Investigation of Gauge/Gravity Correspondence Including Higher Derivative Corrections

Yoshifumi Hyakutake (Osaka Univ. → KEK)

Based on arXiv:0811.3102
with Masanori Hanada, Jun Nishimura and Shingo Takeuchi

and arXiv:0805.2005 ;JHEP 0807:066

with Kyosuke Hotta, Takahiro Kubota and Hiroaki Tanida,

1. Introduction

String theory offers an interesting framework to investigate gauge theory and gravity theory. Especially D-branes play important roles to connect these two theories.

Typical example is realized by taking decoupling limit of D3-branes.

type IIB string theory on $AdS_5 \times S^5$
$\Leftrightarrow$ 4d $\mathcal{N}=4$ $SU(N)$ Super Yang-Mills

Maldacena

$$L \sim (g_s N)^{\frac{1}{4}}\sqrt{\alpha'} \sim \lambda^{\frac{1}{4}}\sqrt{\alpha'}$$

λ : 't Hooft coupling

Supergravity approximation is valid when $1 \ll \lambda,\ 1 \ll N$

In order to investigate finite λ and N region, we have to include higher derivative corrections to the supergravity.

α' correction $\sim$ $\frac{1}{\lambda^{1/2}}$ correction

g_s correction $\sim$ $\frac{1}{N}$ correction

$$\frac{\alpha'}{L^2} \sim \frac{1}{\lambda^{1/2}}$$

$$g_s^2 \frac{\alpha'^4}{L^8} \sim \frac{1}{N^2}$$

Gubser, Klebanov, Tseytlin

The purpose of my talk is to investigate the gauge/gravity correspondence including higher derivative corrections.

α' correction

Decoupling limit of D0-branes in type IIA :

near horizon geometry of black 0-brane
$\Leftrightarrow$ supersymmetric matrix quantum machanics
at large 't Hooft coupling

g_s correction

M5-branes wrapping on 4-cycles in Calabi-Yau 3-fold :
After reducing to 3 dimensions

AdS_3 in Topologically Massive Gravity (TMG)
$\Leftrightarrow$ CFT_2 with $c_L \neq c_R$

Plan

1. Introduction

2. Higher derivative corrections in string theory

3. α'^3 corrections to black hole thermodynamics from supersymmetric matrix quantum mechanics

Masanori Hanada, YH, Jun Nishimura and Shingo Takeuchi

4. Brown-Henneaux's canonical approach to topologically massive gravity

Kyosuke Hotta, YH, Takahiro Kubota and Hiroaki Tanida

5. Summary

2. Higher Derivative Corrections in String Theory

Higher derivative corrections in string theories are considerably investigated in various ways

- String scattering amplitude — Gross, Witten; Gross, Sloan
- Non linear sigma model — Grisaru, Zanon
- Superfield method
- Duality
- Noether's method … and so on

By combining all these results, we find that corrections start from α'^3 order, and a part of bosonic terms in type IIA is written as

$$
\begin{aligned}
\mathcal{L} &= e^{-2\phi}R + \cdots && \text{SUGRA} \\
&+ \frac{\zeta(3)\alpha'^3}{2^8\cdot 4!}e^{-2\phi}\left(t_8t_8R^4 + \tfrac{1}{4\cdot 2!}\epsilon_{10}\epsilon_{10}R^4\right) + \cdots && \text{tree} \\
&+ \frac{\pi^2\alpha'^3}{3\cdot 2^8\cdot 4!}g_s^2\left(t_8t_8R^4 - \tfrac{1}{4\cdot 2!}\epsilon_{10}\epsilon_{10}R^4 - \tfrac{1}{6}\epsilon_{10}t_8BR^4\right) + \cdots && \text{1-loop}
\end{aligned}
$$

t_8 : tensor with 8 indices , ϵ_{10} : 10D antisymmetric tensor, B : NS 2-form field

The complete structure of higher derivative terms will be determined by local supersymmetry. In fact, known terms can be derived completely.

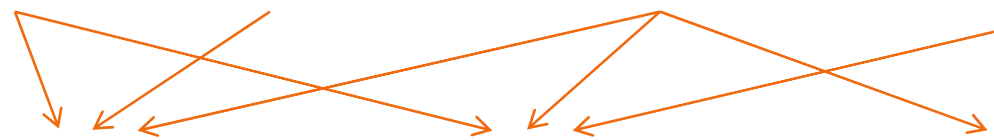
Ogushi, Hyakutake

Local supersymmetry transformation (neglect flux dependence):

$$\delta e^a{}_\mu = \bar{\epsilon}\gamma^a\psi_\mu, \qquad \delta\psi_\mu = 2D_\mu\epsilon, \qquad \delta B_{\mu\nu} = -3\bar{\epsilon}\gamma_{11}\gamma_{[\mu}\psi_{\nu]}$$

Cancellation (neglect flux dependence):

$$[R^4]_7, \qquad [BR^4]_2, \qquad [R^3\psi\psi_2]_{92}, \qquad [R^2\psi_2 D\psi_2]_{25}$$



$$[R^4\epsilon\psi]_{116}, \qquad [R^2 DR\epsilon\psi_2]_{88}, \qquad [R^3\epsilon D\psi_2]_{40} \qquad = 0$$

Solution is given by

$$a\Big(t_8t_8R^4 + \tfrac{1}{4\cdot 2!}\epsilon_{10}\epsilon_{10}R^4\Big) + b\Big(t_8t_8R^4 - \tfrac{1}{4\cdot 2!}\epsilon_{10}\epsilon_{10}R^4 - \tfrac{1}{6}\epsilon_{10}t_8BR^4\Big)$$

