Measurement of Direct Photon Emission
in $K^+ \rightarrow \pi^+\pi^0\gamma$ Decay

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Abstract

Experimental study of the direct photon emission in the radiative $K\pi 2$ decay, $K^+ \rightarrow \pi^+ \pi^0 \gamma$, is a test of low-energy QCD theories as Chiral Perturbation Theory. A new measurement of the $K^+ \rightarrow \pi^+ \pi^0 \gamma$ decay with the data set taken in 1998 by the E787 experiment at the Brookhaven National Laboratory was performed. The number of the observed $K^+ \rightarrow \pi^+ \pi^0 \gamma$ events is 20571, and the best fit to the decay spectrum gives a branching ratio for direct photon emission of $BR(\Delta E) = (3.5 \pm 0.6(stat)_{-0.8}^{+0.6}(sys)) \times 10^{-6}$ in the $\pi^+$ kinetic energy from 55 MeV to 90 MeV. The new result is consistent with the previous E787 result from the 1995 data set. By combining these, $BR(\Delta E) = (3.9 \pm 0.5(stat)_{-0.8}^{+0.6}(sys)) \times 10^{-6}$ is obtained from E787. The results are consistent with those predicted by Chiral Perturbation Theory.
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Chapter 1

Introduction

Particle physics describes the principle of nature on the most fundamental level. The Standard Model, which consists of Quantum Chromodynamics (QCD) and Electroweak theories, provides us with good explanations up to the energy scale of a few hundred GeV. QCD is a theory of strong interactions and is able to explain the behavior of quarks whose energy is larger than 1 GeV.

The standard model is a chiral quantum-field theory, and the structure introduces chiral anomalies to several physics processes. The origin and mathematical properties in the theory have been intensively studied. On the other hand, the number of experimental studies in the past is limited. It has been known that there are some good examples of chiral anomaly in the low energy processes.

QCD has passed many tests in high energy experiments in $e^+ e^-$ colliders and hadron colliders, thanks to the asymptotic freedom. In the low energy scale, where quarks are confined within baryons and mesons and cannot be isolated, QCD is unable to describe their interactions directly. Many effective-field theories of QCD have been proposed, and one of the most promising ones is Chiral Perturbation theory (ChPT) [1, 2, 3]. The effective Lagrangian in the framework of ChPT is described with the fields of pseudo-scalar mesons, which are regarded as Goldstone bosons due to spontaneous breaking of chiral symmetry in the QCD vacuum, in the powers of the momentum. An example of hadronic decays in ChPT is shown in Fig. 1.1. ChPT has succeeded in providing good explanation about hadronic interactions of pions. In order to confirm the validity of ChPT, studies of kaon decay are important because the kaon mass, 493 MeV/$c^2$, is smaller than the cutoff energy for QCD and is larger than the masses of $u$ and $d$ quarks.

1.1 The $K^+ \rightarrow \pi^+ \pi^0 \gamma$ Decay

Charged kaons decay to the final state of $\pi^+ \pi^0 \gamma$ with a branching ratio of $(2.75 \pm 0.15) \times 10^{-4}$ [4]. The motivation to study the $K^+ \rightarrow \pi^+ \pi^0 \gamma$ decay is to understand the component of Direct Emission (DE) [5, 6], which is the process of radiating photon directly from
the kaon due to hadronic interactions. Another component in the decay mode, Internal Bremsstrahlung (IB), is a radiative correction to the $K^+ \to \pi^+\pi^0(K\pi2)$ decay by QED, in which the photon is emitted from the charged particles in the initial or final state ($K^+$ or $\pi^+$). Although $K^+ \to \pi^+\pi^0$ is suppressed by the $\Delta I = \frac{1}{2}$ rule\cite{7}, the radiative-correction process is still large; the amplitude of the DE component is small while the IB component is dominant in the $K^+ \to \pi^+\pi^0\gamma$ decay. The branching ratio for the IB component of the $K^+ \to \pi^+\pi^0\gamma$ decay, $2.61 \times 10^{-4}$, is calculated reliably by QED. The diagrams for the IB and DE components are shown in Fig.1.2.

