Random Matrix Theory and Strong Interactions

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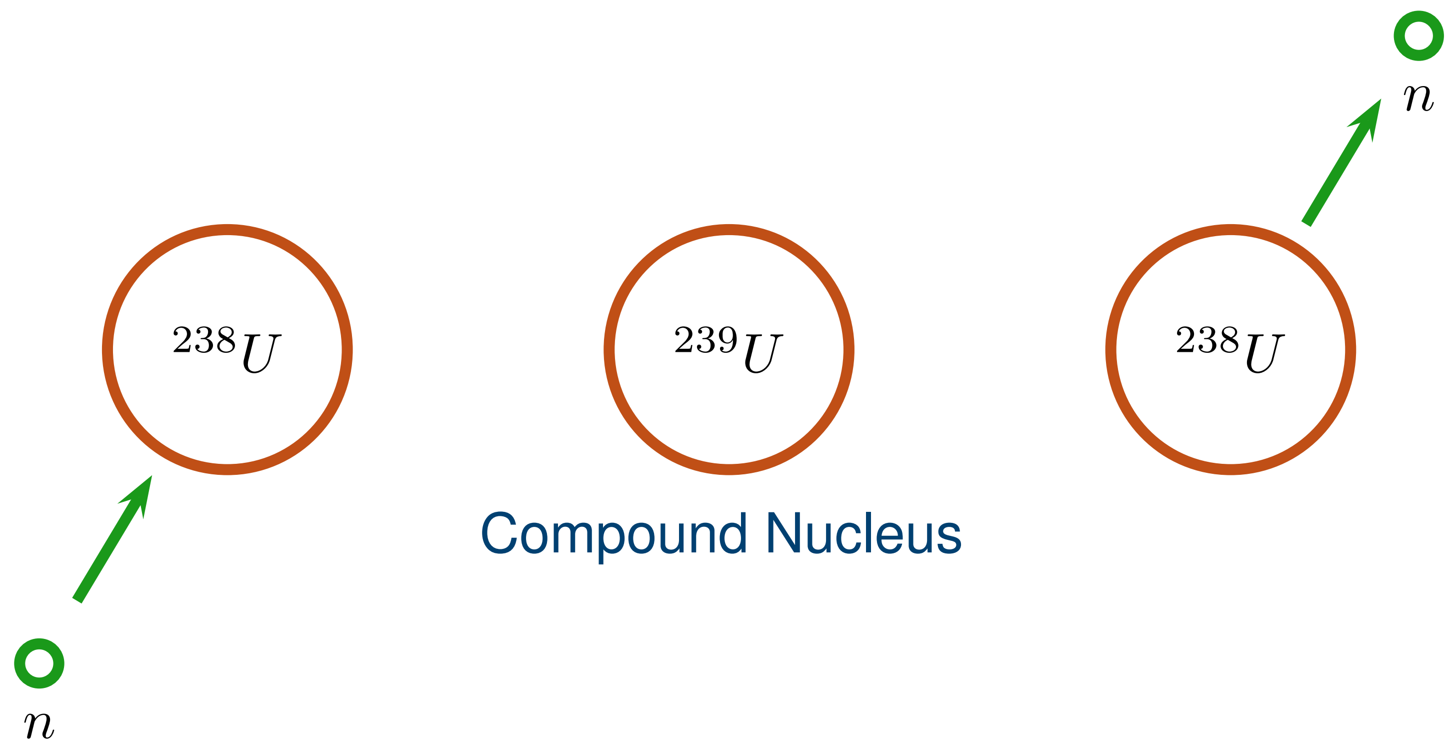
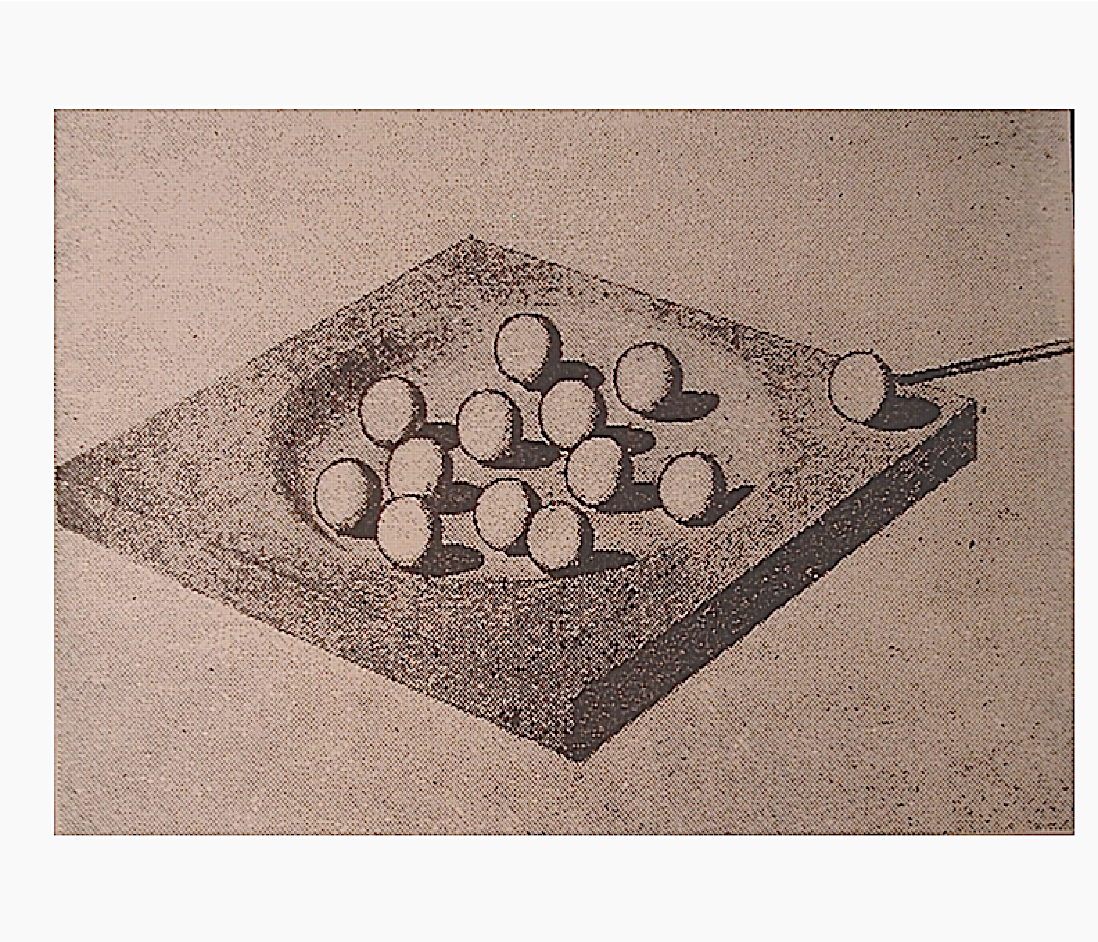
Compound Nucleus Scattering



- Formation and decay of a compound nuclear are independent.
- Because the system is chaotic, all information on its formation get lost.

