Automatic Hermiticity

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1. Introduction

1.1 Motivation

Quantum theory : Feynman path integral
the integrand $\exp\left(\frac{i}{\hbar}S\right)$ is complex

Could S be complex?

Let's seek the possibility

$\rightarrow$ Complex action theory (CAT)
expected to give falsifiable predictions

1.2 Complex action theory

many interesting suggestions made on

- LHC (H.B. Nielsen and M. Ninomiya, 2008)
- Cosmological Constant (H.B.N. and M.N., 2007)
- Higgs mass (H.B.N. and M.N., 2007)
- quantum mechanical philosophy (H.B.N. and M.N., 2009, 2010)
- fine-tuning problems (H.B.N., 2010, H.B.N. and M.N., 2070)
- black holes (H.B.N., 2009)
- De Broglie-Bohm particle and a cut-off in loop diagrams (H.B.N., M. S. Mankoc Borstnik, K.N. and G. Moultaka, 2010)

1.3 future-included or not

They studied a future-included version of the CAT

In contrast to them, we consider a future-not-included version

We study a system defined by a non-hermitian Hamiltonian H, which is correlated to the complex action

cf.) PT symmetric formalism
C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998)
C. M. Bender, S. Boettcher and P. Meisinger, J. Math. Phys. **40**, 2201 (1999)

1.4 Non-hermitian Hamiltonian

What happens?

- time development operator $\exp(-iH(t-t_0))$ is not unitary
  - probability conservation is not held
- eigenvalues are not real
- eigenstates are not orthogonal
  - unphysical transition can be measured

It looks impossible to obtain a reasonable theory, but we can obtain it effectively via two procedures Speculated in

- S. Chadha, C. Litwin and H. B. Nielsen, unpublished
- C. D. Froggatt and H. B. Nielsen, “Origin of Symmetries”, World Scientific Pub. Co. Inc., 1985

1.5 Two procedures

- a proper inner product I_Q
  - eigenstates get orthogonal w.r.t. it
  - H gets normal w.r.t. it (Q-normal)
- "a long time" mechanism
  - states with the highest imaginary part of e.v. of H get favored
  - the anti-hermitian part of H is suppressed
  - hermitian H is obtained

2. Physical significance of an inner product

2.1 Born rule

If a system prepared in $|i\rangle$ at t_i time-develops into $|i(t_f)\rangle$ at time t_f, the probability we measure it in a state $|f\rangle$ is

$$P = |\langle f|i(t_f)\rangle|^2$$

P depends on an inner product of the Hilbert space

2.2 Diagonalization of H

$H = PDP^{-1}$, $D|e_i\rangle = \lambda_i|e_i\rangle$, $\lambda_i(i = 1, \ldots)$: complex

$|e_i\rangle(i = 1, \ldots)$: an orthonormal basis s.t.
$\langle e_i|e_j\rangle = \delta_{ij}$

$$H|\lambda_i\rangle = \lambda_i|\lambda_i\rangle,\ |\lambda_i\rangle = P|e_i\rangle$$

$|\lambda_i\rangle$: eigenstates of H, but not orthogonal in the usual inner product I

$$I(|\lambda_i\rangle, |\lambda_j\rangle) \equiv \langle\lambda_i|\lambda_j\rangle \neq \delta_{ij}$$

A transition from $|\lambda_i\rangle$ to $|\lambda_j\rangle(i \neq j)$ fast in time Δt

$$|I(|\lambda_j\rangle, \exp\left(-\frac{i}{\hbar}H\Delta t\right)|\lambda_i\rangle)|^2 \neq 0,$$

Such a transition should be prohibited

3. A proper inner product and hermitian conjugate

3.1 A proper inner product I_Q

Motivated to define a proper inner product

$$I_Q(|f\rangle, |i\rangle) \equiv \langle f|_Q\, i\rangle,$$

s.t. $I_Q(|\lambda_i\rangle, |\lambda_j\rangle) = \delta_{ij}$

$$I_Q(|\psi_2\rangle, |\psi_1\rangle) = \langle \psi_2|_Q \psi_1\rangle \equiv \langle \psi_2|Q|\psi_1\rangle$$

Q is some operator chosen appropriately

($Q = 1$ for hermitian H)

I is defined to satisfy $\langle \psi_1(t)|\psi_2(t)\rangle = \langle \psi_2(t)|\psi_1(t)\rangle^*$