The DE component is composed of the "magnetic" and "electric" transitions. The magnetic transition is described within ChPT at the order of $p^4(O(p^4))$\cite{8}, where $p$ is the momentum of a hadron, and is further divided into two amplitudes: one is the "reducible" amplitude and another is the "direct" amplitude. The "reducible" amplitude($M_{reducible}$) is described with two coupling constants $G_8$ and $f$, where $f = 93$ MeV, and the kaon mass($m_k$) as\cite{8}

$$M_{reducible} = -\frac{G_8 m_k^3}{2\pi^2 f}.$$  

(1.1)
Integrated over the whole Dalitz plot, the amplitude (Eq.1.1) gives rise to the branching ratio for the DE component of the $K^+ \to \pi^+ \pi^0 \gamma$ decay as [8]

$$BR(K^+ \to \pi^+ \pi^0 \gamma)_M = 8 \times 10^{-6} \left( \frac{G_8}{9 \times 10^{-6} \text{ GeV}^{-2}} \right). \quad (1.2)$$

$|M|$ is therefore obtained from the DE branching ratio, if the reducible amplitude is dominant in the magnetic transition, as

$$|M_{\text{reducible}}| = -\frac{eM_K^3}{2\pi^2 f} \times \sqrt{\frac{BR(K^+ \to \pi^+ \pi^0 \gamma)_M}{8 \times 10^{-6}}} \times 9 \times 10^{-6} \text{ GeV}^{-2}. \quad (1.3)$$

The "direct" amplitude ($M_{\text{direct}}$) has theoretical uncertainties derived from chiral anomaly. A process called the vector meson exchange (VMD) can also occur at $O(p^6)$. However, the direct amplitude at the $O(p^6)$ effect and the VMD amplitude ($M_{\text{VMD}}$) are expected to cancel out each other in the Weak Deformation model[8] of ChPT.

ChPT does not provide us with clear predictions about the electric transition. The electric transition has the same parity as, and interferes with, the IB component, while the magnetic transition does not interfere with IB. If the electric component exists, an additional new component called Interference (INT) would exist[5, 6]. Theories are not predictable about the amount of the INT component, and it is important, if possible, to determine the amplitude of the INT component experimentally.

Physics beyond the standard model, such as a new CP-violating process predicted by SuperSymmetry (SUSY)[9], can be studied with the $K^+ \to \pi^+ \pi^0 \gamma$ and $K^- \to \pi^- \pi^0 \gamma$ decays through the charge asymmetry defined as

$$\frac{\Gamma(K^+ \to \pi^+ \pi^0 \gamma) - \Gamma(K^- \to \pi^- \pi^0 \gamma)}{\Gamma(K^+ \to \pi^+ \pi^0 \gamma) + \Gamma(K^- \to \pi^- \pi^0 \gamma)}, \quad (1.4)$$

which would turn out to be unequal to zero. To do this study, both of the $K^+ \to \pi^+ \pi^0 \gamma$ and $K^- \to \pi^- \pi^0 \gamma$ decays have to be measured simultaneously, which is beyond the scope of the study in this thesis.

### 1.2 Decay Width and W

The decay width $\Gamma$ of the $K^+ \to \pi^+ \pi^0 \gamma$ decay can be described, by normalizing it to the decay width of the IB component and using a kinematic variable $W$, as [10]:

$$\frac{\partial^2 \Gamma}{\partial T_+ \partial W} = \frac{\partial^2 \Gamma_{IB}}{\partial T_+ \partial W} (1 + 2 \frac{m_{\pi^+}}{m_K} Re(\frac{E}{eA})W^2 + \frac{m_{\pi^+}^4}{m_K^2} (|\frac{E}{eA}|^2 + |\frac{M}{eA}|^2)W^4), \quad (1.5)$$

where $m_{\pi^+}$ and $m_K$ are the masses of $\pi^+$ and $K^+$, $T_+$ is the kinetic energy of $\pi^+$ in the kaon rest frame, and $A$ is the decay amplitude of the $K^+ \to \pi^+ \pi^0$ decay. $E$ and $M$ are the
Figure 1.3: Angle $\theta_{\pi^+\gamma}$ between $\pi^+$ and the radiated photon in the $K^+ \rightarrow \pi^+\pi^0\gamma$ decay.

