The static 3-Quark potential in the SU(3) lattice QCD simulation

T. T. Takahashi
H. Matsufuru
Y. Nemoto
H. Suganuma

Research Center for Nuclear Physics
Osaka University

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1. Introduction

Quark Confinement Feature in Hadrons

- Regge trafectory for hadrons
- Heavy Quarkonium Features
- Lattice QCD Monte Carlo simulations

Quark Confinement Potential for
Quark-Antiquark(Q-$\overline{\mathrm{Q}}$)System

Linear Potential

$$V_{\mathrm{Q}\overline{\mathrm{Q}}}(r) \sim \sigma r$$

$\sigma = 0.89 \sim 1.0$ GeV/fm

$\sim$String tension

Color-Electric Flux-Tube Picture for Hadrons

Linear Q-$\overline{\mathrm{Q}}$ potential

Flux-Tube Picture for Quark Confinement

○ Color-Electric flux is squeezed as a one-dimensional tube or string

- Strong-Coupling QCD
  (Expansion on $\frac{1}{g^2}$ in the lattice formalism)
- Dual Superconductor picture
- Lattice QCD simulation

Q-$\overline{\mathrm{Q}}$ potential has been well studied, and its feature is directly related to Meson properties!

Then, how about 3Q potential V_{QQQ} ??

$V_{\mathrm{QQQ}} = \sigma(a+b+c)$?

Triangle Flux configuration

$V_{\mathrm{QQQ}} = \sigma L_{\mathrm{total}}$?

Minimal Flux Length configuration including ' junction point'

- 3Q potential V_{QQQ} is directly responsible to baryon properties

Minimal Length Configuration of Flux Tubes for 3 Quark System

In the flux-tube picture, the total tube length is minimized in the ground-state configuration.
For fixed 3 quarks, there are 2 categories for minimal-length flux.

Type-1 configuration Type-2 configuration

(a) If each angle of the ' quark triangle' does not exceed 120°, this ' Y-type' flux configuration has minimal length of total flux. Here, a junction appears at the ' Fermat point', and total flux-tube length is given as

$$L(a,b,c) = \left[\frac{a^2+b^2+c^2}{2} + \frac{\sqrt{3}}{2}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}\right]^{\frac{1}{2}}$$

(b) If an angle of the quark triangle exceeds 120°, the flux tubes appear along two shorter sides.

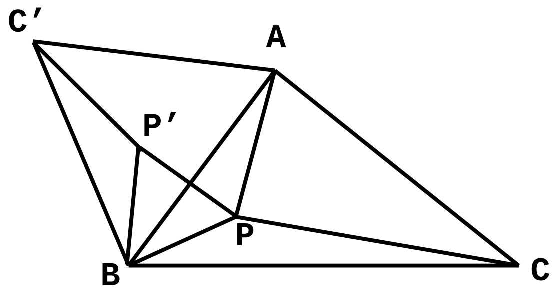
$$L(a,b,c) = a+b+c-\mathrm{Min}(a,b,c)$$

Proof of Minimal Flux Condition

For case (a)

This problem is equivalent to find the point P which realizes the minimization of $L = \overline{\mathrm{AP}} + \overline{\mathrm{BP}} + \overline{\mathrm{CP}}$ for fixed A,B and C.

To begin with, we define the point C' where $\triangle$BAC' is equilateral triangle.Taking an arbitary point P inside $\triangle$ABC, let us consider the point P' so as to make $\triangle$BPP' be an equilateral triangle.



Then, one easily finds $\triangle$BPA$\equiv$ $\triangle$BP'C',
and therefore $\overline{\mathrm{AP}} = \overline{\mathrm{C'P'}}$ and

$$L \equiv \overline{\mathrm{AP}} + \overline{\mathrm{BP}} + \overline{\mathrm{CP}} = \overline{\mathrm{CP'}} + \overline{\mathrm{P'P}} + \overline{\mathrm{PC}}$$

Since C' is a fixed point independent of P, L is minimized when P and P' lie on a straight line CC'. In this case, $\angle$APB=$\angle$BPC=$\angle$CPA=120°.

For case (b), we cannot take the point P lying on a straight line CC'. Then L is minimized when P concides with a vertex point between 2 shorter sides.