Short Summary

- Higher derivative corrections start from α'^3 order.

$$\alpha' \text{ correction : } \quad \alpha'^3 e^{-2\phi} \mathcal{R}^4$$

- g_s corrections contain topological term.

$$g_s \text{ correction : } \quad \alpha'^3 g_s^2 (\mathcal{R}^4 + BR^4)$$

$$B \wedge \mathrm{tr}(R \wedge R) \wedge \mathrm{tr}(R \wedge R)$$

Uplift to 11D

$$\to \underline{A} \wedge \mathrm{tr}(R \wedge R) \wedge \mathrm{tr}(R \wedge R) \sim \underline{F \wedge \mathrm{tr}(R \wedge R)} \wedge I_{GCS}$$

3-form

Integrating out this part gives Gravitational Chern-Simons term in 3 dimensions.

3. α'^3 corrections to black hole thermodynamics from supersymmetric matrix quantum mechanics

Let us consider the system of D0-branes in type IIA superstring theory, which provides a particularly simple example of gauge-gravity duality.

N D0-branes with open strings excited

Decoupling limit

Near horizon geometry of non-extremal black 0-brane	$U(N)$ super matrix quantum mechanics at finite temperature
Valid when $1 \ll \lambda, \ 1 \ll N$	Strongly coupled

In order to test the gauge-gravity duality, we need to know the gauge theory at strongly coupled region.

Hanada, Nishimura, Takeuchi

Anagnostopoulos, Hanada, Nishimura, Takeuchi

➡ Monte Carlo simulation by using a non-lattice regularization

Gravity theory

Type IIA supergravity action

$$\mathcal{S}_{(0)} = \frac{1}{16\pi G_N}\int d^{10}x\sqrt{-g}\left\{e^{-2\phi}\left(R+4\partial_\mu\phi\partial^\mu\phi\right)-\frac{1}{4}G_{\mu\nu}G^{\mu\nu}\right\}$$

Near horizon geometry of non-extremal black 0-brane

$$ds^2 = \alpha'\left(-fH^{-\frac{1}{2}}dt^2+f^{-1}H^{\frac{1}{2}}dU^2+H^{\frac{1}{2}}U^2d\Omega_8^2\right), \quad e^\phi = \alpha'^{-\frac{3}{2}}H^{\frac{3}{4}}$$

$$H = \frac{2^4 15\pi^5\lambda}{U^7}, \quad f = 1-\frac{U_0^7}{U^7} \qquad \lambda = (2\pi)^{-2}\alpha'^{-\frac{3}{2}}g_s N$$

Itzhaki, Maldacena, Sonnenschein Yankielowicz

Hawking temperature, entropy and internal energy are calculated as

$$T = \frac{1}{4\pi}H^{-\frac{1}{2}}f'\Big|_{U=U_0} = c_2\lambda^{\frac{1}{3}}\left(\frac{U_0}{\lambda^{\frac{1}{3}}}\right)^{\frac{5}{2}} \qquad c_2 = 7/(2^4 15^{\frac{1}{2}}\pi^{\frac{7}{2}})$$

$$\frac{S}{N^2} = \frac{1}{N^2}\frac{\mathcal{A}}{4G_N} = c_3\left(\frac{T}{\lambda^{\frac{1}{3}}}\right)^{\frac{9}{5}} \qquad c_3 = 4^{\frac{13}{5}}15^{\frac{2}{5}}(\pi/7)^{\frac{14}{5}}$$

$$\frac{1}{N^2}\frac{E}{\lambda^{\frac{1}{3}}} = c_1\left(\frac{T}{\lambda^{\frac{1}{3}}}\right)^{\frac{14}{5}} \qquad c_1 = \frac{9}{14}c_3 = 7.407\cdots$$

Supergravity approximation is valid when

$$\frac{\rho^2}{\alpha'} \equiv \frac{4\pi^{\frac{5}{2}} 15^{\frac{1}{2}}}{147} \left(\frac{\lambda}{U_0^3}\right)^{\frac{1}{2}} \gg 1 \quad \Longleftrightarrow \quad T/\lambda^{\frac{1}{3}} \ll 1$$

$$g_s e^{\phi} \equiv \frac{2^5 15^{\frac{3}{4}} \pi^{\frac{23}{4}}}{N} \left(\frac{\lambda}{U_0^3}\right)^{\frac{7}{4}} \ll 1 \quad \Longleftrightarrow \quad T/\lambda^{\frac{1}{3}} \gg N^{-\frac{10}{21}}$$

Let us take account of the α'^3 correction to the supergravity.
Then we should modify following things.

action, solution, location of the horizon, temperature, entropy

As a result, the internal energy is modified as

$$\frac{1}{N^2} \frac{E}{\lambda^{\frac{1}{3}}} = c_1 \left(\frac{T}{\lambda^{\frac{1}{3}}}\right)^{\frac{14}{5}} - C \left(\frac{T}{\lambda^{\frac{1}{3}}}\right)^{\frac{23}{5}}$$

$c_1 = 7.407\cdots,$
C : some constant

Note that this result is understood by the dimensional analysis

$$(\alpha'/\rho^2)^3 \sim (\lambda/U_0^3)^{-3/2} \sim (T/\lambda^{\frac{1}{3}})^{\frac{9}{5}}$$