The amplitudes of the electric and magnetic transitions which compose the DE component in $K^+ \rightarrow \pi^+\pi^0\gamma$, respectively. The variable $W$, which is useful in the $K^+ \rightarrow \pi^+\pi^0\gamma$ study, is defined with the four-momenta of $K^+$, $\pi^+$ and photon ($p$, $p^+$ and $q$, respectively) as

$$W^2 = \frac{(p \cdot q)(p^+ \cdot q)}{m^2_{K^+} m^2_{\pi^+}}. \quad (1.6)$$

The amplitudes of the IB, INT and DE components are described at the first, second and third terms in Eq.1.5, respectively. The INT component and the DE component of $K^+ \rightarrow \pi^+\pi^0\gamma$ are in proportional to the second and the fourth power of $W$, respectively. The DE component can therefore be separated from the IB component in the region of larger $W$ in the $K^+ \rightarrow \pi^+\pi^0\gamma$ spectrum. In the kaon rest frame, $W$ can be written with the energy of photon($E_\gamma$), the energy and momentum of $\pi^+$ ($E_{\pi^+}$ and $P_{\pi^+}$), and the angle between $\pi^+$ and $\gamma$ ($\theta_{\pi^+\gamma}$) in Fig.1.3 as:

$$W^2 = \frac{E^2_\gamma (E_{\pi^+} - P_{\pi^+} \cos \theta_{\pi^+\gamma})}{m_K m^2_{\pi^+}}. \quad (1.7)$$

We can easily understand with Eq.1.7 that the DE component of $K^+ \rightarrow \pi^+\pi^0\gamma$ tends to have larger $E_\gamma$ and larger $\theta_{\pi^+\gamma}$, and to have larger $W$, while the IB component tends to have smaller $E_\gamma$ and smaller $\theta_{\pi^+\gamma}$, and to have smaller $W$.

The main purpose of the study in this thesis is to determine the DE component with respect to the IB component, experimentally in the $W$ spectrum in the $K^+ \rightarrow \pi^+\pi^0\gamma$ decay and compare it with the predictions of Chiral Perturbation Theory.

1.3 Previous Experiments

The branching ratio for the IB component of $K^+ \rightarrow \pi^+\pi^0\gamma$ is calculated by QED, with the branching ratio of $K^+ \rightarrow \pi^+\pi^0$, as $2.61 \times 10^{-4}$. This value is consistent with the experimental results[4]. The region of the $\pi^+$ kinetic energy $55 \text{ MeV} < T_{\pi^+} < 90 \text{ MeV}$ is adopted in order to avoid backgrounds from $K^+ \rightarrow \pi^+\pi^0$, whose $\pi^+$ kinetic energy is $108 \text{ MeV}(P_{\pi} = 205 \text{ MeV/c})$, and from $K^+ \rightarrow \pi^+\pi^0\pi^0$, whose $\pi^+$ kinetic energy is less than $53 \text{ MeV}(P_{\pi} \leq 133$
Figure 1.4: Spectra of charged particles from $K^+$ decays at rest. The IB component of $K^+ \rightarrow \pi^+ \pi^0\gamma$ is peaked at 205 MeV/c. The DE component of $K^+ \rightarrow \pi^+ \pi^0\gamma$ has a broad peak at around 140 MeV/c.

The momentum spectra on charged particles from $K^+$ decays at rest are shown in Fig.1.4.

The first observation of the DE component was reported by Abrams et al. [11] at BNL-AGS in the year of 1972. The experiment detected in-flight kaon decays and obtained 2100 events of $K^\pm \rightarrow \pi^\pm \pi^0\gamma$. They used the W spectrum and compared the measured spectrum with the IB and DE spectra from the Monte Carlo simulation, and obtained the branching ratio $BR(DE) = (1.56 \pm 0.35 \pm 0.5) \times 10^{-5}$.

Smith et al.[12] searched for the CP-violation in $K^\pm \rightarrow \pi^\pm \pi^0\gamma$ with the in-flight decays. They assumed that the difference between the measured value of the branching ratio for $K^+ \rightarrow \pi^+ \pi^0\gamma$ and the theoretically-calculated value of the branching ratio for the IB component is due to the DE component. Their result was consistent with zero.