On I_Q we impose $\langle \psi_1(t)|_Q \psi_2(t)\rangle = \langle \psi_2(t)|_Q \psi_1(t)\rangle^*$

$$\rightarrow Q^\dagger = Q$$

3.2 Q-hermitian conjugate $\dagger_Q$

We define $\dagger_Q$ for

- some operator A

$$\langle\psi_2|_Q A|\psi_1\rangle^* = \langle\psi_1|_Q A^{\dagger_Q}|\psi_2\rangle$$
$$\rightarrow A^{\dagger_Q} = Q^{-1}A^\dagger Q$$

- kets and bras

$$|\psi_1\rangle^{\dagger_Q} \equiv \langle\psi_1|_Q,\ (\langle\psi_2|_Q)^{\dagger_Q} \equiv |\psi_2\rangle$$

$\rightarrow$ We can manipulate $\dagger_Q$ like a usual $\dagger$

When A satisfies $A^{\dagger_Q} = A$, we call A Q-hermitian

* A similar inner product is studied also in
F. G. Scholtz, H. B. Geyer and F. J. W. Hahne, Ann. Phys. 213 (1992) 74.

4. Q-normality of the Hamiltonian

4.1 Q-normality of H

$$P = (|\lambda_1\rangle, |\lambda_2\rangle, \ldots)\ ,\ \text{“}P^{\dagger_Q}\text{”} \equiv \begin{pmatrix} \langle\lambda_1|_Q \\ \langle\lambda_2|_Q \\ \vdots \end{pmatrix}$$

“$P^{\dagger_Q}$”$P = \mathbf{1} \rightarrow P$ is Q-unitary

$$\text{“}P^{\dagger_Q}\text{”}HP = D$$

The (i, j)-component : $\langle\lambda_i|_Q H|\lambda_j\rangle = \lambda_i\delta_{ij}$

Taking the complex conjugate, we obtain

$$\langle\lambda_j|_Q H^{\dagger_Q}|\lambda_i\rangle = \lambda_i^*\delta_{ij}$$

This is written in the operator form as

“$P^{\dagger_Q}$”$H^{\dagger_Q}P = D^\dagger \rightarrow [H, H^{\dagger_Q}] = P[D, D^\dagger]P^{-1} = 0$

4.2 Decomposition of H and D

$$H = H_{Qh} + H_{Qa},$$

$$H_{Qh} \equiv \frac{H+H^{\dagger Q}}{2},\ H_{Qa} \equiv \frac{H-H^{\dagger Q}}{2}$$

$$D = D_R + iD_I,$$

where the diagonal components of D_R and D_I are the real and imaginary parts of that of D respectively,

$$H_{Qh} = PD_RP^{-1},\ H_{Qa} = iPD_IP^{-1}$$

5. Normalization of $|\psi\rangle$ and expectation value

5.1 $|\psi(t)\rangle_N$ and $\mathcal{O}_{QH}$

$|\psi(t)\rangle$ obeying the Schrödinger eq.

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

$$|\psi(t)\rangle_N \equiv \frac{1}{\sqrt{\langle\psi(t)|_Q\ \psi(t)\rangle}}|\psi(t)\rangle$$

The expectation value of $\mathcal{O}$:

$$\overline{\mathcal{O}}_Q(t) \equiv {}_N\langle\psi(t)|_Q\mathcal{O}|\psi(t)\rangle_N = {}_N\langle\psi(t_0)|_Q\mathcal{O}_{QH}(t-t_0)|\psi(t_0)\rangle_N$$

$\mathcal{O}_{QH}(t-t_0)$: operator in the Heisenberg picture

$$\mathcal{O}_{QH}(t-t_0) \equiv \frac{\langle\psi(t_0)|_Q\psi(t_0)\rangle}{\langle\psi(t)|_Q\psi(t)\rangle}e^{\frac{i}{\hbar}H^{\dagger Q}(t-t_0)}\mathcal{O}e^{-\frac{i}{\hbar}H(t-t_0)}$$

5.2 What do $|\psi(t)\rangle_N$ and $\mathcal{O}_{QH}$ obey?

$|\psi(t)\rangle_N$ obeys

$$i\hbar\frac{d}{dt}|\psi(t)\rangle_N = H_{Qh}|\psi(t)\rangle_N + \left(H_{Qa} - {}_N\langle\psi(t)|_Q H_{Qa}|\psi(t)\rangle_N\right)|\psi(t)\rangle_N$$

$\mathcal{O}_{QH}$ obeys

$$i\hbar\frac{d}{dt}\mathcal{O}_{QH} = [\mathcal{O}_{QH}, H_{Qh}] + \{\mathcal{O}_{QH}, H_{Qa} - {}_N\langle\psi(t)|_Q H_{Qa}|\psi(t)\rangle_N\}$$

