

CFT Duals for Extreme Black Holes

Tatsuma Nishioka

(Kyoto, IPMU)

based on

- arXiv:0811.4393 with T. Hartman, K. Murata and A. Strominger
- arXiv:0902.1001 with G. Compere and K. Murata

March 19, 2009 @ KEK workshop

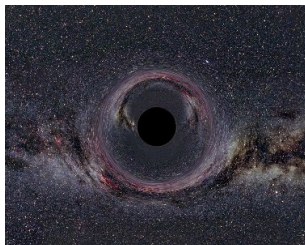
Introduction

What is a black hole?

- solution of the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

- *"a region of space in which the gravitational field is so powerful that nothing, including electromagnetic radiation (e.g. visible light), can escape its pull after having fallen past its event horizon"* (from Wikipedia)
- It seems to exist in our universe!



Hawking temperature

Consider a Schwarzschild black hole

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2^2$$

Near the horizon $r = 2M + \rho^2/8M$ (and after changing $t \rightarrow -i\tau$)

$$ds^2 = \frac{\rho^2}{16M^2} d\tau^2 + d\rho^2 + (2M)^2 d\Omega_2^2$$

To avoid a conical singularity at $\rho = 0$, we obtain the Hawking temperature

$$\tau \sim \tau + \frac{1}{T_H}, \quad T_H = \frac{1}{8\pi M}$$

Then, we can consider the thermodynamics of the black holes!

Black hole horizon and Entropy

If a small mass dM is added to the black hole, the entropy increases

$$dS_{BH} = \frac{dM}{T_H} = d(4\pi M^2) \quad \Rightarrow \quad S_{BH} = \pi(2M)^2$$

Then the black hole has an **entropy** given by the Bekenstein-Hawking area law (Bekenstein '73, Hawking '74)

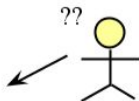
$$S_{BH} = \frac{\text{Area(Horizon)}}{4}$$

This law is applicable to the other black holes including Kerr and Reissner-Nordstrom black holes

How to interpret black hole entropy?

Naively,

- Black hole is “one particular” solution in general relativity
- Entropy is defined as $\log(\text{degeneracy})$
- $S_{BH} = 0$??



We don't know inside the black hole

⇒ Is there fundamental degrees of freedom inside it?

- it is mysterious that the entropy is proportional to the **area** of the black hole, **not its volume**
- its (microscopic) origin remains to be fully understood

Various approach

For specific case, there are several explanations

- Counting BPS states (SUSY BH) (Strominger-Vafa '96)
- Attractor mechanism (Extremal) (Ferrara-Kalosh-Strominger '95, Sen '05
Goldstein-Iizuka-Jena-Trivedi '05)
- AdS₃/CFT₂ (BTZ) (Strominger '97)
- Near horizon conformal symmetry (Carlip '98 '99)
- OSV conjecture $Z_{BH} = |Z_{top}|^2$ (Ooguri-Strominger-Vafa '04)
- Entanglement entropy (Extremal) (Azeyanagi-TN-Takayanagi '07)
- ⋮

Remarkably, the **extremality** plays an important role even though the approaches are quite different

The Kerr/CFT correspondence

Recently, a new duality called **the Kerr/CFT correspondence** was proposed between the extreme Kerr black hole in four-dimension and a two-dimensional CFT

(Guica-Hartman-Song-Strominger '08)

The prescription to obtain the dual CFT is

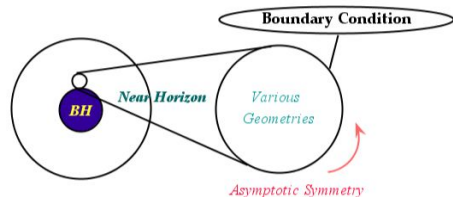
- 1 take the **near horizon** limit of the **extremal** Kerr black hole
- 2 determine the asymptotic **"boundary condition"** in order that the **"Virasoro algebra"** appears
- 3 evaluate the central charge c of this Virasoro algebra
- 4 define the dual temperature T_L analogous to the Frolov-Thorn vacuum