Bohr's Billiard



Bohr, Nature 1934

Compound Nucleus Cross Section

- Hauser-Feshbach formula:

$$\sigma = \frac{T_i \, T_f}{\sum_k T_k}$$

with T_k the transmission coefficient for channel k defined by $T_k = 1 - |S_{kk}|^2$.

- To some extent, a compound nucleus has no hair, as is the case for a black hole.

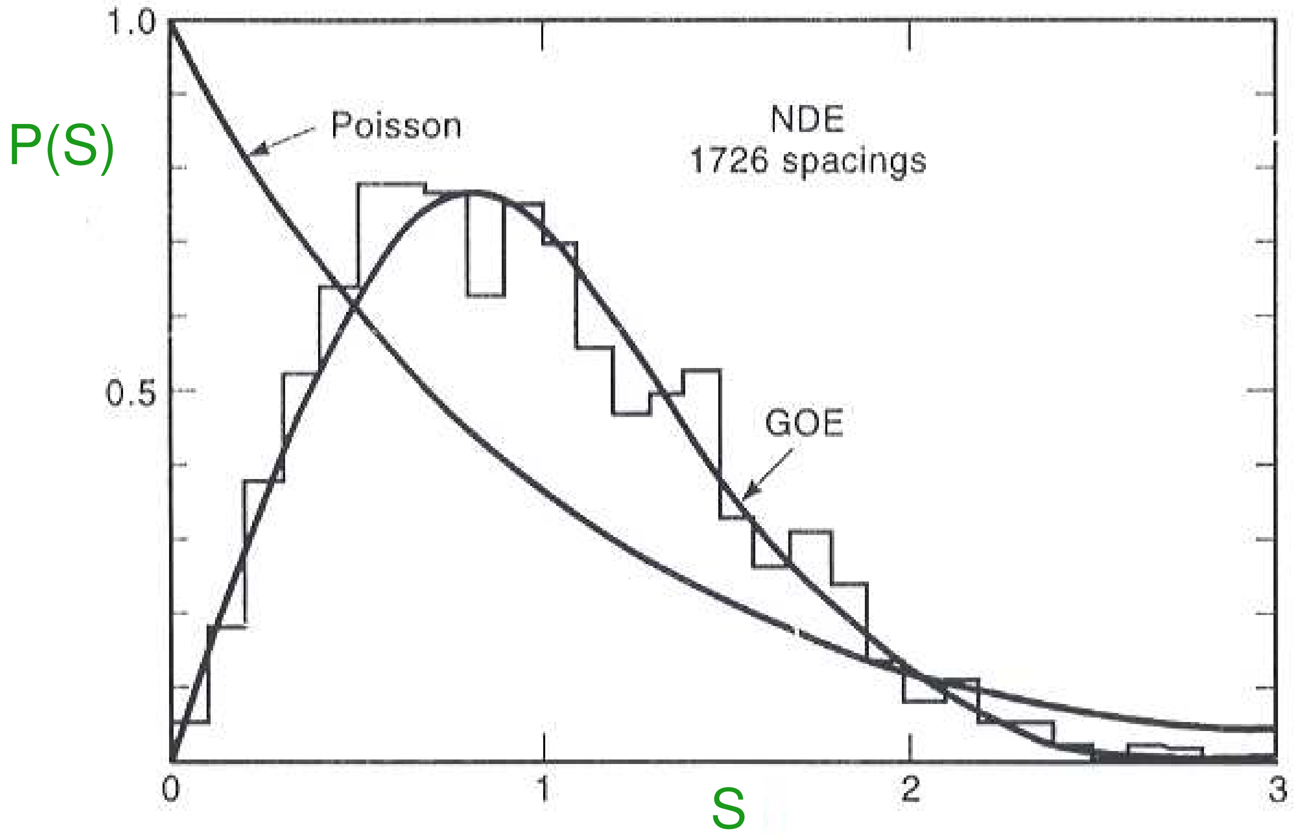
- Most likely a compound nucleus saturates the quantum bound on chaos obtained recently by Maldacena, Shenkar and Stanford. Black holes are believe to saturate this bound as well.

Quantum Hair of a Compound Nucleus

Total cross section versus energy (in eV).

Garg-Rainwater-Petersen-Havens,1964

Nuclear Data Ensemble



Nearest neighbor spacing distribution of an ensemble of different nuclei normalized to the same average level spacing.

Bohigas-Haq-Pandey, 1983

Random Matrix Theory

H $\qquad T \qquad A \qquad P(H)$

Hamiltonian

Anti-Unitary Symmetry

Anti-Commutator Symmetry

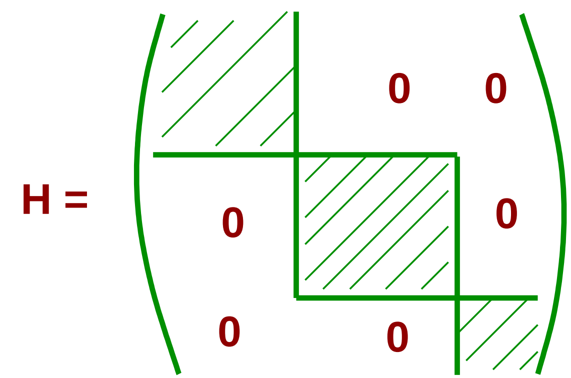
Probability Distribution

Example: Time reversal invariant system,

$$T\psi = \psi^* \qquad (T^2 = 1)$$

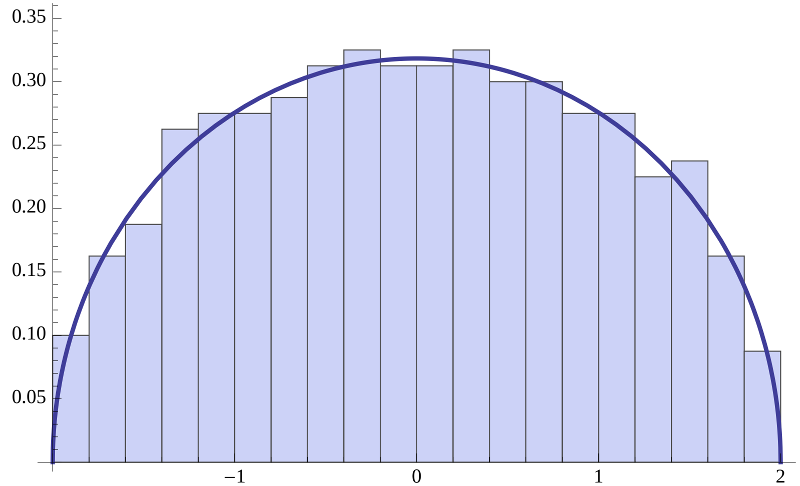
$$H^* = H\ , \qquad P(H) = e^{-N\,\mathrm{Tr}H^\dagger H}$$