Modifications

Action :

$$\mathcal{S} = \mathcal{S}_{(0)} + \mathcal{S}_{(3)}, \quad \mathcal{S}_{(3)} = \frac{1}{16\pi G_{\mathrm{N}}} \int d^{10}x \sqrt{-g} \left\{ \alpha'^3 e^{-2\phi} \mathcal{R}^4 + \cdots \right\}$$

Solution :

$$H = \frac{2^4 15 \pi^5 \lambda}{U^7} \left(1 + H_{(3)}\right), \quad f = 1 - \frac{U_0^7}{U^7} + f_{(3)}$$

$$H_{(3)} = \left(\frac{U_0}{\lambda^{\frac{1}{3}}}\right)^{\frac{9}{2}} \tilde{H}\left(\frac{U_0}{U}\right), \quad f_{(3)} = \left(\frac{U_0}{\lambda^{\frac{1}{3}}}\right)^{\frac{9}{2}} \tilde{f}\left(\frac{U_0}{U}\right)$$

Horizon :

$$\frac{U_0}{U_{\mathrm{H}}} = 1 + \frac{\tilde{f}(1)}{7} \left(\frac{U_{\mathrm{H}}}{\lambda^{\frac{1}{3}}}\right)^{\frac{9}{2}}$$

temperature :

$$T = \frac{1}{4\pi} H^{-\frac{1}{2}} f' \Big|_{U=U_{\mathrm{H}}} = c_2 \lambda^{\frac{1}{3}} \left(\frac{U_{\mathrm{H}}}{\lambda^{\frac{1}{3}}}\right)^{\frac{5}{2}} \left\{1 + c_4 \left(\frac{U_{\mathrm{H}}}{\lambda^{\frac{1}{3}}}\right)^{\frac{9}{2}}\right\}$$

$$c_4 = \tilde{f}(1) - \frac{1}{7}\tilde{f}'(1) - \frac{1}{2}\tilde{H}(1)$$

$$c_5 = -\frac{9}{5}c_4 + s(1)$$

entropy :

$$\frac{S}{N^2} = \frac{1}{N^2} \frac{\tilde{A}}{4G_{\mathrm{N}}} \left\{1 + s(1) \left(\frac{U_{\mathrm{H}}}{\lambda^{\frac{1}{3}}}\right)^{\frac{9}{2}}\right\} = c_3 \left(\frac{T}{\lambda^{\frac{1}{3}}}\right)^{\frac{9}{5}} \left\{1 + c_5 \left(\frac{T}{c_2 \lambda^{\frac{1}{3}}}\right)^{\frac{9}{5}}\right\}$$

Gauge theory

The worldvolume theory of N D0-branes is given by the $U(N)$ supersymmetric MQM defined by the action

$$S = \frac{N}{\lambda} \int_0^\beta dt\, \mathrm{tr} \left\{ \frac{1}{2}(D_t X_i)^2 - \frac{1}{4}[X_i, X_j]^2 + \frac{1}{2}\psi_\alpha D_t \psi_\alpha - \frac{1}{2}\psi_\alpha (\gamma_i)_{\alpha\beta}[X_i, \psi_\beta] \right\}$$

$$\beta = T^{-1}$$

In the Monte Carlo simulation, we fix the gauge by the static diagonal gauge

$$A(t) = \frac{1}{\beta}\mathrm{diag}(\alpha_1, \cdots, \alpha_N) \qquad -\pi < \alpha_a \leq \pi$$

