Topological Field Theories in Higher Dimensions and QP Structures

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目的　　**高次元(≥ 3) TFTを作る**

Physics

§1. 作り方 例：2次元

Izawa '99

2次元manifold Σ を考える。 local coordinate σ^μ $(\mu = 1, 2)$.$A_{\mu,i}(\sigma)$: ゲージ場、$\phi^i(\sigma)$: スカラー場

$$S_0 = \int_\Sigma d^2\sigma \epsilon^{\mu\nu} A_{\mu,i} \partial_\nu \phi^i = -\frac{1}{2} \int_\Sigma d^2\sigma \epsilon^{\mu\nu} F_{\mu\nu,i} \phi^i$$

とすると、この理論は$U(1)$ゲージ対称性 $\delta_0 A_{\mu,i} = \partial_\mu \epsilon_i$, $\delta_0 \phi^i = 0$, を持つ。(abelian BF theory)

consistent な相互作用項S_Iを決める。

$S = S_0 + S_I$, $\delta = \delta_0 + \delta_I$ とすると、Sのゲージ変換の満たすべき条件は、$\delta S = 0$ (off shell) かつ $\{\delta_\epsilon, \delta_{\epsilon'}\} = \delta_{[\epsilon,\epsilon']} +$ (equations of motion)。

Batalin-Vilkovisky 形式

1. ゲージパラメータ ϵ^i を FP ghost c^i にかえる。c^i: anticommuting boson
(ゴースト数 gh Φ. $\quad \mathrm{gh}\ A_{\mu,i} = \mathrm{gh}\ \phi^i = 0$, $\mathrm{gh}\ c^i = 1$)
2. ゲージ変換 δ_0 を BRST 変換に読みかえ、$\delta_0^2 = 0$ を課す。これより $\delta_0 c = 0$.
3. それぞれの field Φ に対して、統計が逆の antifield Φ^* を導入する。ghost number は $\mathrm{gh}\ \Phi + \mathrm{gh}\ \Phi^* = -1$ と定義する。
$A_{\mu,i} \to A^{\mu * i}, \quad (\mathrm{gh}\ A^{\mu * i} = -1)$, $\phi^i \to \phi_i^*, \quad (\mathrm{gh}\ \phi_i^* = -1)$, $c_i \to c^{*i}, \quad (\mathrm{gh}\ c^{*i} = -2)$.
4. antibracket を導入する。

$$\{F, G\} \equiv \sum_{\Phi} \int \left(F \frac{\overleftarrow{\partial}}{\partial \Phi(\sigma)} \frac{\overrightarrow{\partial}}{\partial \Phi^*(\sigma')} G - F \frac{\overleftarrow{\partial}}{\partial \Phi^*(\sigma)} \frac{\overrightarrow{\partial}}{\partial \Phi(\sigma')} G \right) \delta^2(\sigma - \sigma').$$

5. Batalin-Vilkovisky (BV) action S_0 を以下のように構成する。

$$S_{BV0} = S_0 + (-1)^{\mathrm{gh}\Phi} \int_{\Sigma} \Phi^* \delta_0 \Phi + O(\Phi^{*2}).$$

$O(\Phi^{*2})$ は $\{S_{BV0}, S_{BV0}\} = 0$ となるように決める。$\{S_{BV0}, S_{BV0}\} = 0$ を classical master equation という。今の場合、

$$S_{BV0} = \int_{\Sigma} d^2\sigma \epsilon^{\mu\nu} A_{\mu,i} \partial_\nu \phi^i + \int_{\Sigma} d^2\sigma A^{\nu * i} \partial_\nu c_i$$

6. $\{S_{BV0}, S_{BV0}\} = 0$ より、ゲージ不変性 $\longrightarrow \delta_0 S_{BV0} = \{S_{BV0}, S_{BV0}\} = 0.$
ゲージ代数が閉じる $\longrightarrow \delta_0^2 F = \{S_{BV0}, \{S_{BV0}, F\}\} = \frac{1}{2}\{\{S_{BV0}, S_{BV0}\}, F\} = 0.$

変形理論 $S = S_{BV0} + gS_1 + g^2 S_2 + \cdots$, として、$\{S, S\} = 0$ の解 S_a を順に求めていく。(descent equations を解く)