$N \times N$ matrix



Wigner Semi-Circle

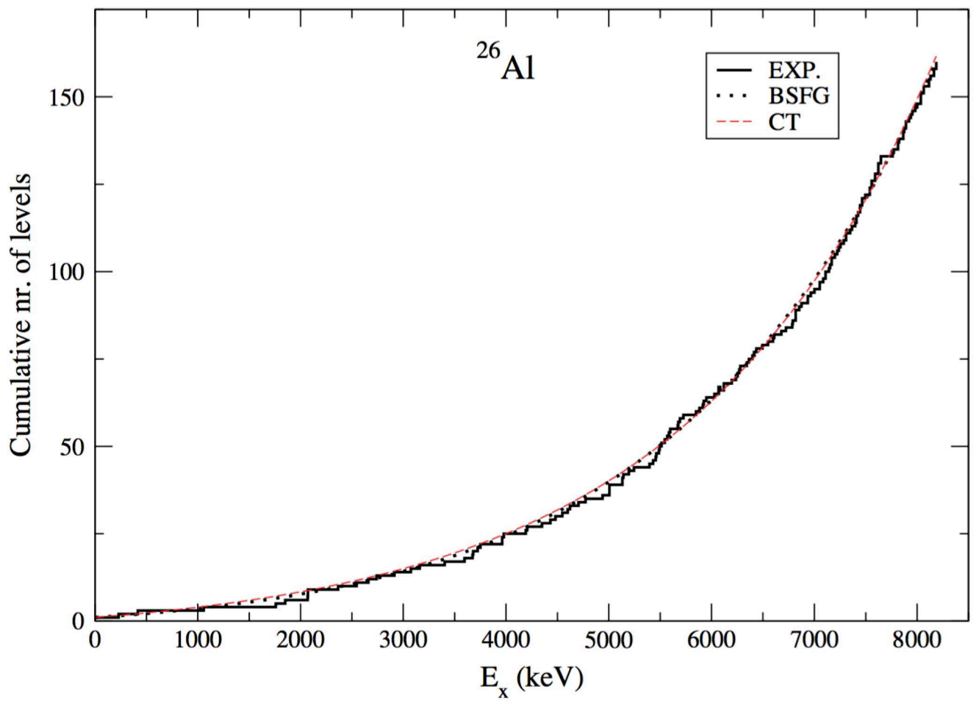
If the matrix elements are independent and have the same distribution, the eigenvalues are distributed according to as semi-circle in the limit of very large matrices



This is the case for a wide range of probability distributions which for convenience is usually taken to be a Gaussian, and a semicircular eigenvalue distribution is found for all 10 classes of random matrices.

Motivation for the Two-Body Random Ensemble

- The nuclear level density behaves as $e^{\alpha\sqrt{E}}$.
- The nuclear interaction is mainly a two-body interaction.
- Random matrix theory describes the level spacings, but it is and N-body interaction with a semicircular level density.



Two Body Random Ensemble

$$H = \sum_{\alpha\beta\gamma\delta} W_{\alpha\beta\gamma\delta} a^\dagger_\alpha a^\dagger_\beta a_\gamma a_\delta.$$

The labels of the fermionic creation and annihilation operators run over N single particle states. The Hilbert space is given by all many particle states containing m particles with $m = 0, 1, \cdots, N$.

The dimension of the Hilbert space is: $\sum \binom{N}{m} = 2^N$.

- $W_{\alpha\beta\gamma\delta}$ is Gaussian random.
- The Hamiltonian is particle number conserving.
- The matrix elements of the Hamiltonian are strongly correlated.

The Sachdev-Ye-Kitaev (SYK) Model

The two-body random ensemble from nuclear physics also has become known as the SYK model. However, being familiar with the history, we will only reserve this name for the two-body random ensemble with Majorana fermions Sachdev-Ye-1993,Kitaev-2015

$$H = \sum_{\alpha<\beta<\gamma<\delta} W_{\alpha\beta\gamma\delta}\chi_\alpha\chi_\beta\chi_\gamma\chi_\delta.$$

The fermion operators satisfy the commutation relations

$$\{\chi_\alpha, \chi_\beta\} = \delta_{\alpha\beta},$$

nd can be represented by γ -matrices.

The two-body matrix elements are taken to be Gaussian distributed.

Spectrum and Partition Function

The partition function of N fermions with Hamiltonian H is given by

$$Z(\beta) = \mathrm{Tr} e^{-\beta H} = \int dE \rho(E) e^{-\beta E}.$$

The spectral density is thus given by the Laplace transform of the partition function.

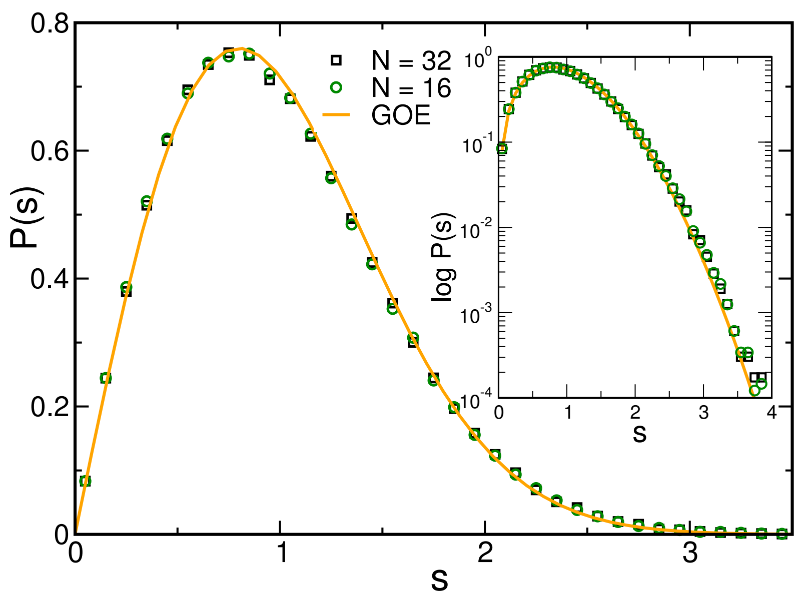
$$\rho(E) \quad = \quad \int_{r-i\infty}^{r+i\infty} e^{\beta E} Z(\beta) = \int_{r-i\infty}^{r+i\infty} e^{\beta E} e^{-\beta N E_0 + S + \frac{c}{2\beta}}$$