Sketch of the following discussion

The boundary condition determines the family of the geometries

$$\mathcal{L}_\xi g \sim 0$$

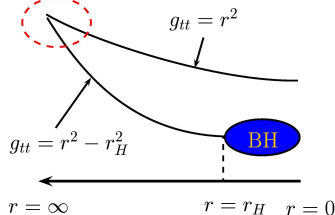
ξ : Asymptotic Symmetry Group



We require

- ASG includes the Virasoro algebra (not too strong)
- Conserved charge is finite (not too weak)

Boundary Condition



Purpose

We will see that the statistical entropy computed by using the Cardy formula agrees with the black hole entropy

$$S_{CFT} = \frac{\pi^2}{3} c T_L = S_{BH}$$

We can obtain the (in a sense) microscopic interpretation of the black hole entropy

Can we apply this strategy to more general black holes ?

The answer is **yes**, and we can construct the dual CFT thanks to the **extremality**

Table of Contents

- Introduction
- Kerr-Newman-(A)dS-black hole
 - Near horizon limit of the extremal black hole
- Asymptotic symmetry group
 - Boundary conditions
 - Central charge
- The Extreme Black Hole/CFT correspondence
- Reissner-Nordstrom-AdS black hole
 - Limit of Kerr-Newman-AdS black hole
 - Another description: embedding into 5D space
- Summary and further application

Generalization to Kerr-Newman-(A)dS black hole

To illustrate the construction of the dual CFT to the extremal black hole, we consider **the Kerr-Newman-(A)dS black hole**

This is the most general solution in the four-dimensional Einstein-Maxwell theory

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(R + \frac{6}{\ell^2} - \frac{1}{4} F^2 \right)$$

Notice that

- Once we set the electric and magnetic charges to zero, we obtain **the Kerr black hole**
- Also we obtain **the Reissner-Nordstrom black hole** in the limit of zero angular momentum

Kerr-Newman-(A)dS black hole

The metric is given by (Caldarelli-Cognola-Klemm '99)

$$ds^2 = -\frac{\Delta_r}{\rho^2} \left(dt - \frac{a}{\Xi} \sin^2 \theta d\hat{\phi} \right)^2 + \frac{\rho^2}{\Delta_r} d\hat{r}^2 \\ + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta}{\rho^2} \sin^2 \theta \left(a dt - \frac{\hat{r}^2 + a^2}{\Xi} d\hat{\phi} \right)^2$$

with

$$\Delta_r = (\hat{r}^2 + a^2) \left(1 + \frac{\hat{r}^2}{\ell^2} \right) - 2M\hat{r} + q^2, \quad \Delta_\theta = 1 - \frac{a^2}{\ell^2} \cos^2 \theta,$$

$$\rho^2 = \hat{r}^2 + a^2 \cos^2 \theta, \quad \Xi = 1 - \frac{a^2}{\ell^2}, \quad q^2 = q_e^2 + q_m^2$$

Thermodynamic quantities

The angular velocity of the horizon and the entropy are

$$\Omega_H = \frac{\Xi a}{(r_+^2 + a^2)}, \quad S = \pi \frac{r_+^2 + a^2}{\Xi},$$
$$\Omega_H^\infty = \Omega_H + \frac{a}{\ell^2} = \frac{a(1 + r_+^2/\ell^2)}{r_+^2 + a^2}$$

The Hawking temperature is

$$T_H = \frac{r_+(1 + a^2/\ell^2 + 3r_+^2/\ell^2 - (a^2 + q^2)/r_+^2)}{4\pi(r_+^2 + a^2)}$$

The physical mass, angular momentum, and electric and magnetic charges are

$$M_{\text{ADM}} = \frac{M}{\Xi^2}, \quad J = \frac{aM}{\Xi^2}, \quad Q_e = \frac{q_e}{\Xi}, \quad Q_m = \frac{q_m}{\Xi}$$