* The effect of H_{Qa} remains, though it seems to disappear in the classical limit
* With the second procedure, we shall find that it disappears

6. The mechanism for suppressing H_{Qa}

6.1 The time development of $|\psi(t)\rangle$

We expand $|\psi'(t)\rangle \equiv P^{-1}|\psi(t)\rangle$ as

$|\psi'(t)\rangle = \sum_i a_i(t)|e_i\rangle$

Since $|\psi'(t)\rangle$ obeys $i\hbar\frac{d}{dt}|\psi'(t)\rangle = D|\psi'(t)\rangle$,

$$\begin{aligned} |\psi(t)\rangle &= Pe^{-\frac{i}{\hbar}D(t-t_0)}|\psi'(t_0)\rangle \\ &= \sum_i a_i(t_0)e^{\frac{1}{\hbar}(\mathrm{Im}\lambda_i - i\mathrm{Re}\lambda_i)(t-t_0)}|\lambda_i\rangle \end{aligned}$$

* $\mathrm{Im}\lambda_i$ is correlated to $H_{Qa} = iPD_IP^{-1}$
Imagine that some of $\mathrm{Im}\lambda_i$ take the maximum value B (the corresponding subset of $\{i\} \equiv A$)

If a long time has passed, i.e. for large $t - t_0$, the states with $\mathrm{Im}\lambda_i|_{i\in A}$ contribute most in the sum

6.2 Automatic hermiticity

We introduce a diagonalized Hamiltonian $\tilde{D}_R$:

$$\langle e_i|\tilde{D}_R|e_j\rangle \equiv \begin{cases} \langle e_i|D_R|e_j\rangle = \delta_{ij}\mathrm{Re}\lambda_i & \text{for} \quad i \in A \\ 0 & \text{for} \quad i \notin A \end{cases}$$

$H_{\text{eff}} \equiv P\tilde{D}_R P^{-1}$ obeys $H_{\text{eff}}^{\dagger_Q} = H_{\text{eff}}$, $H_{\text{eff}}|\lambda_i\rangle = \mathrm{Re}\lambda_i|\lambda_i\rangle$
We also introduce $|\tilde{\psi}(t)\rangle \equiv \sum_{i\in A} a_i(t)|\lambda_i\rangle$

$$\begin{aligned} |\psi(t)\rangle &\simeq e^{\frac{1}{\hbar}B(t-t_0)} \sum_{i\in A} a_i(t_0) e^{-\frac{i}{\hbar}\mathrm{Re}\lambda_i(t-t_0)}|\lambda_i\rangle \\ &= e^{\frac{1}{\hbar}B(t-t_0)} e^{-\frac{i}{\hbar}H_{\text{eff}}(t-t_0)}|\tilde{\psi}(t_0)\rangle = |\tilde{\psi}(t)\rangle \end{aligned}$$

We have effectively obtained a Q-hermitian Hamiltonian H_{eff} after a long time, though our theory is described by the non-hermitian H at first !

6.3 $|\tilde{\psi}(t)\rangle_N$ and $\tilde{\mathcal{O}}_{QH}$

The normalized state

$$|\psi(t)\rangle_N \simeq \frac{1}{\sqrt{\langle\tilde{\psi}(t)|_Q\ \tilde{\psi}(t)\rangle}}|\tilde{\psi}(t)\rangle \equiv |\tilde{\psi}(t)\rangle_N$$

obeys the Schrödinger eq. $i\hbar\frac{\partial}{\partial t}|\tilde{\psi}(t)\rangle_N = H_{\text{eff}}|\tilde{\psi}(t)\rangle_N$

The expectation value is given by

$$\overline{\mathcal{O}}_Q(t) \simeq {}_N\langle\tilde{\psi}(t)|_Q\mathcal{O}|\tilde{\psi}(t)\rangle_N = {}_N\langle\tilde{\psi}(t_0)|_Q\tilde{\mathcal{O}}_{QH}(t-t_0)|\tilde{\psi}(t_0)\rangle_N$$

$\tilde{\mathcal{O}}_{QH}(t-t_0) \equiv e^{\frac{i}{\hbar}H_{\text{eff}}(t-t_0)}\mathcal{O}e^{-\frac{i}{\hbar}H_{\text{eff}}(t-t_0)}$ obeys the Heisenberg eq. $\frac{d}{dt}\tilde{\mathcal{O}}_{QH}(t-t_0) = \frac{i}{\hbar}[H_{\text{eff}}, \tilde{\mathcal{O}}_{QH}(t-t_0)]$

6.4 Assumption of a local form of $H_{\rm eff}$

If H is written in a local form like $H = \frac{1}{2m}p^2 + V(q)$, does the locality remain even after H becomes hermitian?