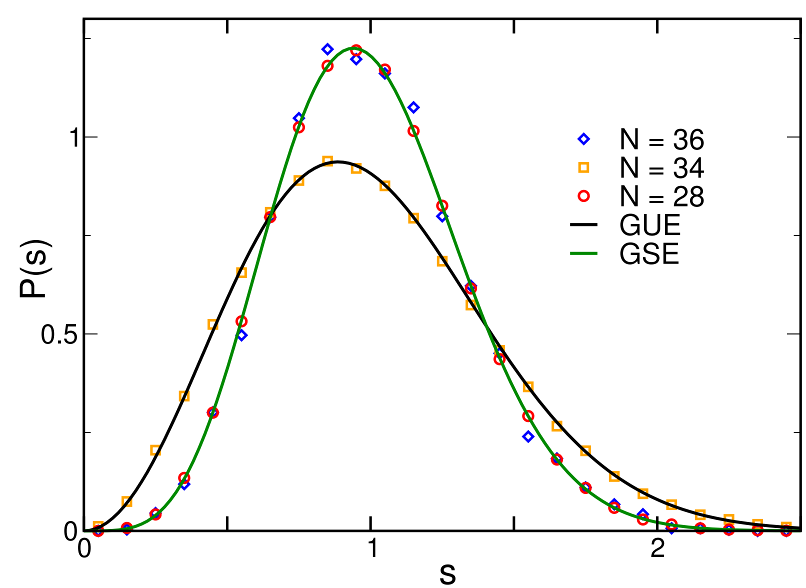
Doing a saddle point approximation one obtains

$$\rho(E) = \left(\frac{1}{2c}\right)^{1/4} \frac{1}{(E-E_0)^{3/4}} e^{\sqrt{2c(E-E_0)}}.$$

This is the Bethe formula for the nuclear level density. Bethe-1936

Nearest Neigbor Spacing Distribution





Garcia-Garcia-JV-2016

This model saturates the quantum bound on chaos. Kitaev-2015, Maldacena-Shenker-Stanford-2015.

This is also the case for black holes which explains the current interest in the SYK model.

Cotler-Hanada-Polchinsky-Saad-Shenkar-Stanford-Streicher-Tezuka-2016

Random Matrix Behavior of Spectra

- Level repulsion
- Spectral rigidity: the variance of the number of eigenvalues in an interval containing n eigenvalues on average behaves as $\log n$.
- Eigenvalues of a random matrix behave as a Wigner crystal.

Chiral Random Matrix Theory

Motivation

Chiral Random Matrix Theory

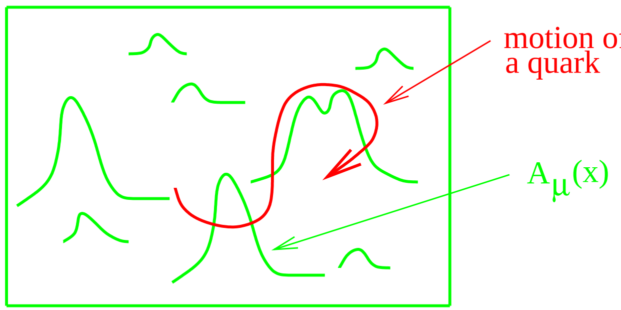
Banks-Casher Formula

Chiral Symmetry Breaking

Random Matrix Theory in QCD

- What we learned from this is that if a system is chaotic, even weakly chaotic, the eigenvalues of the corresponding quantum system are behave according to random matrix theory.

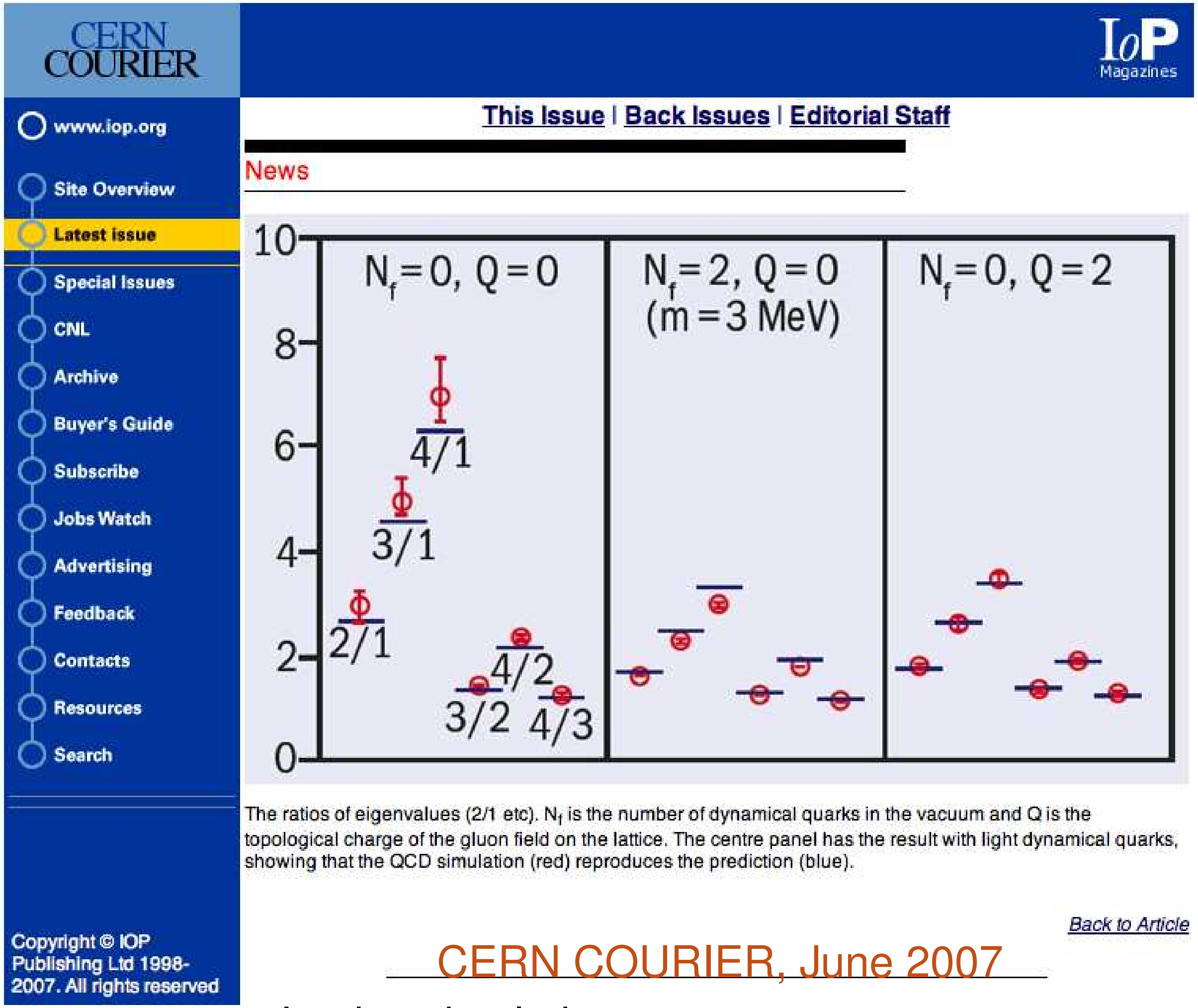
- The remarkable thing is that the microscopic theory for QCD in chaotic as well. If we interpret the Lagrangian of the Euclidean field theory as the Hamiltonian in 4+1 dimensions, the quarks move in a “random” gauge potential which is necessarily chaotic.