Static 3Q Potential

- Long-distance behavior
  $\sim$ Quark Confinement

3Q potential is expected to obey the Flux-Tube picture

$$V_{\mathrm{QQQ}}^{\mathrm{conf}} = \sigma L_{\mathrm{min}}$$

L_{min} : minimal length of Flux-tubes linking 3 quarks

- Short-distance behavier
  $\sim$ Perturbative QCD

QCD exhibits the asymptotic freedom

At short distances, perturbative QCD is applicable. At the leading order of P-QCD, one-gluon-excange is dominant and the inter-quark potential becomes sum of 2-body Coulomb-type as

$$V_{\mathrm{coulomb}} = \sum_{i<j} \frac{-A}{|\vec{r_i} - \vec{r_j}|}$$

To summarize,the 3Q potential is expected to take the form of

$$V_{\mathrm{QQQ}}^{\mathrm{total}} = -A\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + \sigma L_{\mathrm{min}} + C$$

2. Method

Expectaion Value of Operator

The vacuum expectation value of operator $\mathcal{O}$ is given as

$$< \mathcal{O} >= \frac{\int D\bar{\psi} D\psi DA \mathcal{O} \exp(iS_{\rm QCD})}{\int D\bar{\psi} D\psi DA \exp(iS_{\rm QCD})}$$

But, the analytical calculation of the path-integral is difficult in a nonperturbative manner. So, we adopt numerical simulation based on the Monte Carlo method.

To perform the numerical estimation, it is useful to take the Euclidean metric

Minkowski $\to$ Euclidean

Then we get $< \mathcal{O} >$ as

$$< \mathcal{O} >= \frac{\int D\bar{\psi} D\psi DA \mathcal{O} \exp(-S)}{\int D\bar{\psi} D\psi DA \exp(-S)}$$

Here

$$S \equiv \int L_{_{Euclid}} d^4 x.$$

This form is equivalent to that of statistical mechanics.

Lattice QCD Simulation

We define QCD on the lattice by discritizing space-time.

In this formalism, gauge field $A_\mu(x)$ is represented by 'Link Variable' $U_\mu(s) \equiv \exp(iagA_\mu(s))$, with the lattice spacing a and the gauge coupling constant g. All the physical quantities can be expressed as link-variable products.

$$A_\mu(x) \to U_\mu(s)$$

$$F_{\mu\nu}(x) \to U_\mu(s)U_\nu(s+\mu)U_\mu^\dagger(s+\nu)U_\nu^\dagger(s)$$

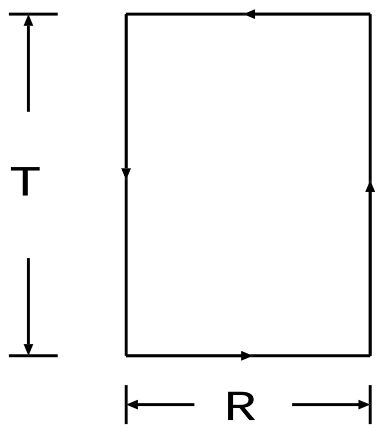
$$S_{\rm QCD} = \int d^4x \frac{1}{2}{\rm tr}\left(F_{\mu\nu}(x)F_{\mu\nu}(x)\right)$$

$$\to S_{\rm latt} = \beta \sum_{s,\mu<\nu} {\rm tr}\left(1 - \frac{1}{2N_c}(P_{\mu\nu} + P_{\mu\nu}^\dagger)\right) \quad \beta \equiv \frac{2N_c}{g^2}$$

• QCD gauge configurations are generated using the Monte Carlo method with random numbers weighted by $\exp(-S)$

$\to$ Statistical Mechanics

Measurement of Q-$\overline{\mathrm{Q}}$ Potential



The Q-$\overline{\mathrm{Q}}$ potential is obtained from the Wilson loop, a product of link variables along the rectangular contour corresponding to the quark color current $j_\mu^a(x)$.