Gauge field

Now we consider the Einstein-Maxwell theory, there is a gauge field

The gauge field and field strength are

$$A = -\frac{q_e \hat{r}}{\rho^2} \left(d\hat{t} - \frac{a \sin^2 \theta}{\Xi} d\hat{\phi} \right) - \frac{q_m \cos \theta}{\rho^2} \left(a d\hat{t} - \frac{\hat{r}^2 + a^2}{\Xi} d\hat{\phi} \right),$$

$$F = -\frac{q_e(\hat{r}^2 - a^2 \cos^2 \theta) + 2q_m \hat{r} a \cos \theta}{\rho^4} \left(d\hat{t} - \frac{a \sin^2 \theta}{\Xi} d\hat{\phi} \right) \wedge d\hat{r}$$

$$+ \frac{q_m(\hat{r}^2 - a^2 \cos^2 \theta) - 2q_e \hat{r} a \cos \theta}{\rho^4} \sin \theta d\theta \wedge \left(a d\hat{t} - \frac{\hat{r}^2 + a^2}{\Xi} d\hat{\phi} \right)$$

Extreme limit

In the extreme limit ($T_H \rightarrow 0$), the inner and outer horizons degenerate to a single horizon at r_+

The extremality condition is

$$a^2 = \frac{r_+^2(1 + 3r_+^2/\ell^2) - q^2}{1 - r_+^2/\ell^2}$$
$$M = \frac{r_+[(1 + r_+^2/\ell^2)^2 - q^2/\ell^2]}{1 - r_+^2/\ell^2}$$

and the entropy at extremality is

$$S(T_H = 0) = \frac{\pi(2r_+^4/\ell^2 + 2r_+^2 - q^2)}{1 - 2r_+^2/\ell^2 - 3r_+^4/\ell^4 + q^2/\ell^2}$$

Near horizon limit

To take the near horizon limit, we introduce new coordinates

(Bardeen-Horowitz '99)

$$\hat{r} = r_+ + \epsilon r_0 r, \quad \hat{t} = tr_0/\epsilon, \quad \hat{\phi} = \phi + \Omega_H \frac{tr_0}{\epsilon}$$

In the limit of $\epsilon \rightarrow 0$, the metric becomes

$$ds^2 = \Gamma(\theta) \left[-r^2 dt^2 + \frac{dr^2}{r^2} + \alpha(\theta) d\theta^2 \right] + \gamma(\theta) (d\phi + k r dt)^2$$

where

$$\Gamma(\theta) = \frac{\rho_+^2 r_0^2}{r_+^2 + a^2}, \quad \alpha(\theta) = \frac{r_+^2 + a^2}{\Delta_\theta r_0^2}, \quad \gamma(\theta) = \frac{\Delta_\theta (r_+^2 + a^2)^2 \sin^2 \theta}{\rho_+^2 \Xi^2}$$

and we have defined

$$\rho_+^2 = r_+^2 + a^2 \cos^2 \theta, \quad r_0^2 = \frac{(r_+^2 + a^2)(1 - r_+^2/\ell^2)}{1 + 6r_+^2/\ell^2 - 3r_+^4/\ell^4 - q^2/\ell^2}, \quad k = \frac{2ar_+ \Xi r_0^2}{(r_+^2 + a^2)^2}$$

Near horizon limit: gauge field

The field strength becomes

$$F = f(\theta)kdr \wedge dt + f'(\theta)(d\theta \wedge d\phi + krd\theta \wedge dt)$$

and the near horizon gauge field is

$$A = f(\theta)(d\phi + kr dt)$$

with

$$f(\theta) = \frac{(r_+^2 + a^2)[q_e(r_+^2 - a^2 \cos^2 \theta) + 2q_m a r_+ \cos \theta]}{2\rho_+^2 \Xi a r_+}$$

Isometry

- The Kerr-Newman-(A)dS black hole has the complicated metric
- But it becomes **fairly simple form** once we take the near horizon limit of the extremal black hole
- The isometry is $U(1) \times SL(2, R)$ ($U(1) : \phi$, $SL(2, R) : \text{AdS}_2$ part)

We will calculate the central charge for this general form for simplicity

Surprisingly, this simple form appears as the near horizon limit of the extreme black hole in the fairly general gravity theory as we will see later