$A_i \equiv dx^\mu A_{\mu,i}$, $A^{+i} \equiv dx^\mu \epsilon_{\mu\nu} A^{\nu * i}$, $\phi_i^+ = *\phi_i^*$, $c^{+i} = *c^{*i}$, として、

一般解

$$S_1 = \int \frac{1}{2} f^{ij}(\phi)(A_i A_j - 2\phi_i^+ c_j) + \frac{\partial f^{ij}}{\partial \phi^k}\left(\frac{1}{2} c^{+k} c_i c_j - A^{+k} A_i c_j\right) + \frac{1}{4}\frac{\partial^2 f^{ij}}{\partial \phi^k \partial \phi^l} A^{+k} A^{+l} c_i c_j,$$
$$S_a = 0 \qquad (a = 2, 3, \cdots).$$

$f^{ij}(\phi)$ は ϕ の関数で

$$\frac{\partial f^{ij}}{\partial \phi^m}(\phi) f^{mk}(\phi) + \frac{\partial f^{jk}}{\partial \phi^m}(\phi) f^{mi}(\phi) + \frac{\partial f^{kl}}{\partial \phi^m}(\phi) f^{mj}(\phi) = 0$$

$\Phi^* = 0$ とおくと、

$$S = \int_\Sigma d^2\sigma \left(\epsilon^{\mu\nu} A_{\mu,i} \partial_\nu \phi^i + \frac{1}{2}\epsilon^{\mu\nu} f^{ij}(\phi) A_{\mu,i} A_{\nu,j}\right).$$

The Poisson sigma model or nonlinear gauge theory N. I. and Izawa '93, Schaller and Strobl '94

Theorem 1. *kinetic term* が *2*次元 *abelian BF* 理論のとき、もっとも一般的な拡張は *Poisson sigma model* である。

性質

1. $f^{ij}(\phi) = f^{ij}{}_k \phi^k$ のとき、2次元 nonabelian BF 理論
2. ゲージ対称性

$$\delta\phi^i = -f^{ij}(\phi)\epsilon_j, \qquad \delta A_{\mu,i} = \partial_\mu \epsilon_i + \frac{1}{2}\frac{\partial f^{jk}(\phi)}{\partial \phi^i} A_{\mu,j}\epsilon_k.$$

3. gauge algebra は Lie algebroid　　Levin, Olshanetsky '00

$$f_1{}^m{}_j \frac{\partial f_1{}^i{}_k}{\partial \phi^m} - f_1{}^m{}_k \frac{\partial f_1{}^i{}_j}{\partial \phi^m} + f_1{}^i{}_m f_2{}^m{}_{jk} = 0,$$

$$f_1{}^m{}_{[i} \frac{\partial f_2{}^l{}_{jk]}}{\partial \phi^m} + f_2{}^l{}_{m[i} f_2{}^m{}_{jk]} = 0,$$

Take $f_1{}^{ij} = f^{ij}(\phi)$, $f_{2i}{}^{jk} = \frac{\partial f^{jk}}{\partial \phi^i}(\phi)$.

4. tree (disc) open string amplitude は Poisson 多様体 M の変形量子化（*積）と一致する。(Kontsevich 公式)　　Kontsevich '97, Cattaneo, Felder '99

§4. Supefield 形式

よく見ると、Poisson sigma model の BV action $S = S_{BV0} + S_1$ は、superfield で書ける。(field を再定義して $g = 1$ とする。) supercoordinate θ^μ $(\mu = 1, 2)$ を導入する。(degree=ghost number: $|\theta| = 1$.) superfield を

$$\boldsymbol{\phi}^i(\sigma, \theta) \equiv \phi^i + \theta^\mu A_\mu^{+i} + \frac{1}{2}\theta^\mu\theta^\nu c_{\mu\nu}^{+i}, \qquad |\boldsymbol{\phi}| = 0,$$

$$\boldsymbol{A}_i(\sigma, \theta) \equiv -c_i + \theta^\mu A_{\mu,i} + \frac{1}{2}\theta^\mu\theta^\nu \phi_{\mu\nu,i}^{+}, \qquad |\boldsymbol{A}| = 1,$$