Bolotov et al.[13] performed an in-flight decay experiment. They used both the W and $E_\gamma$ spectra for fitting, and obtained $BR(DE) = (2.05 \pm 0.46^{+0.39}_{-0.23}) \times 10^{-5}$.

BNL-E787[14] is an experiment with kaon decays at rest, and measured the DE component of $K^+ \rightarrow \pi^+ \pi^0\gamma$ from the data taken in 1995. 20k events of $K^+ \rightarrow \pi^+ \pi^0\gamma$ were obtained. Fitting of the W spectrum was performed, and the branching ratio of the DE component was $(4.7 \pm 0.8 \pm 0.3) \times 10^{-6}$[14], which is significantly smaller than those from the
Table 1.1: Previous $K^+ \to \pi^+ \pi^0 \gamma$ experiments.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Beam</th>
<th>$T_{\pi^+}$ MeV</th>
<th>number of events (including IB)</th>
<th>BR(DE) in $10^{-5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>BNL '72 [11]</td>
<td>$K^\pm$ in-flight</td>
<td>55-90</td>
<td>2100</td>
<td>1.56 ± 0.35 ± 0.5</td>
</tr>
<tr>
<td>CERN '76 [12]</td>
<td>$K^\pm$ in-flight</td>
<td>55-90</td>
<td>2461</td>
<td>2.3 ± 3.2</td>
</tr>
<tr>
<td>IHEP '87 [13]</td>
<td>$K^-$ in-flight</td>
<td>55-90</td>
<td>140</td>
<td>2.05 ± 0.46 ± 0.39</td>
</tr>
<tr>
<td>PDG '98 [18]</td>
<td></td>
<td></td>
<td></td>
<td>1.8 ± 0.4</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>BNL '00 [14]</td>
<td>$K^+$ stopped</td>
<td>55-90</td>
<td>19836</td>
<td>0.47 ± 0.08 ± 0.03</td>
</tr>
<tr>
<td>PDG '02 [15]</td>
<td></td>
<td></td>
<td></td>
<td>0.47 ± 0.09</td>
</tr>
<tr>
<td>KEK '03 [16]</td>
<td>$K^+$ stopped</td>
<td>35-90</td>
<td>4434</td>
<td>0.61 ± 0.25 ± 0.19</td>
</tr>
<tr>
<td>(KEK '03 [16])</td>
<td>(55-90)</td>
<td></td>
<td></td>
<td>(0.32 ± 0.13 ± 0.10)</td>
</tr>
<tr>
<td>PDG '04 [4]</td>
<td></td>
<td></td>
<td></td>
<td>0.44 ± 0.08</td>
</tr>
<tr>
<td>ISTRA '04 [17]</td>
<td>$K^-$ in-flight</td>
<td>55-90</td>
<td>930</td>
<td>0.37 ± 0.39 ± 0.10</td>
</tr>
</tbody>
</table>

previous three experiments. BNL-E787 also provided a constraint on the INT component with respect to the IB component, ($-0.4 \pm 1.6$) %, with the confidence-level interval of 68%.

KEK-E470 is another stopped-kaon experiment for measuring the branching ratio of the DE component [16]. They adopted the region of the $\pi^+$ kinetic energy from 35 MeV to 90 MeV. They obtained 4434 events after imposing all the offline cuts, and the major backgrounds $K^+ \to \pi^+ \pi^0 \pi^0$ was estimated to be 1.2% of the sample. The DE branching ratio was determined as $(6.1 \pm 2.5 \pm 1.9) \times 10^{-6}$. The branching ratio to the region of $\pi^+$ kinetic energy from 55 MeV to 90 MeV was $(3.2 \pm 1.3 \pm 1.0) \times 10^{-6}$, which is consistent with the result from BNL-E787.

The ISTARA group at IHEP in Russia recently reported the branching ratio with $K^-$ decays in flight [17]. The result for the DE component, $(0.37 \pm 0.39 \pm 0.10) \times 10^{-5}$, is consistent with the previous E787 result, even though the result is also consistent with zero.

A summary of the experimental results of $K^+ \to \pi^+ \pi^0 \gamma$ is in Tab.1.1 and Fig.1.5.