Asymptotic Symmetry Group and Boundary Conditions

The asymptotic symmetry group (ASG) of a spacetime is

- A symmetry which obeys **the boundary conditions** in the diffeomorphism

Example: AdS₃ $ds^2 = -(1 + \frac{r^2}{l^2})dt^2 + \frac{dr^2}{1 + \frac{r^2}{l^2}} + \frac{r^2}{l^2}d\phi^2$

Choose the boundary condition as (Brown-Henneaux '86)

$$h_{\mu\nu} \sim O \begin{pmatrix} 1 & 1/r^2 & 1 \\ & 1/r^2 & 1/r^2 \\ & & 1 \end{pmatrix}$$

- This allows the BTZ black hole
- The ASG is the $\text{Vir}_L \times \text{Vir}_R$ algebras
- The **central charges** are $c_L = c_R = 3l/2$

How to choose boundary conditions?

For the general form

$$ds^2 = \Gamma(\theta) \left[-r^2 dt^2 + \frac{dr^2}{r^2} + \alpha(\theta) d\theta^2 \right] + \gamma(\theta) (d\phi + kr dt)^2$$
$$A = f(\theta) (d\phi + kr dt)$$

we choose the boundary condition such that

- the ASG includes the Virasoro algebra
- the charges is finite

like the Brown-Henneaux's case

The appropriate boundary conditions determine the family of the geometries in which the charges are finite

Boundary conditions

Such a boundary condition is (in the basis (t, ϕ, θ, r))

$$h_{\mu\nu} \sim \mathcal{O} \begin{pmatrix} r^2 & 1 & 1/r & 1/r^2 \\ & 1 & 1/r & 1/r \\ & & 1/r & 1/r^2 \\ & & & 1/r^3 \end{pmatrix}$$

For the gauge field we impose the boundary condition

$$a_\mu \sim \mathcal{O}(r, 1/r, 1, 1/r^2) .$$

(This condition is important to obtain the Virasoro symmetry uniquely)

ASG

The most general diffeomorphisms which preserve the boundary conditions are

$$\zeta_\epsilon = \epsilon(\phi)\partial_\phi - r\epsilon'(\phi)\partial_r$$

The gauge field transforms under ζ_ϵ as

$$\delta_\epsilon A = f\epsilon'(d\phi - kr dt)$$

This **does not** satisfy the boundary condition, so we must **add a compensating $U(1)$ gauge transformation** to restore $\delta A_\phi = O(1/r)$

$$\Lambda = -f(\theta)\epsilon(\phi)$$

Under the combined gauge + diffeomorphism transformation,

$$\delta_\epsilon A = -krf(\theta)\epsilon'(\phi)dt - f'(\theta)\epsilon(\phi)d\theta$$

Conserved charge

We focus on **the Einstein-Maxwell theory**. There are two symmetries under which the action is invariant

- **Diffeomorphism**: $\delta_\zeta g_{\mu\nu} = \mathcal{L}_\zeta g_{\mu\nu}$, $\delta_\zeta A_\mu = \mathcal{L}_\zeta A_\mu$
- **$U(1)$ gauge symmetry**: $\delta_\Lambda A_\mu = \partial_\mu \Lambda$

The associated charge $Q_{\zeta,\Lambda}$ is defined by

$$\delta Q_{\zeta,\Lambda} = \frac{1}{8\pi} \int_\infty \left(k_\zeta^{grav}[h; g] + k_{\zeta,\Lambda}^{gauge}[h, a; g, A] \right)$$

where we denote the infinitesimal field variations by $a_\mu = \delta A_\mu$ and $h_{\mu\nu} = \delta g_{\mu\nu}$