と定義すると BV action は

$$S = \int d^2\sigma d^2\theta \left(\boldsymbol{A}_i \boldsymbol{d}\boldsymbol{\phi}^i + \frac{1}{2} f^{ij}(\boldsymbol{\phi}) \boldsymbol{A}_i \boldsymbol{A}_j \right)$$

ここで、$\boldsymbol{d} \equiv \theta^\mu \partial_\mu$, antibracket は

$$\{F, G\} \equiv \int d^2\sigma d^2\theta \left(F \frac{\overleftarrow{\partial}}{\partial \boldsymbol{\phi}^i} \frac{\overrightarrow{\partial}}{\partial \boldsymbol{A}_i} G - F \frac{\overleftarrow{\partial}}{\partial \boldsymbol{A}_i} \frac{\overrightarrow{\partial}}{\partial \boldsymbol{\phi}^i} G \right)$$

(super Poisson bracket of degree 1) BRST 変換は

$$\delta \boldsymbol{\phi}^i = \{S, \boldsymbol{\phi}^i\} = \boldsymbol{d}\boldsymbol{\phi}^i + f^{ij}(\boldsymbol{\phi}) \boldsymbol{A}_j, \qquad \delta \boldsymbol{A}_i = \{S, \boldsymbol{A}_i\} = \boldsymbol{d}\boldsymbol{A}_i + \frac{1}{2} \frac{\partial f^{jk}}{\partial \boldsymbol{\phi}^i}(\boldsymbol{\phi}) \boldsymbol{A}_j \boldsymbol{A}_k,$$

§5. 3次元

The most general deformation of a BF type topological field theory in three dimensions:
(ϕ: scalar, A_1: 1-form, B_2: 2-form) N.I. '00

$$
\begin{aligned}
S = S_0 + S_1 &= \int_X \left[-B_{2,i} d\phi^i + \frac{k_{ab}}{2} {A_1}^a d{A_1}^b \right] \\
&+ \int_X \left[{f_1}^i{}_a(\phi) {A_1}^a B_{2,i} + \frac{1}{6} f_{2abc}(\phi) {A_1}^a {A_1}^b {A_1}^c \right],
\end{aligned}
$$

Properties

1. the Chern-Simons theory as a special case.
2. the Rozansky-Witten theory Qiu, Zabzine, '09
3. Gauge Algebra

$$
\begin{aligned}
& k^{ab} {f_1}^i{}_a {f_1}^j{}_b = 0, \\
& \frac{\partial {f_1}^i{}_b}{\partial \phi^j} {f_1}^j{}_c - \frac{\partial {f_1}^i{}_c}{\partial \phi^j} {f_1}^j{}_b + k^{ef} {f_1}^i{}_e f_{2fbc} = 0, \\
& \left({f_1}^i{}_d \frac{\partial f_{2abc}}{\partial \phi^i} - {f_1}^i{}_c \frac{\partial f_{2dab}}{\partial \phi^i} + {f_1}^i{}_b \frac{\partial f_{2cda}}{\partial \phi^i} - {f_1}^i{}_a \frac{\partial f_{2bcd}}{\partial \phi^i} \right) \\
& \qquad + k^{ef} (f_{2eab} f_{2cdf} + f_{2eac} f_{2dbf} + f_{2ead} f_{2bcf}) = 0,
\end{aligned}
$$

 Courant '90, Liu, Weinstein, Xu '96, Roytenberg '01

Theorem 2. *A gauge algebra of a topological field theory in three dimensions=Courant algebroid*
N.I. '02, Hofman, Park '02, Roytenberg '06

• A bilinear bracket of vector fields X, Y and 1-forms ξ, η, which is diffeomorphism invariant and satisfies the Jacobi identity: $[X+\xi, Y+\eta]_C = [X,Y] + L_X\eta - i_Y d\xi$

§6. 4次元

N.I., Uchino '10

superfields
$\boldsymbol{x}^i$: including a scalar $x^i(\sigma)$. $\boldsymbol{q}^a$: including a 1-form ${q_\mu}^a(\sigma)$.
$\boldsymbol{p}_a$: including a 2-form $p_{\mu\nu,a}(\sigma)$. $\boldsymbol{\xi}_i$: including a 3-form $\xi_{\mu\nu\lambda,i}(\sigma)$.

