalon poles} \quad \text{IR renormalon poles} \quad \text{neutral bion poles} \]

\[ \rightarrow \quad \text{semiclassical realization of IR renormalons} \]
• 2d $O(N)$ & principal chiral model have no instantons!

• but: have non-BPS finite action solutions

• negative fluctuation modes
twisted b.c.s $\rightarrow$ fractionalize (Cherman et al, 1308.0127, 1403.1277; GD, Unsal, 1505.07803, Nitta et al, ...): saddles = bions in resurgent structure

\[ \int \mathcal{D}A \, e^{-\frac{1}{g^2} S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{g^2} S[A_{\text{saddle}]} \times (\text{fluctuations}) \times (\text{qzm})} \]

• Yang-Mills, $\mathbb{C}P^{N-1}$, $U(N)$ PCM, $O(N)$, $Gr(N,M)$, ... have ‘unstable’ finite action non-BPS saddles

• what do these mean physically?

resurgence: ambiguous imaginary non-perturbative terms should cancel ambiguous imaginary terms from Borel summation of perturbation theory

open problem: non-BPS saddle classification/fluctuations
Resurgence and Localization

Mariño, 1104.0783; Aniceto, Russo, Schiappa, 1410.5834; Wang, Wang, Huang, 1409.4967; Grassi, Hatsuda, Mariño, 1410.7658, ...)

- certain protected quantities in especially symmetric QFTs can be reduced to matrix models ⇒ resurgent asymptotics

- 3d Chern-Simons on $S^3 \rightarrow$ matrix model

\[ Z_{CS}(N, g) = \frac{1}{\operatorname{vol}(U(N))} \int dM \exp \left[ -\frac{1}{g} \operatorname{tr} \left( \frac{1}{2} (\ln M)^2 \right) \right] \]

- ABJM: $\mathcal{N} = 6$ SUSY CS, $G = U(N)_k \times U(N)_{-k}$

\[ Z_{ABJM}(N, k) = \sum_{\sigma \in S_N} \frac{(-1)^{\epsilon(\sigma)}}{N!} \int \prod_{i=1}^{N} \frac{dx_i}{2\pi k} \prod_{i=1}^{N} 2\operatorname{ch} \left( \frac{x_i}{2} \right) \operatorname{ch} \left( \frac{x_i-x_{\sigma(i)}}{2k} \right) \]

- $\mathcal{N} = 4$ SUSY Yang-Mills on $S^4$

\[ Z_{SYM}(N, g^2) = \frac{1}{\operatorname{vol}(U(N))} \int dM \exp \left[ -\frac{1}{g^2} \operatorname{tr} M^2 \right] \]
what is the origin of resurgent behavior in QM and QFT?

to what extent are (all?) multi-instanton effects encoded in perturbation theory? And if so, why?

• QM & QFT: basic property of all-orders steepest descents integrals

• Lefschetz thimbles: analytic continuation of path integrals
• all-orders steepest descents for contour integrals: 

\textit{hyperasymptotics} \quad \text{(Berry/Howls 1991, Howls 1992)}

\[ I^{(n)}(g^2) = \int_{C_n} dz \, e^{-\frac{1}{g^2} f(z)} = \frac{1}{\sqrt{1/g^2}} e^{-\frac{1}{g^2} f_n} T^{(n)}(g^2) \]

• \( T^{(n)}(g^2) \): beyond the usual Gaussian approximation

• asymptotic expansion of fluctuations about the saddle \( n \):

\[ T^{(n)}(g^2) \sim \sum_{r=0}^{\infty} T_r^{(n)} g^{2r} \]
All-Orders Steepest Descents: Darboux Theorem

- universal resurgent relation between different saddles:

\[ T^{(n)}(g^2) = \frac{1}{2\pi i} \sum_m (-1)^{\gamma_{nm}} \int_0^\infty \frac{dv}{v} \frac{e^{-v}}{1 - g^2v/(F_{nm})} T^{(m)} \left( \frac{F_{nm}}{v} \right) \]