Conserved charge

- The Einstein part (Barnich-Brandt '01)

$$k_{\zeta}^{grav}[h, g] = \frac{1}{4} \epsilon_{\alpha\beta\mu\nu} [\zeta^{\nu} D^{\mu} h - \zeta^{\nu} D_{\sigma} h^{\mu\sigma} + \zeta_{\sigma} D^{\nu} h^{\mu\sigma} \\ + \frac{1}{2} h D^{\nu} \zeta^{\mu} - h^{\nu\sigma} D_{\sigma} \zeta^{\mu} + \frac{1}{2} h^{\sigma\nu} (D^{\mu} \zeta_{\sigma} + D_{\sigma} \zeta^{\mu})] dx^{\alpha} \wedge dx^{\beta}$$

- The Maxwell part (Barnich-Compere '05)

$$k_{\zeta, \Lambda}^{gauge}[\delta\phi, \phi] = \frac{1}{8} \epsilon_{\alpha\beta\mu\nu} [(-\frac{1}{2} h F^{\mu\nu} + 2 F^{\mu\gamma} h_{\gamma}^{\nu} - \delta F^{\mu\nu})(\zeta^{\rho} A_{\rho} + \Lambda) \\ - F^{\mu\nu} \zeta^{\rho} a_{\rho} - 2 F^{\alpha\mu} \zeta^{\nu} a_{\alpha} - a^{\mu} (\mathcal{L}_{\zeta} A^{\nu} + \partial^{\nu} \Lambda)] dx^{\alpha} \wedge dx^{\beta}$$

where $\delta F^{\mu\nu} \equiv g^{\mu\alpha} g^{\nu\beta} (\partial_{\alpha} a_{\beta} - \partial_{\beta} a_{\alpha})$ ($\delta\phi \equiv \delta(g_{\mu\nu}, A_{\mu}) = (h_{\mu\nu}, a_{\mu})$)

Conserved charge

The algebra of the ASG is the Dirac bracket algebra of the charges themselves

$$\begin{aligned} \{Q_{\zeta,\Lambda}, Q_{\tilde{\zeta},\tilde{\Lambda}}\}_{DB} &= (\delta_{\tilde{\zeta}} + \delta_{\tilde{\Lambda}})Q_{\zeta,\Lambda} \\ &= Q_{[(\zeta,\Lambda),(\tilde{\zeta},\tilde{\Lambda})]} + (\text{central term}) \end{aligned}$$

$$(\text{central term}) = \frac{1}{8\pi} \int \left(k_{\zeta}^{grav} [\mathcal{L}_{\tilde{\zeta}}\bar{g}; \bar{g}] + k_{\zeta,\Lambda}^{gauge} [\mathcal{L}_{\tilde{\zeta}}\bar{g}, \mathcal{L}_{\tilde{\zeta}}\bar{A} + d\tilde{\Lambda}; \bar{g}, \bar{A}] \right)$$

where (\bar{g}, \bar{A}) denote the background solution

The central term give us **the central charge for dual CFT** as we will see in the following

Virasoro algebra

We expand the arbitrary function $\epsilon(\phi)$ by the fourier mode $\epsilon_n = -e^{-in\phi}$, and define

$$\zeta_n \equiv \zeta_{\epsilon_n}, \quad \Lambda_n \equiv \Lambda(\epsilon = \epsilon_n)$$

Combining the diffeomorphism and gauge transformation as (ζ_n, Λ_n) , this becomes the Virasoro algebra

$$i[(\zeta_n, \Lambda_n), (\zeta_m, \Lambda_m)] = (n - m)(\zeta_{n+m}, \Lambda_{n+m})$$

without the central charge

But when we calculate the Dirac bracket between the symmetry generators $Q_{\zeta, \Lambda}$, we will obtain **the central charge** from the central term

Central charge

The Dirac brackets between symmetry generators are

$$\begin{aligned}
 i\{Q_{\zeta_\epsilon, \Lambda}, Q_{\zeta_{\tilde{\epsilon}}, \tilde{\Lambda}}\}_{DB} &= iQ_{[(\zeta_\epsilon, \Lambda), (\zeta_{\tilde{\epsilon}}, \tilde{\Lambda})]} \\
 &- \frac{ik}{16\pi} \int d\theta d\phi \sqrt{\frac{\alpha(\theta)\gamma(\theta)}{\Gamma(\theta)}} \left(\underbrace{\Gamma(\theta)\epsilon'\tilde{\epsilon}'' + \gamma(\theta)\epsilon\tilde{\epsilon}'}_{\text{gravity}} + \underbrace{f(\theta)[\Lambda + f(\theta)\epsilon]}_{\text{gauge}} \tilde{\epsilon}' \right) \\
 &- (\epsilon, \Lambda \leftrightarrow \tilde{\epsilon}, \tilde{\Lambda})
 \end{aligned}$$