$$\begin{aligned} S &= S_0 + S_1, \\ S_0 &= \int_{\mathcal{X}} d^4\sigma d^4\theta(\boldsymbol{\xi}_i \boldsymbol{d}\boldsymbol{x}^i - \boldsymbol{p}_a \boldsymbol{d}\boldsymbol{q}^a), \\ S_1 &= \int_{\mathcal{X}} d^4\sigma d^4\theta({f_1}^i{}_a(\boldsymbol{x})\boldsymbol{\xi}_i\boldsymbol{q}^a + \frac{1}{2}{f_2}^{ab}(\boldsymbol{x})\boldsymbol{p}_a\boldsymbol{p}_b + \frac{1}{2}{f_3}^a{}_{bc}(\boldsymbol{x})\boldsymbol{p}_a\boldsymbol{q}^b\boldsymbol{q}^c \\ &\quad + \frac{1}{4!}f_{4abcd}(\boldsymbol{x})\boldsymbol{q}^a\boldsymbol{q}^b\boldsymbol{q}^c\boldsymbol{q}^d). \end{aligned}$$

Gauge Algebra: Lie algebroid up to homotopy (twisted Lie algebroid, Lie 3-algebroid).

N.I., Uchino '10

$$
\begin{aligned}
&{f_1}^i{}_b {f_2}^{ba} = 0, \\
&{f_1}^k{}_c \frac{\partial {f_2}^{ab}}{\partial x^k} + {f_2}^{da} {f_3}^b{}_{cd} + {f_2}^{db} {f_3}^a{}_{cd} = 0, \\
&{f_1}^k{}_b \frac{\partial {f_1}^i{}_a}{\partial x^k} - {f_1}^k{}_a \frac{\partial {f_1}^i{}_b}{\partial x^k} + {f_1}^i{}_c {f_3}^c{}_{ab} = 0, \\
&{f_1}^k{}_{[d} \frac{\partial {f_3}^a{}_{bc]}}{\partial x^k} + {f_2}^{ae} f_{4bcde} - {f_3}^a{}_{e[b} {f_3}^e{}_{cd]} = 0, \\
&{f_1}^k{}_{[a} \frac{\partial f_{4bcde]}}{\partial x^k} + {f_3}^f{}_{[ab} f_{4cde]f} = 0.
\end{aligned}
$$

• nonabelian BF 理論, Topological Yang-Mills 理論、Topological F 理論を含む。

Mathematics

§7. QP Manifold

Definition 1. *A following triple $(\mathcal{M}, \omega, Q)$ is called a (classical) QP-manifold of degree n.*

• Graded manifold $\mathcal{M}$: a graded manifold with nonnegative degrees (N-manifold) of degrees n (a ringed space with a structure sheaf of nonnegatively graded algebras)

• P-structure

A graded symplectic form ω of degree n on $\mathcal{M}$.

• Q-structure

A vector field Q of degree $+1$ on $\mathcal{M}$ satifying $Q^2 = 0$ and compatible with the P-structure, that is, $\mathcal{L}_Q\omega = 0$.

Notes

$\Theta \in C^\infty(\mathcal{M})$ of degree $n+1$ satifying $Q(-) = \{\Theta, -\}$. $Q^2 = 0 \leftrightarrow \{\Theta, \Theta\} = 0$.

§8. AKSZ Construction of Topological Field Theory

Alexandrov, Kontsevich, Schwartz, Zaboronsky '97, Cattaneo, Felder '01, Roytenberg '06

Let X be a manifold in k dimensions.

$(\mathcal{X}, D)$: a differential graded (dg) manifold with a D-invariant nondegenerat measure μ. $\mathcal{X}$: graded manifold such that $\mathcal{X}|_0 = X$, D: differential on $\mathcal{X}$.