### 1.4 Motivation

The BNL-E787 collaboration published the first measurement of BR(DE) in the year of 2000, based on the data taken in 1995. The analysis had much larger number of $K^+ \to \pi^+ \pi^0 \gamma$ events than those in previous experiments. The Particle Data Group (PDG) accepted the BR(DE) of E787-1995 as theirs in the review in 2002 [15]. The experiments being subsequently carried out had smaller number of events than that of BNL-E787, and the results are consistent with the BNL-E787 result.

In the DE component of the $K^+ \to \pi^+ \pi^0 \gamma$ decay, if the ”reducible” amplitude is dominant in the magnetic transition, the absolute value of the amplitude of magnetic transition $|M|$
Figure 1.5: History of the BR(DE) measurement in the region of $\pi^+$ kinetic energy from 55 MeV to 90 MeV. The blue and red lines indicate the branching ratios predicted by $|M| = 1.8 \times 10^{-7}$ and $|M| = 3.2 \times 10^{-7}$, respectively.

in Eq.1.5 is calculated to be $1.8 \times 10^{-7}$. If the effect of VMD amplitude exists, $|M|$ is predicted to be $3.2 \times 10^{-7}$. The result from E787-1995 indicates $|M| = (2.1 \pm 0.2) \times 10^{-7}$, and suggests that the amplitude of the magnetic transition in BR(DE) is mainly from the reducible amplitude. The motivation of the experimental study in this thesis is to measure the amplitude of magnetic transition more precisely and to investigate its origin. E787 has taken a new data set for $K^+ \rightarrow \pi^+ \pi^0 \gamma$ in the year of 1998, which has the last physics run of E787, with modified trigger conditions so as to achieve more acceptance to the DE component and to increase the statistics. The E787-1998 data set is analyzed and BR(DE) is measured in this study.
Chapter 2

E787 experiment

The BNL-E787 experiment at the Brookhaven National Laboratory (BNL) in the United States was designed for the study of the rare decay $K^+ \rightarrow \pi^+ v \bar{v}$[20, 21], and measured kaon decays at rest. The branching ratio for $K^+ \rightarrow \pi^+ v \bar{v}$ is expected to be at the level of $10^{-10}$, while the potential background sources to $K^+ \rightarrow \pi^+ v \bar{v}$ have much-larger branching ratios. The E787 detector was designed to be excellent in rejecting the backgrounds, and allows us to measure the momentum, range and kinetic energy of charged particles from kaon decays precisely. The kinematics of a charged track can therefore be determined redundantly. Photon detectors in the E787 detector, whose primary purpose is to remove the backgrounds accompanying photons from kaon decays, cover the $4\pi$ solid angle. E787 detector allows us to detect photons as well as $\pi^+$, and to measure $E_\gamma$, $E_{\pi^+}$, $P_{\pi^+}$ and $\theta_{\pi^+ \gamma}$ for calculating the kinematic variable $W$ for the $K^+ \rightarrow \pi^+ \pi^0 \gamma$ decay (Section 1.2). E787 detector is therefore appropriate also for the study of $K^+ \rightarrow \pi^+ \pi^0 \gamma$.

2.1 Outline of the E787 Detector

The E787 detector is a cylindrical spectrometer with a 1.0-Tesla magnetic field along the beam. A side view and a schematic end view of the detector are shown in Fig.2.1 and Fig.2.2, respectively. The kaon beam enters the detectors from the left side of Fig.2.1. The beam is identified as a kaon by the Cherenkov and the B4 counters. The kaon slows down in the degrader and comes to rest in the Target located at the center of the detector. The charged particle from the kaon decay at rest in the Target passes though the I-Counter and goes into the Drift Chamber, in which the momentum of the charged track is measured. The charged particle goes into the Range Stack, deposits all the kinetic energy and comes to rest; and the kinetic energy and range of the charged track are measured in the Range Stack. Photons which pass through the Drift Chamber and Range Stack start showering in the outermost detector of E787: the Barrel photon detector, which provides us with information of position, time and energy of the photons. Other photon detectors such as the EndCaps are unused for photon reconstruction in the $K^+ \rightarrow \pi^+ \pi^0 \gamma$ study; these detectors are used to detect and reject.
events with extra particles from kaon decay as the $K^+ \rightarrow \pi^+ \pi^0\pi^0$ decay. The fiducial regions of the $\pi^+$ and the photons are restricted to be in the one-half and the two-third of the solid angle in the active region of the Range Stack and the Barrel photon detector.