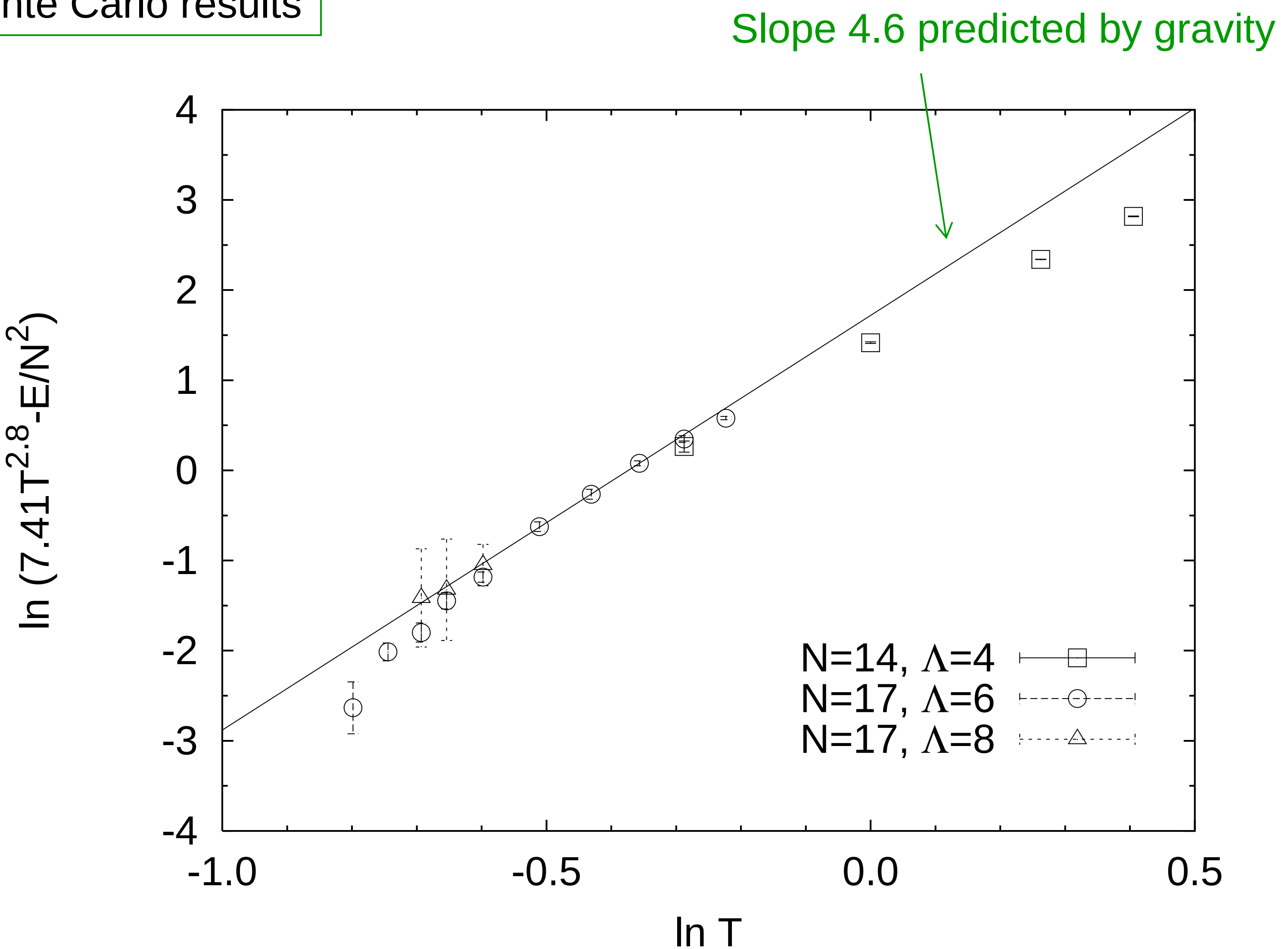
and introduce a UV cutoff Λ

$$X_i^{ab}(t) = \sum_{n=-\Lambda}^{\Lambda} \tilde{X}_{in}^{ab} e^{2\pi i n t/\beta}$$

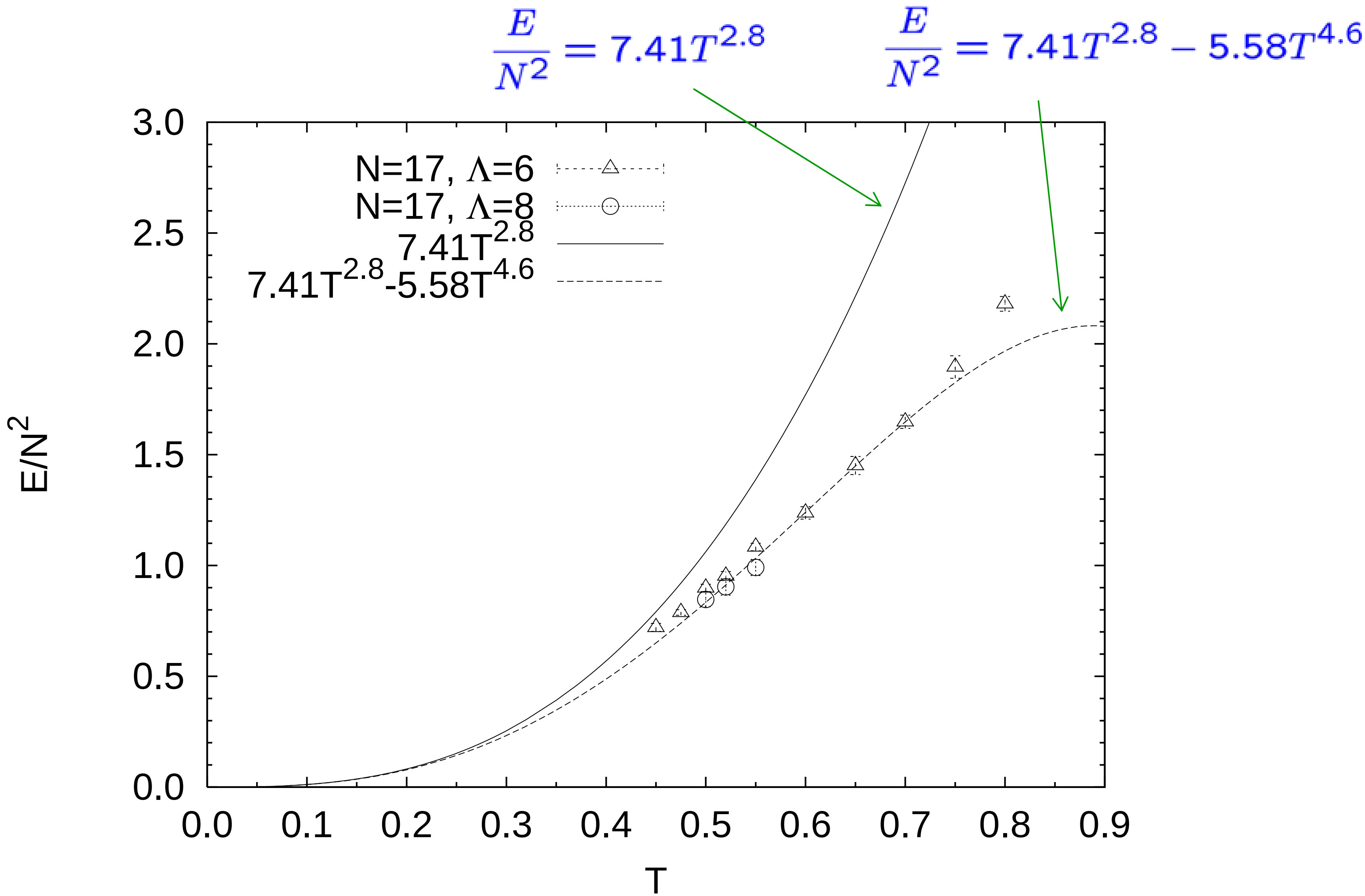
Integration over the fermionic matrices yields a complex Pfaffian

Without loss of generality, we set $\lambda = 1$.