$(\mathcal{M}, \omega, Q)$: QP-manifold of degree n

An *evaluation map* $\mathrm{ev} : \mathcal{X} \times \mathcal{M}^{\mathcal{X}} \longrightarrow \mathcal{M}$ is defined as $\mathrm{ev} : (z, \Phi) \longmapsto \Phi(z)$, where $z \in \mathcal{X}$ and $\Phi \in \mathcal{M}^{\mathcal{X}}$.

A *chain map* $\mu_* : \Omega^\bullet(\mathcal{X} \times \mathcal{M}^{\mathcal{X}}) \longrightarrow \Omega^\bullet(\mathcal{M}^{\mathcal{X}})$ is defined as $\mu_* F = \int_{\mathcal{X}} \mu F$ where $F \in \Omega^\bullet(\mathcal{X} \times \mathcal{M})$ and $\int_{\mathcal{X}} \mu$ is a Berezin integration on $\mathcal{X}$.

Definition 2. [P-structure] *For a graded symplectic form ω on $\mathcal{M}$, we define the graded symplectic form on $\mathrm{Map}(\mathcal{X}, \mathcal{M})$ as $\boldsymbol{\omega} = \mu_* \mathrm{ev}^* \omega$.*

Definition 3. [Q-structure (BV action)] $S = S_0 + S_1$ *on* $\mathrm{Map}(\mathcal{X}, \mathcal{M})$. $S_0 := \iota_{\hat{D}} \mu_* \mathrm{ev}^* \vartheta$, *where* $\omega = -d\vartheta$, *and* $S_1 := \mu_* \mathrm{ev}^* \Theta$.

Theorem 3. [Alexandrov, Kontsevich, Schwartz, Zaboronsky '97] *If $\mathcal{X}$ is a dg manifold with a compatible measure and $\mathcal{M}$ is a QP-manifold, the graded manifold $\mathrm{Map}(\mathcal{X}, \mathcal{M})$ has a QP-structure. i.e.* $\{\Theta, \Theta\} = 0 \Longleftrightarrow \{S, S\} = 0$.

The QP-structure on $\mathrm{Map}(\mathcal{X}, \mathcal{M})$ is equivalent to a BV formalism of a topological field theory in $n+1$ dimensions if $\mathcal{X} = T[1]X$ and $k = n+1$.

Deformation Theory

§9. Local Coordinate

1: $\mathcal{X} = T[1]X$.
supercoordinate (σ^μ, θ^μ) in $n+1$ dimensions (degree $\deg_{\mathcal{X}}(\sigma^\mu, \theta^\mu) = (0, 1)$).
superderivative $\boldsymbol{d} = \theta^\mu \frac{\partial}{\partial \sigma^\mu}$

2: $\mathcal{M}$ is a graded manifold with degree n.
$\boldsymbol{e}_{a(i)}(\sigma, \theta)$ is a section of $T^*[1]X \otimes \boldsymbol{x}^*(\mathcal{M}_i)$ such that $|\boldsymbol{e}_{a(i)}| = \deg_{\mathcal{M}} \boldsymbol{e}_{a(i)} = i$, which is called a **superfield** of degree i, where $i = 0, 1, \cdots, n$.

$$\begin{aligned}
\boldsymbol{e}_{a(i)}(\sigma, \theta) &= \boldsymbol{x}^{a(0)}(\sigma, \theta) = \boldsymbol{q}^{a(0)}(\sigma, \theta) && \text{for } i = 0, \\
&= \boldsymbol{q}^{a(i)}(\sigma, \theta) && \text{for } 1 \leq i \leq \lfloor n/2 \rfloor, \\
&= \boldsymbol{p}_{a(n-i)}(\sigma, \theta) && \text{for } \lfloor n/2 \rfloor < i \leq n-1, \\
&= \boldsymbol{\xi}_{a(0)}(\sigma, \theta) = \boldsymbol{p}_{a(0)}(\sigma, \theta) && \text{for } i = n.
\end{aligned}$$

We expand $\boldsymbol{e}_{a(i)}(\sigma, \theta) = \sum_{k, \mu(k)} \frac{1}{k!} \theta^{\mu(1)} \cdots \theta^{\mu(k)} \boldsymbol{e}^{(k)}_{a(i), \mu(1) \cdots \mu(k)}(\sigma)$.