Here, the Wilson loop $W \equiv \mathrm{tr}\,\Pi_i\, U_{\mu_i}(s_i)$ is gauge invariant, and relates with the potential $V(R)$ such as

$$\begin{aligned} < W > \; &\equiv \; < \mathrm{tr} \prod_i U_{\mu_i}(s_i) > = < \exp(-\textstyle\int j_\mu^a(x) A_\mu^a(x) d^4x) > \\ &= \sum_{n=0}^{\infty} C_n(R) \exp(-V_n(R)T) \end{aligned}$$

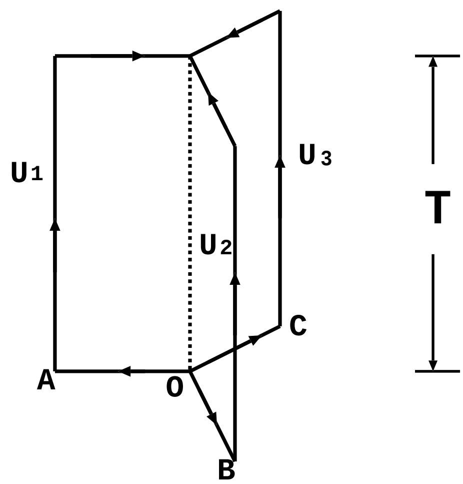
The label n denotes the quantum number, and $n = 0$ corresponds to the ground state. The factor $C_n(R)$ physically means the overlap between the operator W and the n-th state. Here, the normalization condition $\Sigma_{n=0}^{\infty} |C_n(R)|^2 = 1$ is imposed.

If we take the limit $T \to \infty$,

$$< W > \sim C_0 \exp(-V_0(R)T)$$

So, to get the grount-state potential, it is desired to take the large-T Wilson loop as possible.

Measurement of 3Q Potential



Then, how can we get the 3Q potential ? For this purpose, we consider the 3Q operator $W_{3\mathrm{Q}}$ defined on the contour of 3 large staples as shown in the above figure. We denote the product of link-variables along each staple by U_1, U_2 and U_3, which are SU(3) matricies.

Similar to Q-$\overline{\mathrm{Q}}$ case, the 3Q potential can be obtained from the expectation value of the 3Q operator $< W_{3\mathrm{Q}} >$.

$$
\begin{aligned}
< W_{3\mathrm{Q}} > &= < \epsilon^{a_1a_2a_3}\epsilon^{b_1b_2b_3}U_1^{a_1b_1}U_2^{a_2b_2}U_3^{a_3b_3} > \\
&= \sum_{n=0} C_n \exp(-V_n(i,j,k)T)
\end{aligned}
$$

By taking the limit $T \to \infty$, the ground-state potential $V_0(i,j,k)$ can be extracted as

$$
< W_{3\mathrm{Q}} > \sim C_0 \exp(-V_0(i,j,k)T)
$$

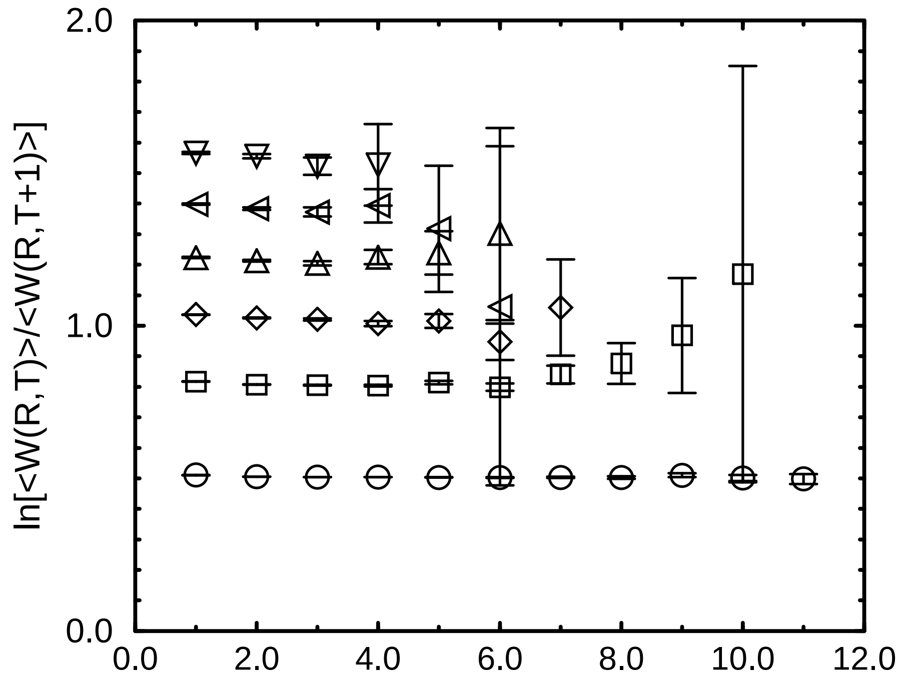
To estimate the ground-state potential, it is desired to take large T.