- Therefore, the eigenvalues of the corresponding Dirac operator are correlated according to random matrix theory.

- This has been confirmed by numerous lattice simulations

Flavor Topology Duality



CERN COURIER, June 2007

chRMT prediction:

In the chiral limit, the average position of the small Dirac eigenvalues only depends on the combination $N_f + Q$.

JV-2000

$$P(\lambda) \sim \lambda^{2N_f}\,\lambda^{2Q+1}$$

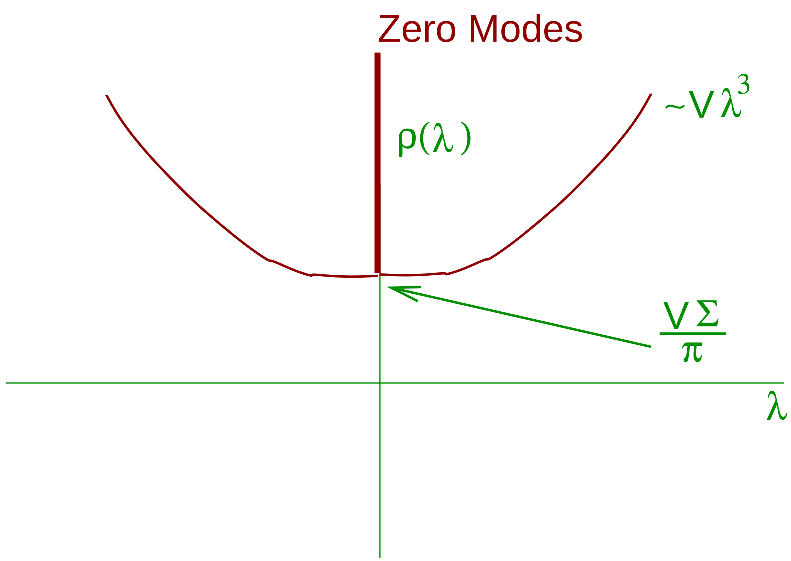
fermion determinant

Jacobian $D_{kl} \to \lambda_k$

Lattice simulations: Fukaya-et al-2007, Giusti-Lüscher-Weisz-Wittig-2003

QCD Dirac Spectra

- The Dirac eigenvalues show level repulsion and spectral rigidity which characterize random matrix theory.
- In other words, the Dirac eigenvalues behave as a Wigner crystal.
- There can be no gap in the Dirac spectrum around zero, or anywhere else in the spectrum.
- According to the Banks-Casher formula, this implies spontaneous breaking of chiral symmetry.



Banks-Casher Formula

$$
\begin{aligned}
\Sigma &= -\langle \bar{\psi}\psi \rangle = \frac{1}{V}\frac{d}{dm}\log Z \\
&= \left\langle \frac{1}{V}\sum_k \frac{1}{i\lambda_k + m} \right\rangle \\
&= \frac{1}{V}\int d\lambda \frac{\rho(\lambda)}{i\lambda + m} = \frac{1}{V}\int d\lambda \frac{\rho(\lambda)m}{\lambda^2 + m^2}
\end{aligned}
$$

For $m \to 0$ we find

$$
\Sigma = \frac{\pi\rho(0)}{V}.
$$

Random Matrix Theory for QCD

$$Z_\nu(m,\mu) = \int dW \det(D+m) e^{-n\Sigma^2 \mathrm{Tr} W^\dagger W}$$

with random matrix Dirac operator JV-Shuryak-1991, JV-1994

$$D = \left(\begin{array}{cc} 0 & iW+\mu \\ iW^\dagger + \mu & 0 \end{array} \right),$$

where μ can be arbitrary complex and W is an $n \times (n+\nu)$ matrix. The model has one parameter, Σ , which is the chiral condensate.

- Imaginary chemical potential Jackon-JV-1994 Schäfer-Weidenmüller-Wettig-1994
- Real chemical potential Stephanov-1996

Chiral Symmetry Breaking

- In all cases we know of the pattern of spontaneous symmetry breaking of QCD or the QCD like theory is the same as the corresponding random matrix theory.
- This is a highly nontrivial result becuase the chiral condensate arises as a consequence of non-perturbative QCD dynamics.
- One of the reasons for this equavalence is the relation between the order parameter and the Dirac spectrum

$$\langle \bar{\psi}_a \psi_a \rangle = \frac{1}{V} \sum_k \frac{1}{i\lambda_k + m_a}$$

- So the condensate is flavor independent and an $SU(N_f)$ subgroup of the flavor group cannot be broken. We can only break the symmetry to the full axial group.

Chiral Lagrangian

At a more technical level QCD at low energy is given by a weakly interacting gas of pions.

If the Compton wave length is much larger than the size of the box, we can neglect the kinetic term and the QCD partition function is given by

$$Z(M) = \int_{U \in SU(N_f)} dU e^{\mathrm{Tr}(M^\dagger U + M U^\dagger)}$$

This partition function is identical to the random matrix partition function in the limit of large matrices.