- exact resurgent relation between fluctuations about \( n^{\text{th}} \) saddle and about neighboring saddles \( m \)

\[ T_r^{(n)} = \frac{(r - 1)!}{2\pi i} \sum_m (-1)^{\gamma_{nm}} \left( \frac{F_{nm}}{(F_{nm})^r} \right) \left[ T_0^{(m)} + \frac{F_{nm}}{(r - 1)} T_1^{(m)} + \frac{(F_{nm})^2}{(r - 1)(r - 2)} T_2^{(m)} + \ldots \right] \]

- universal factorial divergence of fluctuations (Darboux)
- fluctuations about neighboring saddles explicitly related!
$d = 0$ partition function for periodic potential $V(z) = \sin^2(z)$

$$I(g^2) = \int_0^\pi dz \, e^{-\frac{1}{g^2} \sin^2(z)}$$

two saddle points: $z_0 = 0$ and $z_1 = \frac{\pi}{2}$. 
All-Orders Steepest Descents: Darboux Theorem

- large order behavior about saddle $z_0$:

$$T^{(0)}_r = \frac{\Gamma (r + \frac{1}{2})^2}{\sqrt{\pi} \Gamma (r + 1)}$$

$$\sim \frac{(r - 1)!}{\sqrt{\pi}} \left( 1 - \frac{1}{4} \frac{1}{(r - 1)} + \frac{9}{32} \frac{1}{(r - 1)(r - 2)} - \frac{75}{128} \frac{1}{(r - 1)(r - 2)(r - 3)} + \ldots \right)$$

- low order coefficients about saddle $z_1$:

$$T^{(1)}(g^2) \sim i \sqrt{\pi} \left( 1 - \frac{1}{4} g^2 + \frac{9}{32} g^4 - \frac{75}{128} g^6 + \ldots \right)$$

- fluctuations about the two saddles are explicitly related

- could something like this work for path integrals?

- multi-dimensional case is already non-trivial and interesting

Picard/Lefschetz; Pham (1965); Delabaere/Howls (2002); ...
Resurgence in (Infinite Dim.) Path Integrals  (GD, Ünsal, 1401.5202)

- periodic potential: \( V(x) = \frac{1}{g^2} \sin^2(gx) \)

- vacuum saddle point
  \[
  c_n \sim n! \left( 1 - \frac{5}{2} \cdot \frac{1}{n} - \frac{13}{8} \cdot \frac{1}{n(n-1)} - \cdots \right)
  \]

- instanton/anti-instanton saddle point:
  \[
  \text{Im } E \sim \pi e^{-2\frac{1}{2g^2}} \left( 1 - \frac{5}{2} \cdot g^2 - \frac{13}{8} \cdot g^4 - \cdots \right)
  \]

- double-well potential: \( V(x) = x^2(1-gx)^2 \)

- vacuum saddle point
  \[
  c_n \sim 3^n n! \left( 1 - \frac{53}{6} \cdot \frac{1}{3} \cdot \frac{1}{n} - \frac{1277}{72} \cdot \frac{1}{3^2} \cdot \frac{1}{n(n-1)} - \cdots \right)
  \]

- instanton/anti-instanton saddle point:
  \[
  \text{Im } E \sim \pi e^{-2\frac{1}{6g^2}} \left( 1 - \frac{53}{6} \cdot g^2 - \frac{1277}{72} \cdot g^4 - \cdots \right)
  \]
there is even more resurgent structure ...
Uniform WKB & Resurgent Trans-series

\[-\frac{\hbar^2}{2} \frac{d^2}{dx^2} \psi + V(x)\psi = E\psi\]

- weak coupling: degenerate harmonic classical vacua

⇒ uniform WKB: \( \psi(x) = \frac{D_\nu\left(\frac{1}{\sqrt{\hbar}}\varphi(x)\right)}{\sqrt{\varphi'(x)}} \)

- non-perturbative effects: \( g^2 \leftrightarrow \hbar \) \( \Rightarrow \exp\left(-\frac{S}{\hbar}\right) \)

- trans-series structure follows from exact quantization condition \( \rightarrow E(N, \hbar) = \text{trans-series} \)

- Zinn-Justin, Voros, Pham, Delabaere, Aoki, Takei, Kawai, Koike, ...