Monte Carlo results



The deviation of the internal energy E/N^2 from $7.41T^{2.8}$ is plotted against the temperature in the log-log scale.



Fitting the data within $0.5 \leq T \leq 0.7$ (with largest Λ at each T) to $E/N^2 = 7.41T^{2.8} - CT^p$, we obtain $p = 4.58(3)$ and $C = 5.55(7)$.
If we make a one-parameter fit with $p = 4.6$ fixed, we obtain $C = 5.58(1)$.

Short Summary

- Thermodynamic properties of the near horizon limit of non-extremal black 0-brane are studied including α'^3 correction.

$$\frac{E}{N^2} = 7.41 T^{2.8} - C\,T^{4.6}$$

- The power 4.6 is precisely reproduced by Monte Carlo data in gauge theory. $C = 5.58$ is predicted.

Gauge/Gravity correspondence is confirmed beyond supergravity approximation.

4. Brown-Henneaux's canonical approach to TMG

Now we want to investigate the gauge/gravity correspondence including g_s corrections.

As mentioned before, there is a topological term AR^4 which becomes gravitational Chern-Simons (GCS) term after the dimensional reduction.

3D Gravity + GCS = Topologically Massive Gravity (TMG) Deser, Jackiw

$$\mathcal{S}_{\mathrm{TMG}} = \frac{1}{16\pi G_N}\int d^3x\,(\mathcal{L}_{\mathrm{EH}} + \mathcal{L}_{\mathrm{CS}})$$

$$\mathcal{L}_{\mathrm{EH}} = \sqrt{-G}\left(R + \frac{2}{\ell^2}\right)$$

$$\mathcal{L}_{\mathrm{CS}} = \frac{\beta}{2}\sqrt{-G}\epsilon^{IJK}\left(\Gamma^P{}_{IQ}\partial_J\Gamma^Q{}_{KP} + \frac{2}{3}\Gamma^P{}_{IQ}\Gamma^Q{}_{JR}\Gamma^R{}_{KP}\right)$$

The goal of this section is to generalize Brown-Henneaux's canonical approach to TMG.

➡ $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence with $c_L \neq c_R$

Equation of motion for TMG is expressed as

$$\frac{1}{2}G^{IJ}\left(R+\frac{2}{\ell^2}\right)-R^{IJ}=\beta\epsilon^{KL(I}\mathcal{D}_K R_L^{J)}$$

Geometries which satisfy $R_{IJ}=-\frac{2}{\ell^2}G_{IJ}$ become solutions

➡ Global AdS_3 and BTZ black hole exist in TMG.

global AdS_3 :

$$ds^2=-\left(1+\frac{r^2}{\ell^2}\right)dt^2+\left(1+\frac{r^2}{\ell^2}\right)^{-1}dr^2+r^2d\phi^2$$

BTZ black hole :

Banados, Teitelboim, Zanelli

$$ds^2=-N^2dt^2+N^{-2}dr^2+r^2(d\phi+N^\phi dt)^2,$$

$$N^2=\left(\frac{r}{\ell}\right)^2+\left(\frac{4G_Nj}{r}\right)^2-8G_Nm,\quad N^\phi=\frac{4G_Nj}{r^2}$$

Thermodynamic entropy of BTZ

BTZ black hole has inner and outer horizons :

$$r_\pm = \sqrt{2G_N\ell(\ell m + j)} \pm \sqrt{2G_N\ell(\ell m - j)} \qquad \ell m \geq j$$

The entropy of the BTZ black hole is evaluated as

$$
\begin{aligned}
S &= \frac{1}{4G_N}\oint_{r_+} d\phi \sqrt{G_{\phi\phi}} + \frac{\beta}{4G_N}\oint_{r_+} d\phi\, \varepsilon^{JI}\Gamma_{IJ\phi} \\
&= \frac{\pi}{2G_N} r_+ + \frac{\pi\beta}{2G_N\ell} r_- \\
&= \frac{\pi}{2G_N}\left\{\left(1+\frac{\beta}{\ell}\right)\sqrt{2G_N\ell^2\left(m+\frac{j}{\ell}\right)} + \left(1-\frac{\beta}{\ell}\right)\sqrt{2G_N\ell^2\left(m-\frac{j}{\ell}\right)}\ \right\}
\end{aligned}
$$

Solodukhin;
Tachikawa

This is a thermodynamic entropy. Then it is natural to ask whether we can derive the above quantity from the statistical viewpoint.