§10. Gauge Symmetry in $n = 3$

N.I.,Uchino '10

$\mathcal{M} = T^*[3]E[1]$: graded manifold with grading $(0, 1, 2, 3)$

$\omega = d\xi_i \wedge dx^i - dp_a \wedge dq^a$

The corresponding QP-structure on $\mathcal{M}$ is

$$\begin{aligned}\Theta &= f_1{}^i{}_a(x)\xi_i q^a + \frac{1}{2} f_2{}^{ab}(x) p_a p_b + \frac{1}{2} f_3{}^a{}_{bc}(x) p_a q^b q^c + \frac{1}{4!} f_{4abcd}(x) q^a q^b q^c q^d, \\ &= \theta_1 + \theta_2 + \theta_3 + \theta_4, \\ \theta_{13} &:= \theta_1 + \theta_3,\end{aligned}$$

where a local coordinate is (x^i, q^a, p_a, ξ_i) of degree $(0, 1, 2, 3)$ on $\mathcal{M}$.

Operations is defined by derived brackets.

1. $\rho(e)(F) := \{\{\theta_{13}, e\}, F\}$; a bundle map $\rho : E \to TM$,
2. $(\alpha_1, \alpha_2) := \{\{\theta_2, \alpha_1\}, \alpha_2\}$; a C^∞-valued symmetric pairing on E^*,
3. $[e_1, e_2] := \{\{\theta_{13}, e_1\}, e_2\}$; a skewsymmetric bracket on ΓE.
4. $\Omega(e_1, e_2, e_3, e_4) := \{\{\{\{\theta_4, e_1\}, e_2\}, e_3\}, e_4\}$; a 4-form on E.

where $e_k \in \Gamma E$ and $\alpha_k \in \Gamma E^*$.

Algebroids (Anchored almost Lie algebroids)

Definition 4. *Let $E \to M$ be a vector bundle over M, let $[-,-]$ be a skewsymmetric bracket defined on the space of sections ΓE and let $\rho : E \to TM$ a bundle map. The triple $(E, [-,-], \rho)$ is called an algebroid, if the following two conditions are satisfied,*

(A0) $\rho[e_1, e_2] = [\rho(e_1), \rho(e_2)]$,
(A1) $[e_1, Fe_2] = F[e_1, e_2] + \rho(e_1)(F)e_2$,

where $e_1, e_2 \in \Gamma E$ and $F \in C^\infty(M)$.

Theorem 4. *Operations 1.3. become an algebroid.*

• de Rham type derivation on $\Gamma \wedge^{\cdot} E^*$ (from the algebroid structure), $\delta(-) := \{\theta_{13}, -\}$, $\delta\delta \neq 0$ in general.

• a Lie type derivation on ΓE^*, $\mathcal{L}_e(\alpha) := \{\{\theta_{13}, e\}, \alpha\}$.

$3'$. a symmetric bundle map $\partial : E^* \to E$, $(\alpha_1, \alpha_2) = \langle \partial\alpha_1 , \alpha_2 \rangle$,
($\langle - , - \rangle$ is a natural pairing of E and E^*.)

Definition 5. [N.I., Uchino '10] *Let $(E, [-,-], \rho)$ be an algebroid equipped with a symmetric bundle map $\partial : E^* \to E$ and a skewsymmetric 4-form Ω. We call the algebroid a Lie algebroud up to homotopy (twisted Lie algebroid), if the four identities (A2)-(A5) are satisfied.*

(A2) $[[e_1, e_2], e_3] + (\text{cyclic permutations}) = \partial\Omega(e_1, e_2, e_3)$.