Effective Mass Plot

To evaluate the ground-state potential $V_0(R)$, it is convinient to measure the 'Effective mass', defined as

$$V(R,T) \equiv \ln \frac{< W(R,T) >}{< W(R,T+1) >}$$
$$\sim V_0(R) + \frac{C_1(R)}{C_0(R)} e^{-\Delta V(R)T} \left(1 - e^{-\Delta V(R)}\right)$$

Then, from the plot of the 'effective mass' $V(R,T)$ as the function of T, we can check whether ground-state is dominant in $< W >$, and can estimate the appropriate value of T to get the ground-state potential.



Smearing Technique

Smearing is a standard technique which enhances ground-state overlap in the lattice formalism. This is achieved by the replacementof spatial link variables $U_i(s)$ by 'obscured' links $\widetilde{U_i}(s)$ iteratively.

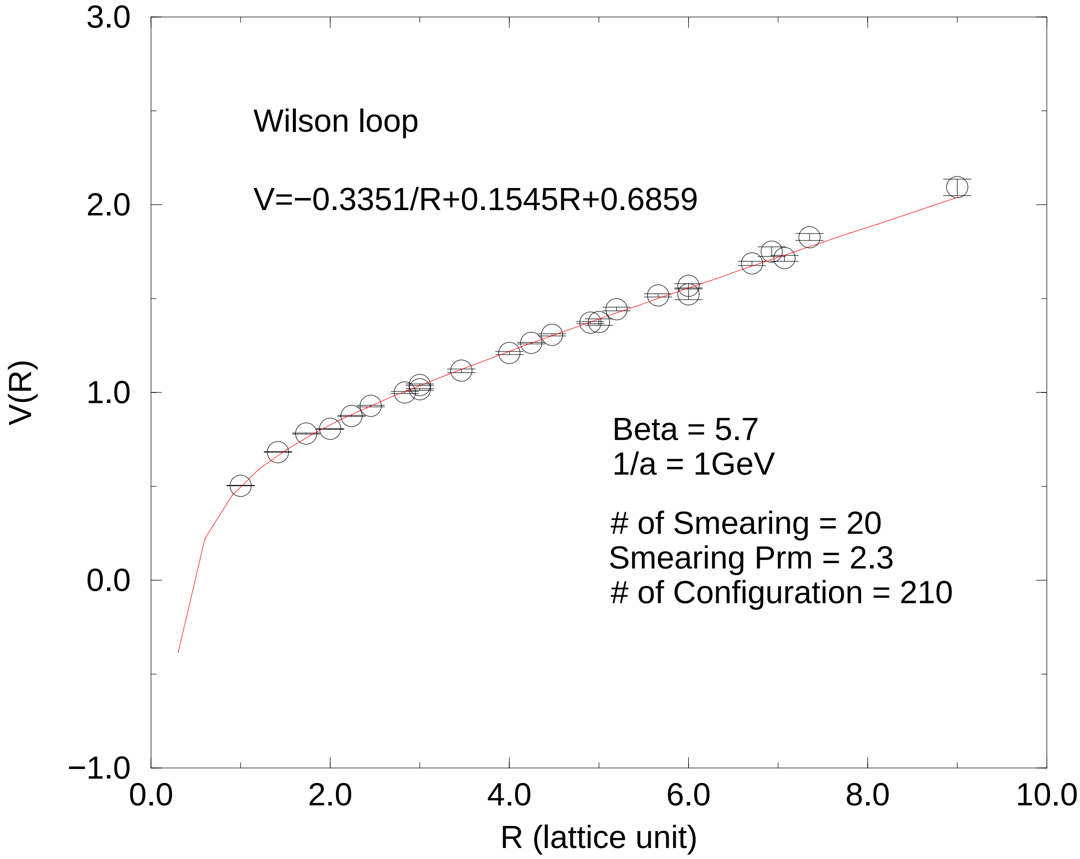
$$U_i(s) \to \widetilde{U_i}(s) \equiv N \left[(\alpha U_i(s) + \sum_{j \neq i} U_j(s) U_i(s+j) U_j^\dagger(s+i) \right]$$

(N is a normarization constant, and α is a parameter.)

We take α=2.3, number of iteration:$N_{\rm smr}$=20 for Q-$\overline{\rm Q}$ potential, and α=2.3, $N_{\rm smr}$=20 for 3Q potential .