Figure 2.1: Side view of the E787 detector.
2.2 Kaon Beam

The E787 detector is located at the Low Energy Separated Beam line III (LESB3 in Fig.2.3)\cite{22} of the Alternating Gradient Synchrotron (AGS) in BNL. AGS accelerates the protons up to 24 GeV and extracts the beam of $20 \times 10^{12}$ protons per spill during 1.3 seconds in every 3.4 seconds. Protons hit a production target of Platinum and produce particles such as kaon and pion. Charged particles emitted forward zero degree are transported to the E787 detector by LESB3. LESB3 is a beam line incorporating two-stage DC separators, and provides us with high purity kaons of 710 MeV/c. The kaon to pion ratio is improved to be 4 or larger. LESB3 has a total length of 19.6 m from the production target to the E787 detector. About $10^7$ kaons per spill reach the E787 detector.

2.3 Beam Counters

Kaons are detected and identified by a Cherenkov counter and two-sets of multi-wire proportional chambers, and are slowed down in a BeO degrader. BeO was chosen because of the small atomic numbers and the high density. An active degrader made of lead glass is placed downstream of the BeO and is used as a pion and photon detector. Hodoscopes of plastic scintillator, called the B4-counter, detect the beam particles after passing through the degraders and measure the energy loss in them. Since a kaon is slowed down, it leaves more energy than a pion. Approximately 25% of the kaons which were identified in the Cherenkov counter reach the kaon stopping Target.
A schematic side view of the Cherenkov counter is shown in Fig.2.4. Kaons and pions with the same momentum pass through an acrylic (lucite) radiator, and Cherenkov light is emitted toward different angles, depending on the velocity. The light from a kaon goes out of the radiator and is reflected to the outer ring of 14 photo-tubes by a parabolic mirror. The Cherenkov counter is used not only for confirming that the incoming particle is kaon, but also for providing the beam time to the delayed-coincidence requirement in the trigger (described later in Tab.2.1).

The beam instruments are also used to confirm that there is no extra incoming particle at the kaon decay time. The multi-wire proportional chambers have three sense-wire planes; the wire spacing is 2.54 mm for the upstream chamber and is 2.40 mm for the down stream one.

The B4-Counter, with a diameter of 12 cm, consists of two layers, and each layer is separated to 8 pieces in the ±45° direction.
2.4 Measurement of Charged Track

2.4.1 Target

The kaon-stopping Target has a diameter of 12 cm and is a bundle 413 plastic scintillating fibers (main fibers) of 5 mm-square in the cross section. The peripheral region of main fibers are plugged with smaller fibers (edge fibers). Each fiber is 3.1 m in length. An end view and a side view of the Target are shown in Fig.2.5 and Fig.2.6, respectively. Fig.2.6 shows the other counters (I-Counter and V-Counter described in subsequent sections) in the Target system. Scintillating light from each main fiber is detected by a photo-tube, and recorded by an ADC and a TDC in order to measure the energy deposit and time in each fiber. Photo-tube signals from the edge fibers are summed up and are also recorded by an ADC and a TDC. The fiber hits in the Target by the incoming kaon and outgoing pion are pattern-recognized and identified offline based on the energy and timing information in each fiber. The track of the stopping kaon, the decay vertex point, and the track of the $\pi^+$ from the $K^+ \rightarrow \pi^+ \pi^0 \gamma$ decay in the Target are reconstructed for further analysis. Details of the algorithm, which is one of the main software developments in this study, are described in Chapter 3.

2.4.2 I-Counter

There is a layer of scintillating counters, called the I-Counter, surrounding the Target in order to detect charged particles coming from kaon decays at rest. The I-Counter is composed of six counters, which are made of 24.2 cm long and 0.703 cm thick plastic scintillators and define the fiducial region of the Target for the kaon stopping position in the beam direction (24 cm from the surface of the Target) by requiring offline that the events should have an I-Counter hit due to the $\pi^+$ from the $K^+ \rightarrow \pi^+ \pi^0 \gamma$ decay. Each of the I-Counter is read
Figure 2.6: Side view of the target system. A charged particle from the kaon decay at rest is requested to pass through the I-Counter scintillators surrounding the Target.

out through a light guide by a photo-tube located outside the magnet. The I-Counter also provides the decay time to the delayed-coincidence requirement in the trigger.