- Misumi, Nitta, Sakai, 2015
Zinn-Justin/Jentschura conjecture: generate *entire trans-series* from just two functions:

(i) perturbative expansion \( E = E_{\text{pert}}(\hbar, N) \)
(ii) single-instanton fluctuation function \( \mathcal{P}(\hbar, N) \)
(iii) rule connecting neighbouring vacua (parity, Bloch, ...)

\[
E(\hbar, N) = E_{\text{pert}}(\hbar, N) \pm \frac{\hbar}{\sqrt{2\pi N!}} \left( \frac{32}{\hbar} \right)^{N+\frac{1}{2}} e^{-\frac{S}{\hbar}} \mathcal{P}(\hbar, N) + \ldots
\]

in fact ... (GD, Ünsal, 1306.4405, 1401.5202) fluctuation factor:

\[
\mathcal{P}(\hbar, N) = \frac{\partial E_{\text{pert}}}{\partial N} \exp \left[ S \int_0^\hbar \frac{d\hbar}{\hbar^3} \left( \frac{\partial E_{\text{pert}}(\hbar, N)}{\partial N} - \hbar + \frac{(N + \frac{1}{2}) \hbar^2}{S} \right) \right]
\]

\( \Rightarrow \) perturbation theory \( E_{\text{pert}}(\hbar, N) \) encodes everything!
Resurgence at work

- fluctuations about $I$ (or $\bar{I}$) saddle are determined by those about the vacuum saddle, to all fluctuation orders

- "QFT computation": 3-loop fluctuation about $I$ for double-well and Sine-Gordon:

Escobar-Ruiz/Shuryak/Turbiner 1501.03993, 1505.05115

$$DW: \quad e^{-\frac{S_0}{\hbar}} \left[ 1 - \frac{71}{72} \hbar - 0.607535 \hbar^2 - \ldots \right]$$

resurgence: \quad e^{-\frac{S_0}{\hbar}} \left[ 1 + \frac{1}{72} \hbar \left( -102N^2 - 174N - 71 \right) \right. \\
\left. + \frac{1}{10368} \hbar^2 \left( 10404N^4 + 17496N^3 - 2112N^2 - 14172N - 6299 \right) + \ldots \right]$$

- known for all $N$ and to essentially any loop order, directly from perturbation theory!

- diagrammatically mysterious ...
all orders of multi-instanton trans-series are encoded in
perturbation theory of fluctuations about perturbative vacuum

\[
\int \mathcal{D}A e^{-\frac{1}{g^2} S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{g^2} S[A_{\text{saddle}}]} \times \text{(fluctuations)} \times \text{(qzm)}
\]
Analytic Continuation of Path Integrals: Lefschetz Thimbles

\[ \int \mathcal{D}A \, e^{-\frac{1}{g^2} S[A]} = \sum_{\text{thimbles } k} \mathcal{N}_k \, e^{-\frac{i}{g^2} S_{\text{imag}}[A_k]} \int_{\Gamma_k} \mathcal{D}A \, e^{-\frac{1}{g^2} S_{\text{real}}[A]} \]

Lefschetz thimble = “functional steepest descents contour”

remaining path integral has real measure:
(i) Monte Carlo
(ii) semiclassical expansion
(iii) exact resurgent analysis

resurgence: asymptotic expansions about different saddles are closely related

requires a deeper understanding of complex configurations and analytic continuation of path integrals ... gradient flow

Stokes phenomenon: intersection numbers \( \mathcal{N}_k \) can change with phase of parameters
Thimbles from Gradient Flow

gradient flow to generate steepest descent thimble:

\[
\frac{\partial}{\partial \tau} A(x; \tau) = -\frac{\delta S}{\delta A(x; \tau)}
\]

- keeps $Im[S]$ constant, and $Re[S]$ is monotonic

\[
\frac{\partial}{\partial \tau} \left( \frac{S - \bar{S}}{2i} \right) = -\frac{1}{2i} \int \left( \frac{\delta S}{\delta A} \frac{\partial A}{\partial \tau} - \frac{\delta S}{\delta A} \frac{\partial A}{\partial \tau} \right) = 0
\]

\[
\frac{\partial}{\partial \tau} \left( \frac{S + \bar{S}}{2} \right) = -\int \left| \frac{\delta S}{\delta A} \right|^2
\]