Validity of Random Matrix Theory

Random matrix theory is valid for Osborn-JV-1996

$$\frac{1}{m_\pi} \gg L \qquad \text{or} \qquad \frac{F_\pi^2}{2m\Sigma} \gg \sqrt{V}$$

In terms of the quark mass

$$m \ll \frac{F_\pi^2}{2\Sigma\sqrt{V}}.$$

The same argument can be applied to the eigevalues of the Dirac operator. Since

$$\lambda_{\min} = \frac{1}{\rho(0)} = \frac{\pi}{\Sigma V},$$

we always have a large number of Dirac eigenvalues described by RMT.

QCD at Nonzero Chemical Potential

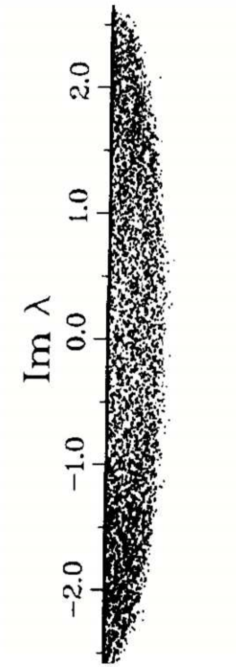
QCD at Nonzero Chemical Potential

Silver Blaze Problem

The Complex Langevin Algorithm

QCD at nonzero chemical potential

- QCD at nonzero chemical potential remains chaotic.
- The Dirac operator becomes non-hermitian
- Therefore the eigenvalues of the Dirac operator behave as random matrix eigenvalues, i.e. as a two-dimensional Wigner crystal.
- The eigenvalue density has no holes.
- Because it is two-dimensional, it is more interesting.



Dirac eigenvalues with Re(λ)> 0 on a $4^3 \times 8$ lattice.

Barbour-Bhilil-Dagotto-Karsch-Moreo-Stone-Wyld-1986

Sign Problem for $\mu \neq 0$

Because the Dirac operator at nonzero μ is nonhermitean, the fermion determinant is complex

$$\det(D + \mu\gamma_0 + m) = e^{i\theta} |\det(D + \mu\gamma_0 + m)|.$$

The *fundamental* problem is that the average phase factor may vanish in the thermodynamic limit, so that Monte-Carlo simulations are not possible (sign problem).

The severity of the sign problem can be measured by the ratio

$$\langle e^{2i\theta} \rangle_{1+1^*} \equiv \frac{\langle \det^2(D + m + \mu\gamma_0) \rangle}{\langle |\det(D + m + \mu\gamma_0)|^2 \rangle} \sim e^{-V(F_{N_f=2} - F_{pq})}.$$

full QCD
partition function

phase quenched
partition function

Splittorff-JV-2006

Phase Quenched Theory

$$Z(m,\mu) = \langle |\det(D+\mu\gamma_0)|^2\rangle = \langle \det(D+\mu\gamma_0)\det(D-\mu\gamma_0)\rangle$$

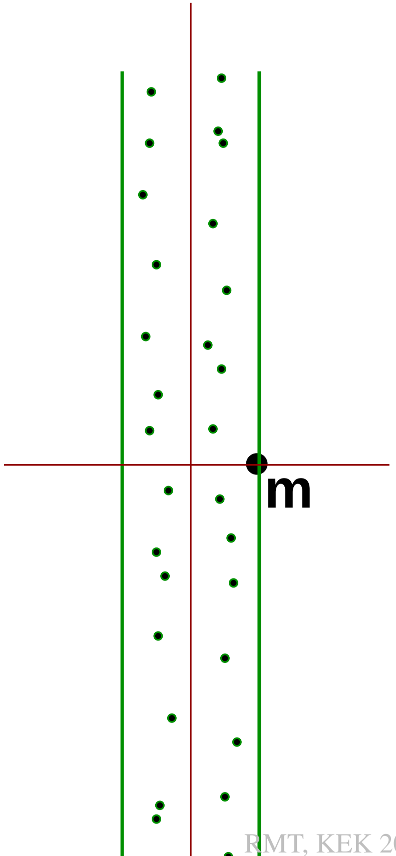
This is QCD at nonzero isospin chemical potential. Pion condensation occurs for $\mu > m_\pi/2$. Alford-Kapustin-Wilczek-1999