It is not clear, but let us assume that $H_{\rm eff}$ has a local expression like $H_{\rm eff} \simeq \frac{1}{2m_{\rm eff}} p_{\rm eff}^2 + V_{\rm eff}(q_{\rm eff})$, and see probability conservation

We define a probability density

$$\rho_{\rm eff} = \tilde{\psi}_Q(q_{\rm eff})^* \tilde{\psi}(q_{\rm eff}) = {}_N\langle \tilde{\psi}(t)|_Q q_{\rm eff}\rangle\langle q_{\rm eff}|\tilde{\psi}(t)\rangle_N,$$

where $\tilde{\psi}(q_{\rm eff}) \equiv \langle q_{\rm eff}|\tilde{\psi}(t)\rangle_N$, $\tilde{\psi}_Q(q_{\rm eff}) \equiv \langle q_{\rm eff}|_Q \tilde{\psi}(t)\rangle_N$

6.5 a conserved probability current density

$i\hbar\frac{\partial}{\partial t}\tilde{\psi}(q_{\text{eff}}) = H_{\text{eff}}\tilde{\psi}(q_{\text{eff}}),\ i\hbar\frac{\partial}{\partial t}\tilde{\psi}_Q(q_{\text{eff}}) = H^*_{\text{eff}}\tilde{\psi}_Q(q_{\text{eff}}),$
$\rightarrow$ a continuity equation

$$\frac{\partial\rho_{\text{eff}}}{\partial t} + \frac{\partial}{\partial q_{\text{eff}}} j_{\text{eff}}(q_{\text{eff}}, t) = 0,$$

where $j_{\text{eff}}(q_{\text{eff}}, t)$ is a probability current density,

$$j_{\text{eff}}(q_{\text{eff}}, t) = \frac{i\hbar}{2m_{\text{eff}}}\left(\frac{\partial}{\partial q_{\text{eff}}}\tilde{\psi}^*_Q\tilde{\psi} - \tilde{\psi}^*_Q\frac{\partial}{\partial q_{\text{eff}}}\tilde{\psi}\right)$$

$\rightarrow$ If H_{eff} has a local expression, we have the probability conservation $\frac{d}{dt}\int\rho_{\text{eff}}dq_{\text{eff}} = 0$

7. A possible misestimation of a past time

- The true past state at an early time t_1:
$$|\psi_{\text{true}}(t_1)\rangle_N = e^{-\frac{i}{\hbar}H(t_1-t_0)}\sqrt{\frac{\langle\psi(t_0)|_Q\psi(t_0)\rangle}{\langle\psi(t_1)|_Q\psi(t_1)\rangle}}|\psi(t_0)\rangle_N$$
- The seeming past state by a historian who lives at a late time t: $|\psi_{\text{historian}}(t_1)\rangle_N =$
$$e^{-\frac{i}{\hbar}H_{\text{eff}}(t_1-t)}e^{-\frac{i}{\hbar}H(t-t_0)}\sqrt{\frac{\langle\psi(t_0)|_Q\psi(t_0)\rangle}{\langle\psi(t)|_Q\psi(t)\rangle}}|\psi(t_0)\rangle_N$$

$|\psi(t_0)\rangle_N$ tends to be hidden from the historian more and more as the time t gets later and later

If our universe had begun with a non-hermitian H at first in some fundamental theory, then we could misestimate the early state

8 Summary and outlook

8.1 Summary

We studied a system described by a non-hermitian H, and effectively obtained the Q-hermitian H_{eff} via

- the proper inner product I_Q
- "a long time" mechanism

$\rightarrow$ At the fundamental level the Hamiltonian does not have to be hermitian

We also

- constructed a conserved probability current density with two kinds of wave functions, assuming that H_{eff} is given in a local form
- pointed out a possible misestimation of a past state by extrapolating back in time with the hermitian H_{eff}

8.2 Outlook

We assumed that the correspondence principle between a quantum regime and a classical one
$\rightarrow$ Desirable to examine it explicitly

At the point where S_I is minimized, we have $\delta S_I \simeq 0$, so that in the region around it S_I is constant practically
$\rightarrow$ little effect of S_I there
$\rightarrow$ consistent with our observation

In the CAT q obtained at the saddle point can be complex in general $\rightarrow$ Extension to complex q
There are many things to be studied