3. Results

Lattice QCD Result of Q-$\overline{\text{Q}}$ Potential

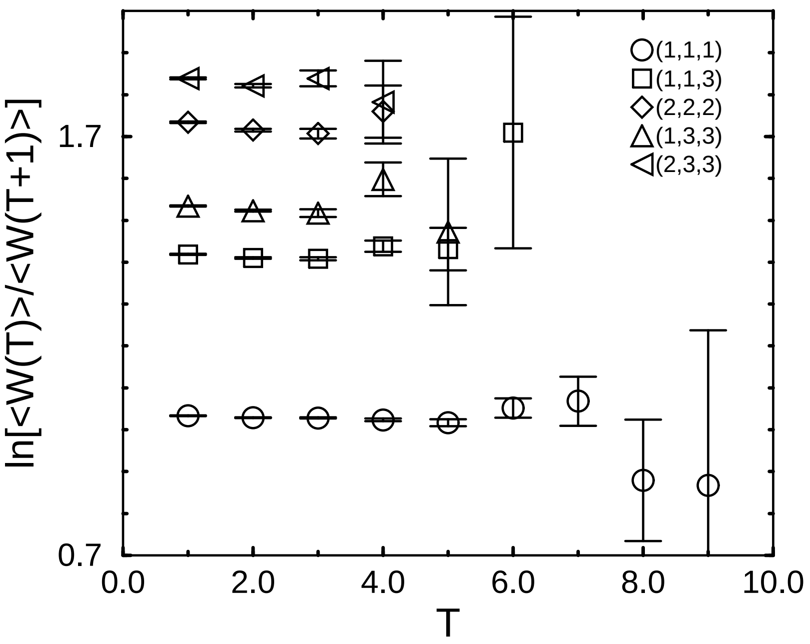


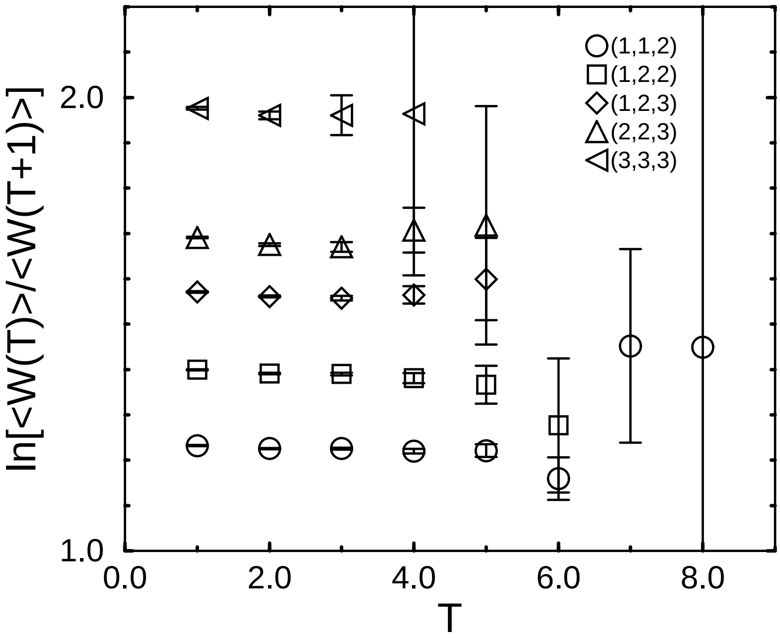
The solid curve denotes a fitting function of