At this point, the quark mass hits the cloud of eigenvalues Toublan-JV-2000

$$\mu^2 = \frac{1}{4}m_\pi^2 = \frac{m\Sigma}{2F^2},$$

Width of the cloud of eigenvalues

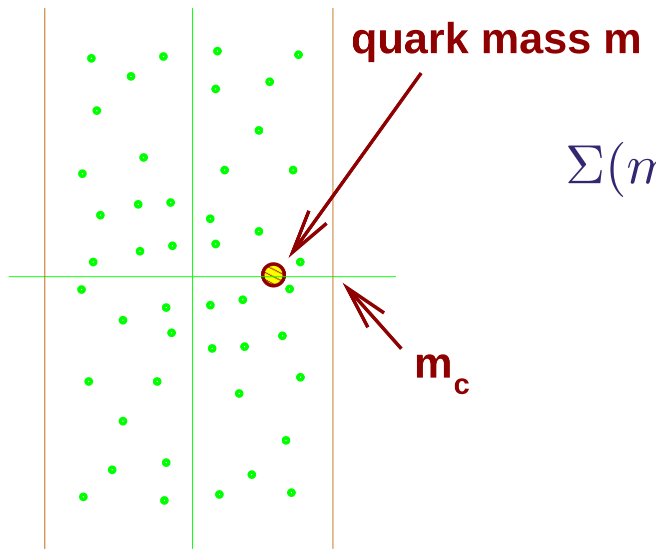
$$\frac{2\mu^2 F^2}{\Sigma}.$$



Silver Blaze Problem

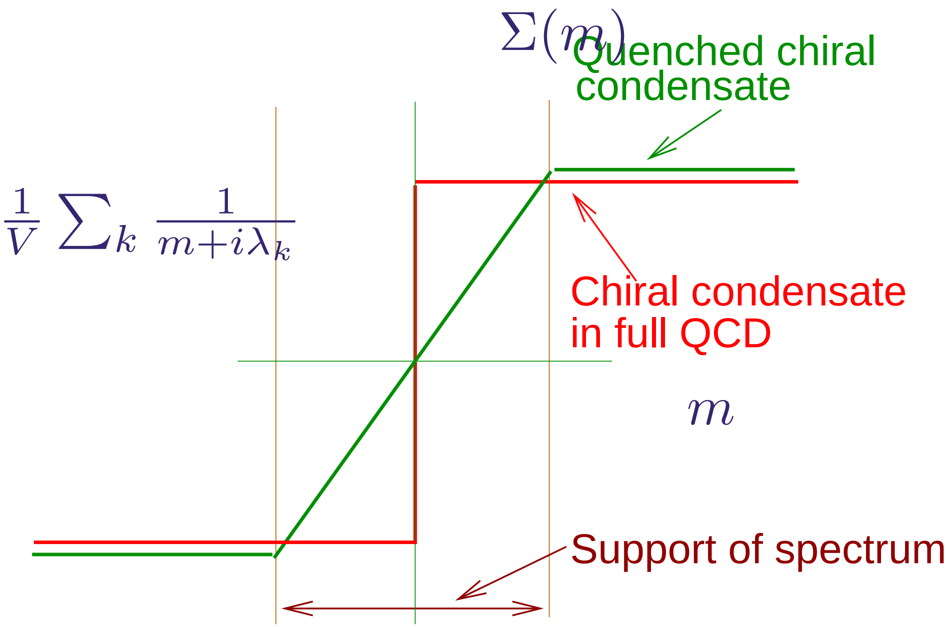
- For full QCD the eigenvalue distribution for each gauge field configuration is the same as that of quenched or phase quenched QCD.
- At low temperature, the contribution of the nucleons to the partition function can be ignored, so that the partition function does not depend on the baryon number chemical potential.
- Yet at $\mu = m_\pi/2$, the quark mass hits the cloud of eigenvalues but the chiral condensate remains constant until $m = 0$.
- Apparently, the transition to the pion condensed phase is nullified by the phase of the fermion determinant.
- The puzzle of how this can happen is know as the Silver Blaze problem. Cohen

Dirac Spectrum and Chiral Condensate



Scatter plot of Dirac eigenvalues

$$\Sigma(m) = \frac{1}{V}\sum_k \frac{1}{m+i\lambda_k}$$



- The support of the Dirac spectrum does not depend on the complex phase of the determinant.
- Exponential cancellations can wipe out the critical point and reveal a completely different physical system. This is the case of QCD at nonzero baryon density.

Solution of the Silver Blaze Problem

- Since the phase quenched partition function is exponentially larger (in the volume) than the full QCD partion function, it requires exponential cancellations to nullify the pion condensed phase.
- It was explained in terms of a random matrix theory at nonzero chemical potential.
- The discontinuity of the chiral condensate at $m = 0$ can be obtained from a strongly oscillating spectral density with a period $\sim 1/V$ and an amplitude that grows exponentially with the volume.
- This is a generic mechanism that occurs in nonhermitian theories with a sign problem including QCD at nonzero θ -angle. Osborn-Splittorff-JV-2006,Ravagli-JV-2007,Kanazawa-Wettig-2013,Wettig-JV-2014,Wettig-JV-Kieburg-2017

Analyticity

- Another way to get a μ-independent chiral condensate would be if the theory is equivalent to a theory at zero chemical potential.
- If the integrand of the partition function is an analytical function of the gauge fields, it could be possible that we can deform the integration contour to a field configuration with eigenvalues of the Dirac operator on the imaginary axis.
- In the complex Langevin algorithm gauge fields are complexified which is justified if if the integrand is analytic in the gauge fields.
- When the quark mass is inside the cloud of the integrand of the partition function is not an analytical function of the gauge fields.

The BBKSV Model

$$Z(m,\mu) = \int d\Phi \det \begin{pmatrix} m & e^{\mu}\Phi_1 - e^{-\mu}\Phi_2^\dagger \\ -e^{-\mu}\Phi_1^\dagger + e^{\mu}\Phi_2 & m \end{pmatrix} e^{-2n\Sigma^2 \mathrm{Tr}(\Phi_1\Phi_1^\dagger + \Phi_2\Phi_2^\dagger)},$$

where Φ_1 and Φ_2 are complex $n \times n$ matrices.

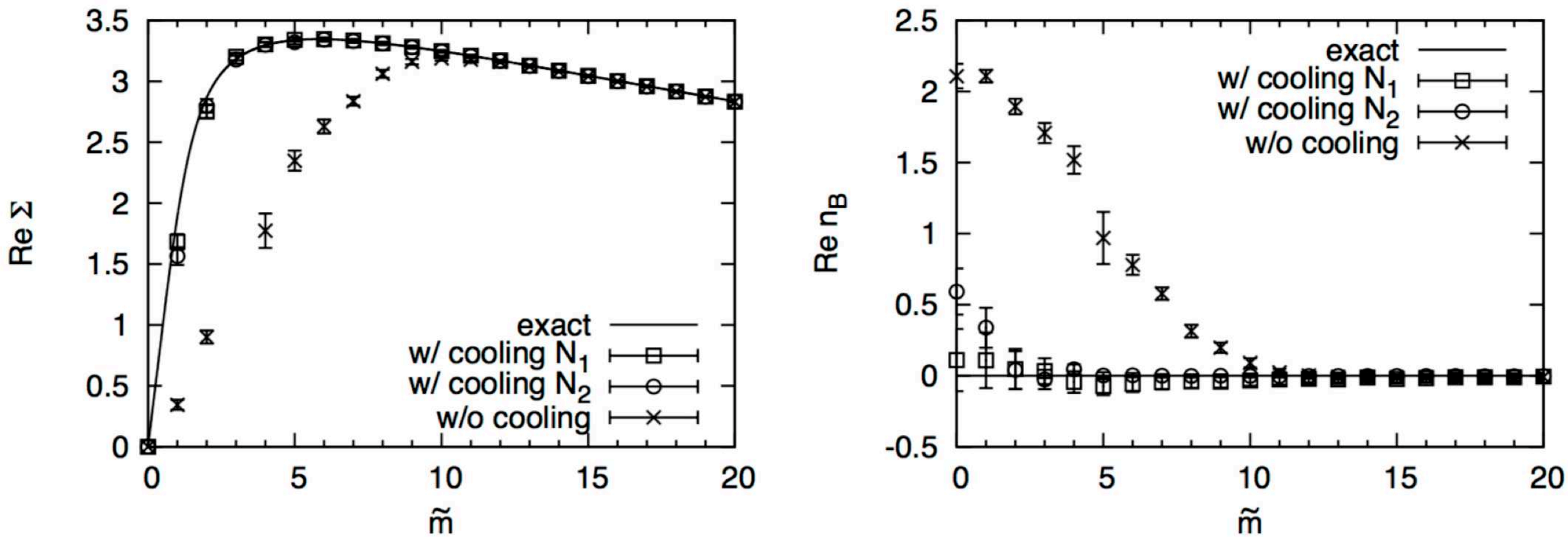
Bloch-Brückmann-Kieburg-Splittorff-JV-2013

For small μ this model reduces to the Osborn model Osborn-2004

The Gaussian integral is only nonzero for terms that have an equal number of factors Φ_i and $\Phi_i^\dagger$ for $i = 1, 2$ so that the partition function does not depend on μ .