- Chern-Simons theory (Witten 2001)
- complex Langevin (Aarts 2013; ... ; Hayata, Hidaka, Tanizaki, 2015)

resurgence: asymptotics about different saddles related
Ghost Instantons: Analytic Continuation of Path Integrals

\[ Z(g^2|m) = \int Dx e^{-S[x]} = \int Dx e^{-\int d\tau \left( \frac{1}{4} \dot{x}^2 + \frac{1}{g^2} s^2 (g x|m) \right)} \]

- doubly periodic potential: real & complex instantons

\[ a_n(m) \sim -\frac{16}{\pi} n! \left( \frac{1}{(S_{\Pi\Pi}(m))^{n+1}} - \frac{(-1)^{n+1}}{|S_{G\bar{G}}(m)|^{n+1}} \right) \]

\[ \text{without ghost instantons} \quad \text{with ghost instantons} \]

\[ \text{complex instantons directly affect perturbation theory, even though they are not in the original path integral measure} \]

(Başar, GD, Ünsal, arXiv:1308.1108)
Necessity of Complex Saddles

(Behtash, GD, Schäfer, Sulejmanpasic, Ünsal (1510.00978), (1510.03435)

\[ g \mathcal{L} = \frac{1}{2} \dot{x}^2 + \frac{1}{2} (W')^2 \pm \frac{g}{2} W'' \]

- \( W = \frac{1}{3} x^3 - x \rightarrow \) tilted double-well
- \( W = \cos \frac{x}{2} \rightarrow \) double Sine-Gordon
- new (exact) complex saddles (= neutral bions)
Necessity of Complex Saddles

(Schäfer, Sulejmanpasic, Ünsal (1510.00978), (1510.03435)

SUSY QM: $g \mathcal{L} = \frac{1}{2} \dot{x}^2 + \frac{1}{2} (W')^2 \pm \frac{g}{2} W''$

• complex saddles have complex action:
  $$S_{\text{complex bion}} \sim 2S_I + i\pi$$

• $W = \cos \frac{x}{2} \rightarrow$ double Sine-Gordon
  $$E_{\text{ground state}} \sim 0 - 2 e^{-2S_I} - 2 e^{-i\pi} e^{-2S_I} = 0 \quad \checkmark$$

• $W = \frac{1}{3} x^3 - x \rightarrow$ tilted double-well
  $$E_{\text{ground state}} \sim 0 - 2 e^{-i\pi} e^{-2S_I} > 0 \quad \checkmark$$

semiclassics: complex saddles required for SUSY algebra
Connecting weak and strong coupling

important physics question:

does weak coupling analysis contain enough information to extrapolate to strong coupling?

... even if the degrees of freedom re-organize themselves in a very non-trivial way?

what about a QFT in which the vacuum re-arranges itself in a non-trivial manner?

classical (Poincaré) asymptotics is clearly not enough:

is resurgent asymptotics enough?
Resurgence in $\mathcal{N} = 2$ and $\mathcal{N} = 2^*$ Theories (Başar, GD, 1501.05671)

\[-\frac{\hbar^2}{2} \frac{d^2\psi}{dx^2} + \cos(x) \psi = u \psi\]

- electric sector (convergent)
- magnetic sector
- dyonic sector (divergent)

- energy: $u = u(N, \hbar)$; ’t Hooft coupling: $\lambda \equiv N \hbar$
- very different physics for $\lambda \gg 1$, $\lambda \sim 1$, $\lambda \ll 1$
- Mathieu, Lamé encode Nekrasov-Shatashvili superpotential
Resurgence of $\mathcal{N} = 2$ SUSY SU(2)