The algebra of the ASG is the Virasoro algebra generated by (ζ_n, Λ_n) with charges Q_n

$$i\{Q_m, Q_n\}_{DB} = (m-n)Q_{m+n} + \frac{c}{12}(m^3 - Bm)\delta_{m+n,0}$$

where B is a constant that can be absorbed by a shift in Q_0

Central charge

The central charge has contributions from k^{grav} and k^{gauge}

$$c = c_{grav} + c_{gauge}$$

We find

$$c_{grav} = 3k \int_0^\pi d\theta \sqrt{\Gamma(\theta)\alpha(\theta)\gamma(\theta)}$$
$$c_{gauge} = 0$$

For the Kerr-Newman-(A)dS black hole

$$c = \frac{12r_+ \sqrt{(3r_+^4/\ell^2 + r_+^2 - q^2)(1 - r_+^2/\ell^2)}}{1 + 6r_+^2/\ell^2 - 3r_+^4/\ell^4 - q^2/\ell^2}$$

Temperature

The extremality constraint requires that any fluctuations satisfy

$$0 = T_H dS = dM_{ADM} - (\Omega_H dJ + \Phi_e dQ_e + \Phi_m dQ_m)$$

For such constrained variations we may write

$$-dI_{gr} = dS = \frac{dJ}{T_L} + \frac{dQ_e}{T_e} + \frac{dQ_m}{T_m}$$

Like GKP-W relation, we identify the density matrix of the bulk with that of the boundary

$$\rho_{gravity} \equiv \rho_{CFT}$$

$$\rho_{gravity} = \exp(-I_{gr}), \quad \rho_{CFT} = \exp\left(-\frac{L_0}{T_L} - \frac{\hat{q}_e}{T_e} - \frac{\hat{q}_m}{T_m}\right)$$

Then we obtain **the temperature T_L of dual CFT**

Entropy

For Kerr-Newman-(A)dS case

$$T_L = \frac{(1 + 6r_+^2/\ell^2 - 3r_+^4/\ell^4 - q^2/\ell^2)[2r_+^2(1 + r_+^2/\ell^2) - q^2]}{4\pi r_+ [(1 + r_+^2/\ell^2)(1 - 3r_+^2/\ell^2) + q^2/\ell^2] \sqrt{(1 - r_+^2/\ell^2)(3r_+^4/\ell^2 + r_+^2 - q^2)}}$$

Assuming the Cardy formula, we obtain the statistical entropy of the CFT

$$S_{CFT} = \frac{\pi^2}{3} c T_L = \frac{\pi(2r_+^4/\ell^2 + 2r_+^2 - q^2)}{1 - 2r_+^2/\ell^2 - 3r_+^4/\ell^4 + q^2/\ell^2}$$

This agrees in precise with the Bekenstein-Hawking entropy of the Kerr-Newman-(A)dS black hole!

Notice that the temperature is rewritten as the surprisingly simple form

$$T_L = \frac{1}{2\pi k}$$

The Extreme Black Hole/CFT correspondence

We treated the KNAdS black hole as the following general form

$$ds^2 = \Gamma(\theta) \left[-r^2 dt^2 + \frac{dr^2}{r^2} + \alpha(\theta) d\theta^2 \right] + \gamma(\theta) (d\phi + kr dt)^2$$

It was shown that the above form is obtained as **the near horizon geometry of the extremal black hole** constructed in the following action (Kunduri-Lucietti-Reall '07)

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(R - \frac{1}{2} f_{AB}(\chi) \partial_\mu \chi^A \partial^\mu \chi^B - V(\chi) - \frac{1}{4} g_{IJ}(\chi) F_{\mu\nu}^I F^{J\mu\nu} \right) + \frac{1}{2} \int h_{IJ}(\chi) F^I \wedge F^J$$