Asymptotically symmetry group

Brown, Henneaux

Let us consider asymptotic behaviors of global AdS3 and BTZ black hole at the boundary ($r \to \infty$). It is easy to see that those satisfy the following boundary condition.

$$G_{tt} = -\frac{r^2}{\ell^2} + \mathcal{O}(1), \quad G_{tr} = \mathcal{O}(r^{-3}), \quad G_{t\phi} = \mathcal{O}(1)$$
$$G_{rr} = \frac{\ell^2}{r^2} + \mathcal{O}(r^{-4}), \quad G_{r\phi} = \mathcal{O}(r^{-3}), \quad G_{\phi\phi} = r^2 + \mathcal{O}(1)$$

This b.c. is preserved under the coordinate transformations of

$$t \to t + \bar{\xi}^t, \quad r \to r + \bar{\xi}^r, \quad \phi \to \phi + \bar{\xi}^\phi$$

$$\bar{\xi}^t = \frac{\ell}{2} e^{inx^\pm}\left(1 - \frac{\ell^2 n^2}{2r^2}\right), \quad \bar{\xi}^r = -i\frac{nr}{2}e^{inx^\pm}, \quad \bar{\xi}^\phi = \pm\frac{1}{2}e^{inx^\pm}\left(1 + \frac{\ell^2 n^2}{2r^2}\right)$$

$n \in \mathbf{Z}$

$x^\pm = t/\ell \pm \phi$

Then Killing vector fields $\xi_n^\pm \equiv \bar{\xi}^I \partial_I$ satisfy commutation relations of

$$\left[\xi_m^\pm, \xi_n^\pm\right] = -i\,(m-n)\,\xi_{m+n}^\pm, \qquad \left[\xi_m^+, \xi_n^-\right] = \mathcal{O}(r^{-4})$$

Thus the asymptotically AdS3 spacetime is endowed with the 2D conformal symmetry on the boundary.

Hamiltonian formalism and central extension

We want to evaluate the central extension of the Virasoro algebras in TMG. In order to do it, we execute the following procedure.

A) Hamiltonian formalism.

B) Calculate the variation of the Hamiltonian, and add surface term to obtain correct equations of motion.

C) From this surface term, we obtain global charge. Possible to evaluate central charges.

A) Hamiltonian formalism.

ADM decomposition of 3D metric.

$$G_{IJ} = \begin{pmatrix} -N^2 + N_k N^k & N_j \\ N_i & g_{ij} \end{pmatrix}$$

$$K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - \mathcal{D}_i N_j - \mathcal{D}_j N_i\right)$$
$$K = g^{ij} K_{ij}$$

Then the Lagrangian of TMG is written as

$$\mathcal{L}_{\mathsf{TMG}} = \sqrt{g}N\left(r + \frac{2}{\ell^2} + K^{ij}K_{ij} - K^2\right) + v^{ij}(\dot{g}_{ij} - 2NK_{ij} - 2\mathcal{D}_i N_j)$$
$$+ \beta\sqrt{g}\epsilon^{mn}\dot{K}_{mk}K_n{}^k + \beta\sqrt{g}N\left(2\epsilon^{mn}\mathcal{D}_k\mathcal{D}_n K_m{}^k - A^{kl}K_{kl}\right) \qquad A_{ij}(\gamma)$$
$$+ \beta\sqrt{g}N^i\Big\{ - 2\epsilon^{mn}K_i{}^l\mathcal{D}_n K_{ml} - \epsilon^{mn}\mathcal{D}_k(K_{ni}K_m{}^k) + \frac{1}{2}\epsilon_{ij}\partial^j r + \mathcal{D}_k A_i{}^k\Big\}$$

Canonical variables conjugate to g_{ij} and K_{ij} are given as

$$\pi^{ij} = v^{ij}, \qquad \Pi^{ij} = \beta\sqrt{g}\epsilon^{ik}K_k{}^j$$

N, N^i : auxiliary fields

Then Hamiltonian is constructed as

$$\mathcal{H}_{\mathsf{TMG}} \cong \sqrt{g}N\Big\{ - r - \frac{2}{\ell^2} - K^{kl}K_{kl} + K^2 - 2\beta\epsilon^{mn}\mathcal{D}_k\mathcal{D}_n K_m{}^k + \left(2g^{-\frac{1}{2}}\pi^{kl} + \beta A^{kl}\right)K_{kl}\Big\}$$
$$+ \sqrt{g}N^i\Big\{2\beta\epsilon^{mn}K_i{}^l\mathcal{D}_n K_{ml} + \beta\epsilon^{mn}\mathcal{D}_k(K_{ni}K_m{}^k) - \frac{1}{2}\beta\epsilon_{ij}\partial^j r - \mathcal{D}_j\left(2g^{-\frac{1}{2}}\pi_i{}^j + \beta A_i{}^j\right)\Big\}$$
$$+ f_{ij}\left(\Pi^{ij} - \beta\sqrt{g}\epsilon^{ik}K_k{}^j\right)$$

B) Add surface term to obtain correct equations of motion.

Variations of the Hamiltonian is evaluated as

$$\delta\mathcal{H}_{\mathsf{TMG}} = (\cdots)\delta N + (\cdots)\delta N^i + (\cdots)\delta g_{ij} + (\cdots)\delta\pi^{ij} + (\cdots)\delta K_{ij} + (\cdots)\delta\Pi^{ij} + (\cdots)\delta f_{ij} - \boxed{\partial_l\{\cdots\}}$$

Correct equations of motion can be obtained iff the total derivative part is cancelled. Thus we define a new generator for each Killing vector $\bar{\xi}^I$ as

Regge, Teitelboim

$$\mathcal{H}[\xi] = \int d^2x\mathcal{H}_{\mathsf{TMG}}[\xi] + Q[\xi]$$

$$(\xi^0, \xi^r, \xi^\phi) = (N\bar{\xi}^t, \bar{\xi}^r + N^r\bar{\xi}^t, \bar{\xi}^\phi + N^\phi\bar{\xi}^t)$$

where is defined so as to cancel the total derivative part.