$$V(R) = -\frac{A}{R} + \sigma R + C$$

The string tension is estimated as $\sigma \sim 0.85\text{GeV/fm}$

Effective Mass Plot for 3Q state

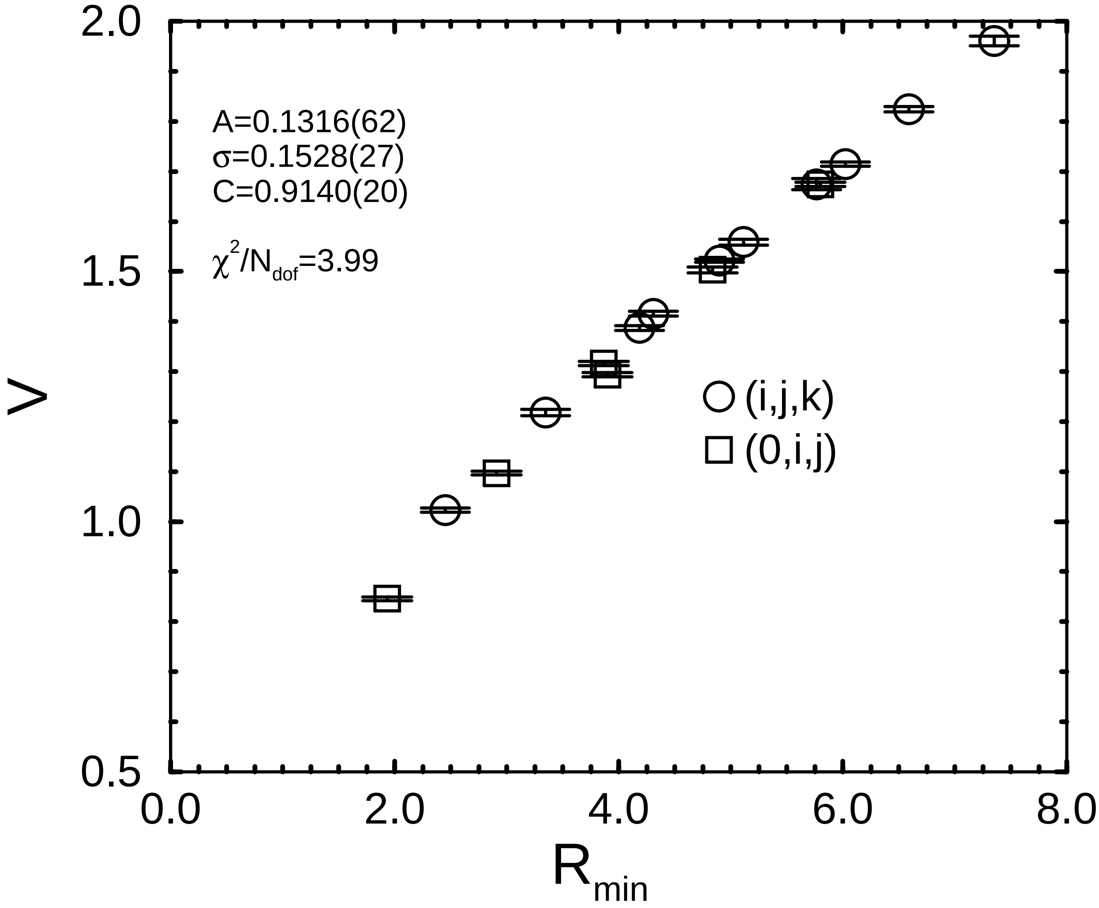




Lattice QCD Result for 3Q Potential

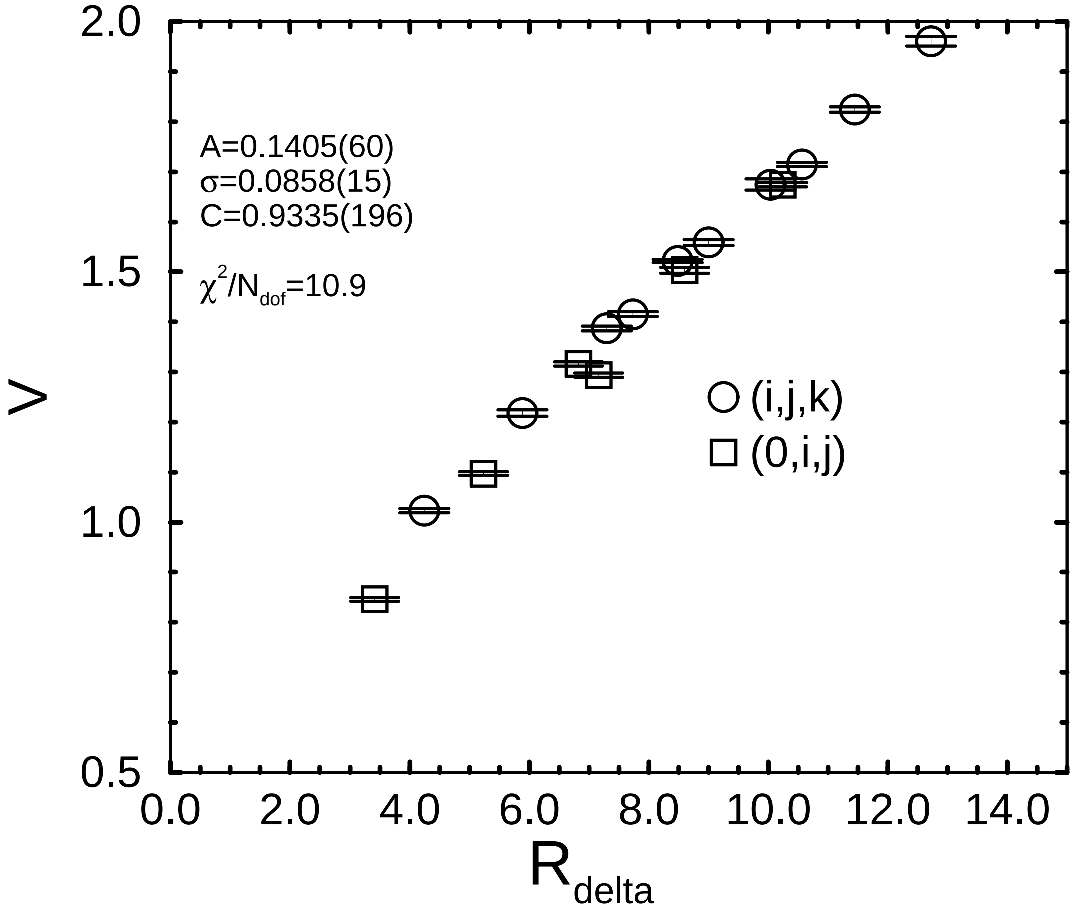
(a) Y-type ansatz

The 3Q potential $V_{\rm QQQ}$ as the function of the minimal length $R_{\rm min}$ of the flux tube linking the 3 quarks. Apart from a constant, $V_{\rm QQQ}$ is described as a linear potential in terms of the minimal linking length $R_{\rm min}$.



(b) Delta-type ansatz

The 3Q potential $V_{\rm QQQ}$ as the function of the perimeter $R_{\rm delta}$ of the quark triangle. The delta-type ansatz is less successful for the fitting $V_{\rm QQQ}$ in comparison with the Y-type ansatz.


A=0.1405(60)
σ=0.0858(15)
C=0.9335(196)
χ²/N_dof=10.9
(i,j,k)
(0,i,j)
V
R_delta
2.0
1.5
1.0
0.5
0.0
2.0
4.0
6.0
8.0
10.0
12.0
14.0