The near horizon scalar fields and gauge fields have the form

$$\chi^A = \chi^A(\theta), \quad A^I = f^I(\theta)(d\phi + kr dt)$$

Construct dual CFT

The Bekenstein-Hawking entropy of such a black hole is

$$S_{grav} = \frac{\pi}{2} \int_0^\pi d\theta \sqrt{\Gamma(\theta)\alpha(\theta)\gamma(\theta)}$$

We would like to explain the black hole entropy as the statistical entropy of dual CFT

- the calculation of the central charge is the same as before
- but we must take the contribution of the **non-gravitational part** such as the scalar fields into account

We derive the expression for the **conserved charges** of the general action following the (improved) covariant phase method (Wald '93, Iyer-Wald '94, Barnich-Compere '07)

Conserved charge

The final results are

$$\mathbf{k}_\xi^{grav} = \frac{1}{8\pi G} (d^{D-2}x)_{\mu\nu} \left\{ \xi^\nu \nabla^\mu h - \xi^\nu \nabla_\sigma h^{\mu\sigma} + \xi_\sigma \nabla^\nu h^{\mu\sigma} + \frac{1}{2} h \nabla^\nu \xi^\mu - h^{\rho\nu} \nabla_\rho \xi^\mu + \frac{1}{2} h^{\sigma\nu} (\nabla^\mu \xi_\sigma + \nabla_\sigma \xi^\mu) \right\},$$

$$\mathbf{k}_{\xi,\Lambda}^F = \frac{1}{16\pi G} (d^{D-2}x)_{\mu\nu} \left[\left\{ -k_{IJ,A}(\chi) F^{J\mu\nu} \delta\chi^A + 2k_{IJ}(\chi) h^{\mu\lambda} F^J{}_{\lambda}{}^\nu - k_{IJ}(\chi) \delta F^{J\mu\nu} - \frac{1}{2} h k_{IJ}(\chi) F^{J\mu\nu} \right\} (A_\rho^I \xi^\rho + \Lambda^I) - k_{IJ}(\chi) F^{J\mu\nu} a_\rho^I \xi^\rho - 2\xi^\mu k_{IJ}(\chi) F^{J\nu\lambda} a_\lambda^I - k_{IJ}(\chi) a^{J\mu} g^{\nu\sigma} (\mathcal{L}_\xi A_\sigma^I + \partial_\sigma \Lambda^I) \right],$$

$$\mathbf{k}_{\xi,\Lambda}^{top} = \frac{1}{16\pi G} dx^\rho \wedge dx^\sigma \left[\{ h_{IJ,A}(\chi) F_{\rho\sigma}^J \delta\chi^A + h_{IJ}(\chi) \delta F_{\rho\sigma}^J \} (A_\lambda^I \xi^\lambda + \Lambda^I) - 2h_{IJ}(\chi) a_\rho^I \nabla_\sigma (A_\lambda^J \xi^\lambda + \Lambda^J) \right],$$

$$\mathbf{k}_\xi^\chi = \frac{1}{8\pi G} (d^{D-2}x)_{\mu\nu} \xi^\nu f_{AB}(\chi) \nabla^\mu \chi^B \delta\chi^A.$$

Central charge

We can calculate the central charge as before

- In KNAdS case (or the Einstein-Maxwell theory), we checked that **the gauge field does not contribute to the central charge**
- Remarkably, **even in the presence of the non-gravitational fields**, the central charge is always given by (Compere-Murata-TN '09)

$$c = c_{grav} = 3k \int_0^\pi d\theta \sqrt{\Gamma(\theta)\alpha(\theta)\gamma(\theta)}$$

as before

Entropy

We saw that the temperature of the KNAdS is given by

$$T_L = \frac{1}{2\pi k}$$

We naively apply this formula to the general cases¹

Using the Cardy formula

$$\begin{aligned} S_{CFT} &= \frac{\pi^2}{3} c_{grav} T_L \\ &= \frac{\pi}{2} \int_0^\pi d\theta \sqrt{\Gamma(\theta)\alpha(\theta)\gamma(\theta)} = \frac{\text{Area}(\text{horizon})}{4} \end{aligned}$$

in agreement with [the Bekenstein-Hawking entropy!](#)