The explicit expression is written as

$$
\begin{aligned}
\delta Q[\xi] = & \int d\phi \Big[\sqrt{g} S^{ijkr} \Big(\xi^0 \mathcal{D}_k \delta g_{ij} - \mathcal{D}_k \xi^0 \delta g_{ij} \Big) + \xi^i (2\pi^{jr} + \beta g^{\frac{1}{2}} A^{jr}) \delta g_{ij} - \frac{1}{2} \xi^r (2\pi^{ij} + \beta g^{\frac{1}{2}} A^{ij}) \delta g_{ij} \\
& + \xi_i \delta (2\pi^{ir} + \beta g^{\frac{1}{2}} A^{ir}) - 2\beta \sqrt{g} \epsilon^{mr} \mathcal{D}_k \xi^0 g^{kl} \delta K_{ml} + 2\beta \sqrt{g} \epsilon^{mn} \xi^0 \mathcal{D}_n (g^{rl} \delta K_{ml}) \\
& + \frac{1}{2} \beta \sqrt{g} S^{ijkr} \Big((\epsilon^{mn} \partial_m \xi_n) \mathcal{D}_k \delta g_{ij} - \mathcal{D}_k (\epsilon^{mn} \partial_m \xi_n) \delta g_{ij} \Big) \\
& - 2\beta \sqrt{g} \epsilon^{mr} \xi^i K_i{}^l \delta K_{ml} - \beta \sqrt{g} \epsilon^{mn} \xi^i (\delta K_{ni} K_m{}^r + K_{ni} \delta K_m{}^r) - 2\beta \sqrt{g} \epsilon^{mn} g^{rl} \xi^0 K_{mo} \delta \gamma^o{}_{nl} \\
& - 2\beta \sqrt{g} \epsilon^{mn} g^{kl} g^{op} \Big\{ - \mathcal{D}_k \xi^0 K_{mo} T^{ijr}_{nlp} + \xi^0 \mathcal{D}_o K_{ml} T^{ijr}_{knp} + 2\xi^0 \mathcal{D}_n K_{o(l} T^{ijr}_{m)kp} \Big\} \delta g_{ij} \\
& + 2\beta \sqrt{g} \epsilon^{mn} \xi^o \Big\{ K_o{}^p K_{mk} g^{kl} T^{ijr}_{npl} + g^{qk} g^{lp} (K_{mk} K_{l(o} T^{ijr}_{n)qp} + K_{no} K_{l(k} T^{ijr}_{m)qp}) \Big\} \delta g_{ij} \\
& + \frac{1}{2} \beta \sqrt{g} \epsilon^{mn} T^{xyk}_{mlo} g^{op} \Big\{ \mathcal{D}_k u_{xy} g^{lq} T^{ijr}_{npq} - u_{xy} \gamma^q{}_{np} g^{ls} T^{ijr}_{kqs} + 2 u_{xy} \gamma^l{}_{q(p} g^{qs} T^{ijr}_{n)ks} \Big\} \delta g_{ij} \\
& - \frac{1}{2} \beta \sqrt{g} \epsilon^{mn} T^{ijr}_{mlo} u_{ij} g^{op} \delta \gamma^l{}_{np} + \frac{1}{2} \beta \sqrt{g} \epsilon^{mr} \xi_m \delta r \Big]
\end{aligned}
$$

$$S^{ijkl} = \frac{1}{2} \left(g^{ik} g^{jl} + g^{il} g^{jk} - 2 g^{ij} g^{kl} \right)$$

C) Global charges are obtained by the surface term.

$Q[\xi]$ defines a conserved quantity for each Killing vector $\bar{\xi}^I$. In particular, for $\bar{\xi} = \partial_t$ and $\bar{\xi} = \partial_\phi$, in the background of BTZ black hole, mass and angular momentum can be obtained

$$M = Q[\bar{\xi} = \partial_t]$$
$$J = Q[\bar{\xi} = \partial_\phi]$$

Algebraic structure of symmetric transformation group is given by the Poisson bracket of generators.

$$\{\mathcal{H}[\xi], \mathcal{H}[\eta]\}_P = \mathcal{H}[[\xi, \eta]] + \delta_\eta Q[\xi]$$

The last term gives the central extension of the algebra

Now it is possible to evaluate mass and angular momentum of BTZ black hole, and central charges at the boundary.

Mass

$$(\xi^0, \xi^r, \xi^\phi) = (N, N^r, N^\phi) \sim \left(\frac{r}{\ell},\ 0,\ \frac{4G_N j}{r^2}\right)$$

$$\begin{aligned} M &= \frac{1}{16\pi G_N}\delta Q[\xi] \\ &= \frac{1}{16\pi G_N}\oint_{r=\infty} d\phi \Big\{ 2\sqrt{g} S^{r\phi r\phi}(-\xi^0 \gamma^r{}_{\phi\phi}\delta g_{rr}) + 2\beta \mathcal{D}_k \xi^0 g^{kl}\delta K_{\phi l} \Big\} \\ &= m + \frac{\beta}{\ell^2} j \end{aligned}$$

Angular mom.

$$(\xi^0, \xi^r, \xi^\phi) = (0, 0, 1)$$

$$\begin{aligned} J &= \frac{1}{16\pi G_N}\delta Q[\xi] \\ &= \frac{1}{16\pi G_N}\oint_{r=\infty} d\phi \Big\{ \xi_i \delta(2\pi^{ir} + \beta g^{\frac{1}{2}} A^{ir}) + \beta\sqrt{g} S^{r\phi r\phi}(-\epsilon^{mn}\partial_m \xi_n \gamma^r{}_{\phi\phi}\delta g_{rr}) \Big\} \\ &= j + \beta m \end{aligned}$$