Table

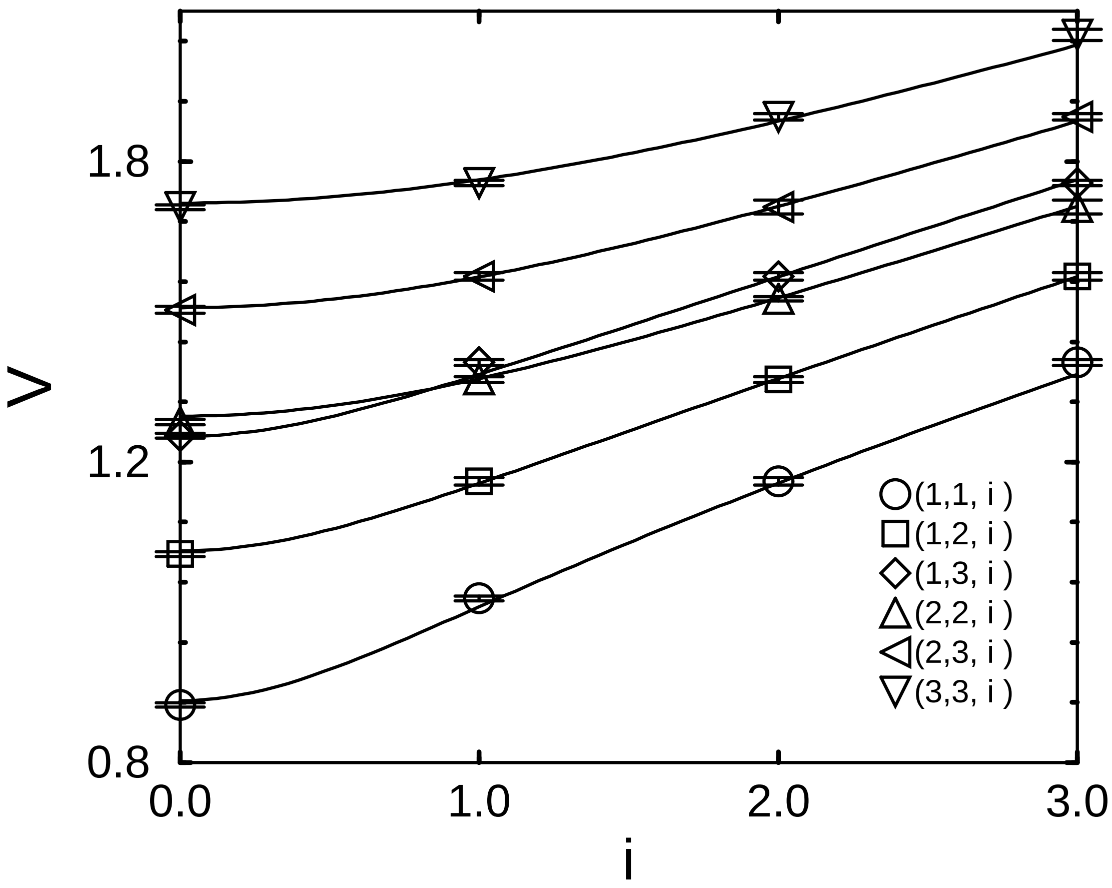
(i,j,k)	$V_{\rm latt}$	$V_{\rm min}$	$V_{\rm latt} - V_{\rm min}$
(1, 1, 1)	1.0231(38)	1.0091	-0.0140
(1, 1, 2)	1.2181(61)	1.2145	-0.0036
(1, 1, 3)	1.4154(49)	1.3958	-0.0196
(1, 2, 2)	1.3870(46)	1.3887	0.0017
(1, 2, 3)	1.5588(60)	1.5580	-0.0008
(1, 3, 3)	1.7141(43)	1.7195	0.0054
(2, 2, 2)	1.5216(33)	1.5230	0.0014
(2, 2, 3)	1.6745(11)	1.6755	0.0010
(2, 3, 3)	1.8242(54)	1.8169	-0.0073
(3, 3, 3)	1.9607(92)	1.9438	-0.0169
(1, 1, 0)	0.8459(36)	0.8529	0.0070
(1, 2, 0)	1.0970(40)	1.1023	0.0053
(1, 3, 0)	1.2935(39)	1.2926	-0.0009
(2, 2, 0)	1.3164(40)	1.3262	0.0098
(2, 3, 0)	1.5032(58)	1.5069	0.0037
(3, 3, 0)	1.6741(40)	1.6808	0.0067

The comparison between the lattice QCD data $V_{\rm latt}$ for the 3Q potential and the fitting function of

$$V_{\rm min} = -A\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) + \sigma R_{\rm min} + C$$

for each 3 quark configuration labeled by (i,j,k). One finds a quite good agreement between $V_{\rm latt}$ and $V_{\rm min}$ within a few % accuracy.

Comparison of lattice QCD data with Theoretical Fitting Curve



The comparison between the lattice QCD data for the 3Q potential and the fitting function

$$V_{\rm QQQ} = -A\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) + \sigma R_{\rm min} + C$$

which is suggested from the flux-tube picture. This fitting with 3 parameters (A,σ,C) seems quite well.

Summary and Conclusion

The static three-quark potential is studied in detail with the SU(3) lattice-QCD Monte Carlo simulation at the quenched level. The potential measurements are done for 16 patterns of three-quark configurations on the SU(3) lattice with $12^3 \times 24$ and $\beta = 5.7$. The ground-state potential for three-quark systems is extracted with the standard smearing technique. Apart from a constant, the ground-state three-quark potential is well fitted by a sum of the two-body Coulomb-type part and the three-body linear part within a few % accuracy. Here, the three-quark confinement potential $V_{\rm 3Q}^{\rm conf}$ is set to be proportional to the minimal length linking the three quarks, and the coefficient almost coincides with the string tension obtained from the Q-$\bar{Q}$ potential. In terms of the hadron flux-tube picture, this lattice QCD result supports the minimal flux-tube formation among the three quarks instead of the triangle flux-tube formation. As an approximation, however, $V_{\rm 3Q}^{\rm conf}$ is almost describable by a sum of effective two-body Q-Q potentials with a reduced string tension by a color factor.

$$V_{\rm QQQ} = -A\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) + \sigma R_{\rm min} + C$$

$$R_{min}(a,b,c) = \left[\frac{a^2+b^2+c^2}{2} + \frac{\sqrt{3}}{2}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}\right]^{\frac{1}{2}}$$