(A3) $\rho\partial = 0$,

(A4) $\rho(e)(\alpha_1, \alpha_2) = (\mathcal{L}_e\alpha_1, \alpha_2) + (\alpha_1, \mathcal{L}_e\alpha_2)$,

(A5) $\delta\Omega = 0$,

Theorem 5. [N.I., Uchino '10] *A (classical) QP-structure of degree 3 on $\mathcal{M} = T^*[3]E[1]$ is a Lie algebroid up to homotopy on E.*

• Higher Courant-Dorfman brackets
$[-, -]_{CD} := \{\{\Theta, -\}, -\}$ is called a Courant-Dorfman (CD) bracket. It is a Loday bracket. $C^2(\mathcal{M}) = \Gamma(E \oplus \wedge^2 E^*)$ is a subalgebra. If $\theta_2 = 0$, the CD-bracket on $E \oplus \wedge^2 E^*$ has the following form

$$[e_1 + \beta_1, e_2 + \beta_2]_{CD} = [e_1, e_2] + \mathcal{L}_{e_1}\beta_2 - i_{e_2}\delta\beta_1 + \Omega(e_1, e_2),$$

where $\beta_1, \beta_2 \in \Gamma \wedge^2 E^*$. Hagiwara '02, Sheng '10

Example in $n = 3$

Topological Yang-Mills Theory. We consider a semi-simple Lie algebra $\mathfrak{g}$ and a nondegenerate Killing form $(-, -)_K$. Consider a vector bundle E with a fibre $\mathfrak{g}$ and $\mathcal{M} = T^*[3]E[1]$.

$\Theta = \theta_2 + \theta_3$, where $\theta_2 = (p, p)_{K-1}$ and $\{\Theta, \Theta\} = 0$.

$$S = \int_{\mathcal{X}} \mu \, (-\boldsymbol{p}_a \boldsymbol{F}^a + k^{ab} \boldsymbol{p}_a \boldsymbol{p}_b).$$

This is equivalent to a topological Yang-Mills theory,

$$S = -\frac{1}{4} \int_{\mathcal{X}} \mu \; k_{ab} \boldsymbol{F}^a \boldsymbol{F}^b.$$

§11. Classical Observables

A classical observable $\mathcal{O}$ is defined as $Q\mathcal{O} = \{S, \mathcal{O}\} = 0$. Since $S = S_0 + S_1$, define $Q_0\mathcal{O} = \{S_0, \mathcal{O}\} = \boldsymbol{d}\mathcal{O}, \quad Q_1\mathcal{O} = \{S_1, \mathcal{O}\}$. Then since $Q_0^2 = 0$, $Q_1^2 = 0$ and $\{Q_0, Q_1\} = 0$ are satisfied, Q_0 and Q_1 define a double complex.

Q-Cohomology

Theorem 6. *For a Q_1 cohomology class $\mathcal{O}$, $\int \mathcal{O}$ is a Q-cocycle since $Q\mathcal{O} = Q_0\mathcal{O} = \boldsymbol{d}\mathcal{O}$.*

We assume that the fiber of $T^*[3]E[1]$ is a direct sum of semi-simple Lie algebras. We consider a special obeservable which is Q_1-cohomology class. Then an invariant polynomial is constructed as $W_N = c_N \mathrm{tr} \; \mathcal{O}^N$, where tr is a trace with respect to the fiber and c_N is a normalization constant. From the theorem, the integration on a k-cycle γ_k,

$$\boxed{\mathcal{O}_k^{(N)} = \int_{\gamma_k} W_N}$$

is a Q-cohomology class.

In the example, a topological Yang-Mills theory, we consider

$$W_2 \quad = \quad -\frac{1}{2}\mathrm{tr}\ (\boldsymbol{p}^a)^2,$$

Set $\boldsymbol{q} = c + A + 0 + 0 + 0, \boldsymbol{p} = -i\phi + i\psi + B + 0 + 0$, and define $F^a = \boldsymbol{d}A^a - \frac{1}{2} f_3{}^a{}_{bc} A^b A^c$. Using the equations of motion,

$$W_2 \quad = \quad -\frac{1}{2}\mathrm{tr}\left[(\phi + \psi + F)^2\right] = -\frac{1}{2}\mathrm{tr}F^2 + i\mathrm{tr}\psi F + \mathrm{tr}\left(\frac{1}{2}\psi^2 - i\phi F\right) - \mathrm{tr}\psi\phi + \frac{1}{2}\mathrm{tr}\phi^2$$

is a Donaldson polynomial invariant. Witten '88, Kanno '89

Reference:

N. I., Donaldson Invariants and Their Generalizations from AKSZ Topological Field Theories. arXiv:1104.2100 [hep-th].