Central charges

$$(\xi^0,\xi^r,\xi^\phi)\sim\left(\frac{r}{2}e^{inx^\pm},-i\frac{nr}{2}e^{inx^\pm},\pm\frac{1}{2}e^{inx^\pm}\right)$$

$$\begin{aligned}\frac{1}{16\pi G_N}\delta_{\eta=\xi_n^+}Q[\xi=\xi_m^+] &= \frac{1}{16\pi G_N}\oint_{r=\infty}d\phi\Big\{\tfrac{1}{\ell}\Big(\tfrac{1}{r}\xi^0+\partial_r\xi^0\Big)\delta_\eta g_{\phi\phi}+\tfrac{r^3}{\ell^3}\xi^0\delta_\eta g_{rr}+\xi_\phi\delta_\eta(2\pi^{\phi r}+\beta g^{\frac{1}{2}}A^{\phi r})\Big\}\\ &+\frac{\beta}{16\pi G_N}\oint_{r=\infty}d\phi\left\{\tfrac{1}{\ell^2}\Big(\tfrac{1}{r}\xi^0+\partial_r\xi^0\Big)\delta_\eta g_{\phi\phi}+\tfrac{r^3}{\ell^4}\xi^0\delta_\eta g_{rr}+2\partial_r\xi^0 g^{rl}\delta_\eta K_{\phi l}\right\}\\ &= -\frac{i}{12}\frac{3\ell}{2G_N}\left(1+\frac{\beta}{\ell}\right)m(m^2-1)\delta_{m,-n}\end{aligned}$$

$$\begin{aligned}\frac{1}{16\pi G_N}\delta_{\eta=\xi_n^-}Q[\xi=\xi_m^-] &= \frac{1}{16\pi G_N}\oint_{r=\infty}d\phi\Big\{\tfrac{1}{\ell}\Big(\tfrac{1}{r}\xi^0+\partial_r\xi^0\Big)\delta_\eta g_{\phi\phi}+\tfrac{r^3}{\ell^3}\xi^0\delta_\eta g_{rr}+\xi_\phi\delta_\eta(2\pi^{\phi r}+\beta g^{\frac{1}{2}}A^{\phi r})\Big\}\\ &+\frac{\beta}{16\pi G_N}\oint_{r=\infty}d\phi\left\{-\tfrac{1}{\ell^2}\Big(\tfrac{1}{r}\xi^0+\partial_r\xi^0\Big)\delta_\eta g_{\phi\phi}-\tfrac{r^3}{\ell^4}\xi^0\delta_\eta g_{rr}+2\partial_r\xi^0 g^{rl}\delta_\eta K_{\phi l}\right\}\\ &= -\frac{i}{12}\frac{3\ell}{2G_N}\left(1-\frac{\beta}{\ell}\right)m(m^2-1)\delta_{m,-n}\end{aligned}$$

Thus we obtain left-right asymmetric central charges.

$$c_L=\frac{3\ell}{2G_N}\left(1+\frac{\beta}{\ell}\right),$$
$$c_R=\frac{3\ell}{2G_N}\left(1-\frac{\beta}{\ell}\right)$$

Statistical entropy of BTZ

Note that $L_0^\pm$ correspond to the isometries $\xi_0^\pm = \partial_\pm = \frac{1}{2}(\ell\partial_t \pm \partial_\phi)$.
Therefore we obtain

$$L_0^+ = \frac{1}{2}(M\ell + J) = \left(1 + \frac{\beta}{\ell}\right)\frac{1}{2}(m\ell + j)$$
$$L_0^- = \frac{1}{2}(M\ell - J) = \left(1 - \frac{\beta}{\ell}\right)\frac{1}{2}(m\ell - j)$$

From Cardy's formula for counting the states in CFT, we obtain the statistical entropy for BTZ black holes.

$$S = 2\pi\sqrt{\frac{1}{6}c_L L_0^+} + 2\pi\sqrt{\frac{1}{6}c_R L_0^-}$$
$$= \frac{\pi}{2G_N}\left(1 + \frac{\beta}{\ell}\right)\sqrt{2G_N\ell\,(m\ell + j)} + \frac{\pi}{2G_N}\left(1 - \frac{\beta}{\ell}\right)\sqrt{2G_N\ell\,(m\ell - j)}$$

This agrees with the previous thermodynamic entropy. We have thus proven the agreement between the macroscopic entropy and the statistical entropy including higher derivative correction.

g_s correction

5. Summary

Higher derivative corrections in string theory or M-theory are derived by imposing local 32 supersymmetry. Found only two candidates.

Tree level : $\alpha'^3 e^{-2\phi} \mathcal{R}^4$

1-loop : $\alpha'^3 g_s^2 (\mathcal{R}^4 + BR^4)$

Gauge/gravity correspondence including α'^3 corrections was tested by using Monte Calro simulation.

$$\frac{E}{N^2} = 7.41 T^{2.8} - C T^{4.6} \qquad C = 5.58$$

AdS/CFT correspondence was confirmed in TMG.

$$c_L = \frac{3\ell}{2G_N}\left(1 + \frac{\beta}{\ell}\right),$$
$$c_R = \frac{3\ell}{2G_N}\left(1 - \frac{\beta}{\ell}\right)$$

There is a warped AdS_3 vacuum in TMG. It is an interesting direction to test warped AdS_3/CFT_2 correspondence. Li, Song, Strominger