¹Recently, this conjecture has been checked under some assumption

Relation to higher dimension

- The four-dimensional action we study is **very general** in its own right (but we exclude the non-abelian gauge field)
- Once we reduce the higher dimensional Einstein-Hilbert action by **torus compactification**, it **always takes that form**
- The interesting example is that we can explain indirectly the entropy of the nontrivial solutions such as the **black rings and saturns** in five-dimension

Reissner-Nordstrom-AdS black hole: limit of KNAdS

In the limit of $J \rightarrow 0$ of the Kerr-Newman-AdS black hole, we obtain **the Reissner-Nordstrom-AdS black hole**, and **reproduce the Bekenstein-Hawking entropy**

$$S_{RN} = \pi r_+^2$$

But the subtleties

- the central charge approaches **zero**
- the temperature goes to **infinity**
- these singular behaviors cancel against each other to reproduce the finite entropy

Below we propose a dual description which does not require a singular temperature and central charge

Another description: embedding into 5D space

Assume that we can embed the Kerr-Newman-AdS black hole into 5D space by combining the $U(1)$ gauge bundle

$$ds^2 = ds_{BH}^2 + (dy + A)^2$$

We shift a gauge field A in order to choose it non-singular in the $a \rightarrow 0$ limit

$$A \rightarrow A - \frac{q_e r_+}{2a} d\phi$$

Setting $a = 0$

$$A = q_e r \frac{\bar{r}_0^2}{r_+^2} dt + q_m \cos \theta d\phi$$

Another description: entropy

Once we embed the RN black hole into 5D dimension

- choose boundary conditions appropriately
- the Virasoro algebra from the gauge fiber direction

$$\zeta^{(y)} = \epsilon(y)\partial_y - r\epsilon'(y)\partial_r .$$

- we can calculate its central charge similar to 4D case

$$c_{(y)} = 6q_e \bar{r}_0^2 \quad (\bar{r}_0^2 \equiv r_0^2|_{a \rightarrow 0})$$

- the temperature conjugate to the electric charge is defined by $T_e dS = dQ_e$ and we find

$$T_e = \frac{r_+^2}{2\pi q_e \bar{r}_0^2}$$

- Using Cardy formula

$$S_{CFT} = \frac{\pi^2}{3} c_{(y)} T_e = \pi r_+^2 = S_{BH}$$

Summary

- The entropy of the Kerr-Newman-(A)dS black hole is reproduced as the statistical entropy of dual CFT
- If we assume the formula for the temperature of CFT, we can apply this idea to the fairly general four-dimensional extremal black holes
- The Reissner-Nordstrom black hole also can be treated, but there is a dual description by embedding it into 5D space

Futher generalization

- Generalization to the higher dimension (Lu-Mei-Pope, Azeyanagi-Terashima-Ogawa, Nakayama, Chow-Cvetic-Lu-Pope, Isono-Tai-Wen '08)
- There are several cycles along which we can construct the Virasoro algebra
- The central charge is exactly (in 5D) (Compere-Murata-TN '09)

$$c_i = \frac{3k_i}{2\pi G_N} \text{Area}(\text{horizon})$$

- The temperature associated with i -th cycle is

$$T_i = \frac{1}{2\pi k_i}$$

- The entropy

$$S_{CFT} = \frac{\pi^2}{3} c_1 T_1 = \frac{\pi^2}{3} c_2 T_2 = S_{BH}$$

agrees with the "general" extreme black hole in five dimension

Open problem

- We can explain the entropy of the extremal black holes by the Cardy formula in dual CFTs
- but we **don't know** the complete spectrum of dual CFT which accounts the statistical entropy
- It is important to embed the extremal black holes into the string theory in order to understand the CFTs
(cf. Azeyanagi-Ogawa-Terashima '08)