- moduli parameter: $u = \langle \text{tr} \Phi^2 \rangle$
- electric: $u \gg 1$; magnetic: $u \sim 1$; dyonic: $u \sim -1$
- $a = \langle \text{scalar} \rangle$, $a_D = \langle \text{dual scalar} \rangle$, $a_D = \frac{\partial W}{\partial a}$
- Nekrasov-Shatashvili twisted superpotential $W(a, \hbar, \Lambda)$:
- Mathieu equation:
  $$-\hbar^2 \frac{d^2 \psi}{dx^2} + \Lambda^2 \cos(x) \psi = u \psi, \quad a \equiv \frac{N \hbar}{2}$$
- Matone relation:
  $$u(a, \hbar) = \frac{i\pi}{2} \Lambda \frac{\partial W(a, \hbar, \Lambda)}{\partial \Lambda} - \frac{\hbar^2}{48}$$
Mathieu Equation Spectrum

\[-\frac{\hbar^2}{2} \frac{d^2 \psi}{dx^2} + \cos(x) \psi = u \psi\]

- small \(N\): divergent, non-Borel-summable \(\rightarrow\) trans-series

\[u(N, \hbar) \sim -1 + \hbar \left[ N + \frac{1}{2} \right] - \frac{\hbar^2}{16} \left[ \left( N + \frac{1}{2} \right)^2 + \frac{1}{4} \right] - \frac{\hbar^3}{16^2} \left[ \left( N + \frac{1}{2} \right)^3 + \frac{3}{4} \left( N + \frac{1}{2} \right) \right] - \ldots\]

- large \(N\): convergent expansion: \(\rightarrow\) ?? trans-series ??

\[u(N, \hbar) \sim \frac{\hbar^2}{8} \left( N^2 + \frac{1}{2(N^2 - 1)} \left( \frac{2}{\hbar} \right)^4 + \frac{5N^2 + 7}{32(N^2 - 1)^3(N^2 - 4)} \left( \frac{2}{\hbar} \right)^8 \right.\]

\[+ \frac{9N^4 + 58N^2 + 29}{64(N^2 - 1)^5(N^2 - 4)(N^2 - 9)} \left( \frac{2}{\hbar} \right)^{12} + \ldots \)
Resurgence of $\mathcal{N} = 2$ SUSY SU(2) (Başar, GD, 1501.05671)

- $N\hbar \ll 1$, deep inside wells: resurgent trans-series

$$u^{(\pm)}(N, \hbar) \sim \sum_{n=0}^{\infty} c_n(N)\hbar^n \pm \frac{32}{\sqrt{\pi} N!} \left(\frac{32}{\hbar}\right)^{N-1/2} e^{-\frac{8}{\hbar}} \sum_{n=0}^{\infty} d_n(N)\hbar^n + \ldots$$

- Borel poles at two-instanton location

- $N\hbar \gg 1$, far above barrier: convergent series

$$u^{(\pm)}(N, \hbar) = \frac{\hbar^2 N^2}{8} \sum_{n=0}^{N-1} \frac{\alpha_n(N)}{\hbar^{4n}} \pm \frac{\hbar^2}{8} \left(\frac{2}{\hbar}\right)^{2N} \frac{(2N-1)(N-1)!}{(2N-1)(N-1)!^2} \sum_{n=0}^{N-1} \frac{\beta_n(N)}{\hbar^{4n}} + \ldots$$

(Basar, GD, Ünsal, 2015)

- Coefficients have poles at $O$(two-(complex)-instanton)

- $N\hbar \sim \frac{8}{\pi}$, near barrier top: “instanton condensation”

$$u^{(\pm)}(N, \hbar) \sim 1 \pm \frac{\pi}{16} \hbar + O(\hbar^2)$$
Conclusions

- Resurgence systematically unifies perturbative and non-perturbative analysis
- Trans-series ‘encode’ all information, and expansions about different saddles are intimately related
- Local analysis encodes more than one might think
- Matrix models, large $N$, strings, SUSY QFT
- IR renormalon puzzle in asymptotically free QFT
- Multi-instanton physics from perturbation theory
- $\mathcal{N} = 2$ and $\mathcal{N} = 2^*$ SUSY gauge theory
- Hydrodynamical equations
- Fundamental property of steepest descents expansion
- Analytic continuation for path integrals