- $Z(m,\mu) = Z(m,0)$.
- $Z(m,\mu + \pi i/2) = Z(m,\mu)$.

Testing the Complex Langevin Algorithm



Test of the complex Langevin algorithm for the BBKSV Model

Nagata-Nishimura-Shimasaki-2016

Complex Langevin works after “gauge cooling” adapted for random matrix theory.

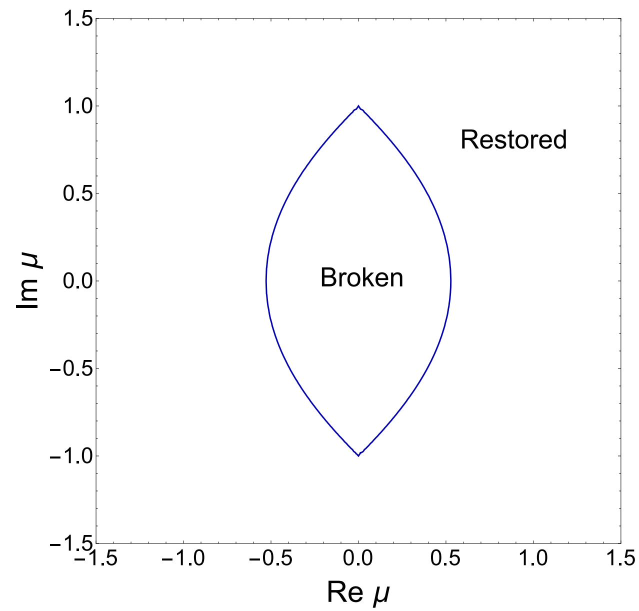
Random Matrix Model with Phase Transition

One major drawback of the BBKSV model is that it has no phase transition contrary to the $D + \mu\gamma_0$ model. This model can be solved analytically, and for $m = 0$, the solution is particularly simple,
Halasz-Jackson-JV-1998

$$Z_\nu(m,\mu) = \int_0^\infty ds s^{\nu+1} I_\nu(2mns\Sigma)(s^2-\mu^2)^n e^{-n\Sigma^2(s^2-\mu^2+m^2)}.$$

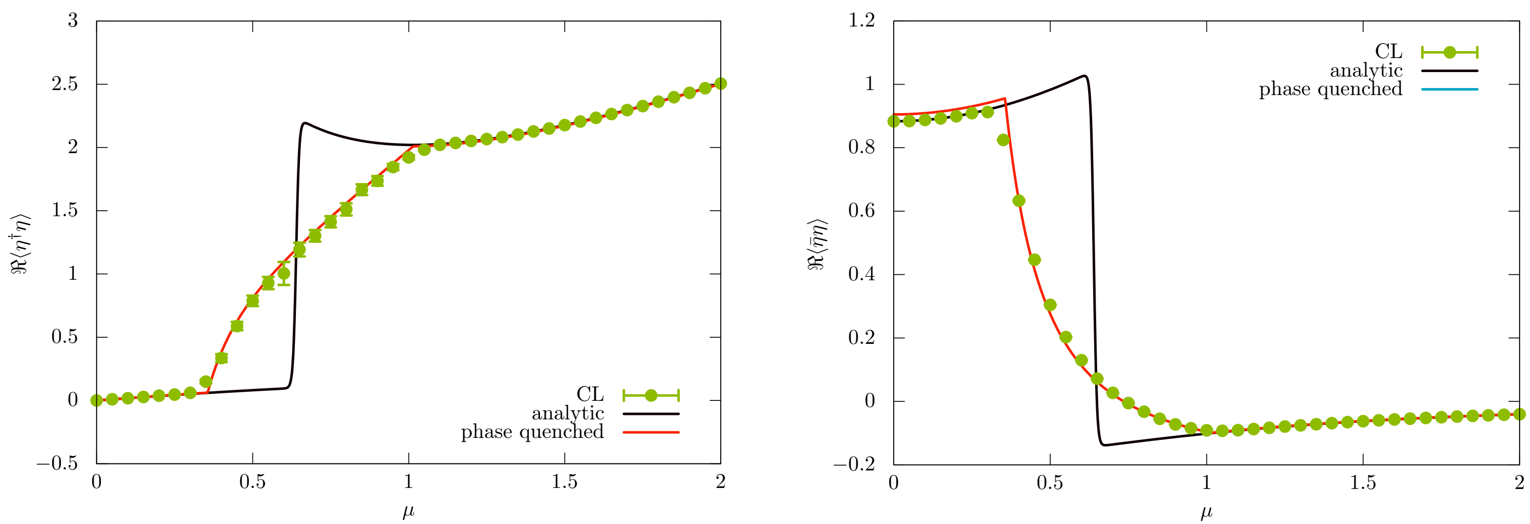
This expression is an analytic function of $\mu \in \mathcal{C}$.

$$D = \begin{pmatrix} 0 & iW + \mu\mathbf{1} \\ iW^\dagger + \mu\mathbf{1} & 0 \end{pmatrix}.$$



Can the complex Langevin algorithm reproduce this phase diagram?

Complex Langevin Simulation



Bloch-Glesaaen-Philipsen-JV-Zafeiropouos-2016

The baryon number (left) and the chiral condensate (right) as a function of the chemical potential for $m = 0.2$.

The complex Langevin algorithm converges to the phase quenched result. The reason is that the Complex Langevin method is probabilistic.

Why do Random Matrix Models Work?

- The random matrix theory has the global symmetries of QCD.
- The pattern of spontaneous symmetry breaking is the same as in QCD.
- In microscopic limit,

$$mV = \text{fixed}, \quad \lambda V = \text{fixed}, \quad \mu^2 V = \text{fixed} \qquad \text{for} \quad V \to \infty$$

  the above random matrix theories coincide with QCD.
- More precisely, in this limit random matrix theory coincides with the ϵ-domain of chiral perturbation theory.
- The mean field limit of the chiral Lagrangian in the p-counting scheme coincides with the ϵ -limit of QCD.

IV. Conclusions

- Chaos pervades determines the fundamental properties of both nuclear physics and QCD.

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- It is one of the great triumphs of Random Matrix Theory that it can describe the low-energy limit of a fundamental quantum field theory.
- Random Matrix theory has contributed greatly to our understanding of the QCD partition function at nonzero chemical potential and the spectral properties of the nonhermitian Dirac operator.
- Without random matrix theory the cancellation mechanism that leads to the Silver Blaze property of the chiral condensate could not have been understood.