2.4.3 V-Counter

There is another layer of scintillating counters called the V-Counter, which surrounds the Target and the light guides of the I-Counter. The V-Counter is installed in order to veto charged particles and photons produced by the kaons which did not come to rest in the fiducial region of the Target.

The reconstructed events should have no other hits in the I-Counter than the hits due to $\pi^+$ from $K^+ \to \pi^+ \pi^0 \gamma$ and have no hits in the V-Counter. The Target, I-Counter and V-Counter are also used to reject events with extra photons from kaon decay by detecting a part of electromagnetic shower in them.

2.4.4 Drift Chamber

A central Drift Chamber (Fig.2.7), which is called the Ultra Thin Chamber (UTC)[23] in the E787 experiment, measures the curvature of charged tracks in the 1.0-Tesla magnetic field to calculate the $\pi^+$ momentum from $K^+ \to \pi^+ \pi^0 \gamma$. The UTC was designed to minimize the amount of materials (and to reduce the multiple scatterings in the Drift Chamber), which in particular benefits the measurement of $\pi^+$ from $K^+ \to \pi^+ \pi^0 \gamma$.

The UTC is cylindrical with the length of 51 cm and with the inner and outer radii of 7.85 cm and 43.3 cm, respectively. The chamber was three active ”Super layers” of axial and sense wires. The super layers has four layers of drift cells in each and are filled with a 50:50 mixture of Argon and Ethane, and are separated by two inactive regions filled with Nitrogen gas. One drift-cell consists of one Tungsten sense-wire of 20 $\mu$m diameter surrounded by
eight Aluminum cathode field-wires of 100 \( \mu m \) diameter. Cathode wires are shared with the adjacent cells, and every other layer is shifted by one-half cell to allow a local resolution of left-right ambiguity. Each of the inner and outer ends of a Super layer have a 25 \( \mu m \) thick Kapton cathode foil with 1200 \( \AA \) thick, 7-mm wide Cupper Strips tilted at 45° with respect to the beam axis (\( z \) coordinate) and with a 1-mm gap between strips. The cathode-strip foils are supported by differential gas pressure between the active and inactive regions of the UTC. The folis provide the position measurements in the \( z \)-coordinate for each track (via measurement of the induced change on the strips), and the polar angle of the charged track is calculated by a straight line fitting in the plane of \( r \) (the direction of the UTC radius) and \( z \). The overall mass in the measurement region of UTC (excluding the inner and outer support tubes as well as the innermost and outermost foils) amounts to \( 2 \times 10^{-3} \) radiation lengths.

The position resolutions in the UTC were estimated to be \( \sigma_{xy} = 150 \mu m \) and \( \sigma_z = 1 \text{mm} \)[23]. In order to check the performance of the UTC, the monochromatic momentum peaks from \( K\pi2 \) and \( K\mu2 \) were reconstructed, and the resolution of the chamber was found to be 0.9%[23].
2.4.5 Range Stack

The charged track comes to the Range Stack after the Drift Chamber and the range and kinetic energy of the $\pi^+$ track which came to rest in it can be measured. The Range Stack is an array of plastic scintillation counters; the radial region between 45.1 cm and 89.6 cm for the Range Stack is segmented into 24 azimuthal sectors and 21 radial layers totaling 0.8 radiation length. Each of the counters is 1.90 cm thick and 182 cm long except for the innermost counters, which are called the T-Counters, with 0.64 cm thick and 52 cm long. The T-Counters define the solid-angle acceptance for the $\pi^+$ track in the Range Stack. Most of the particles come to rest in the Range Stack (except for the muons from $K^+ \rightarrow \mu^+\nu(K\mu)$). The Range Stack counter in the sector and layer where the $\pi^+$ track came to rest is called the "stopping counter".

The scintillation light is read out from both the upstream and downstream ends of each counter. The output pulse shapes of the photo-tubes are recorded by 500-MHz sampling transient digitizers(TDs), each of which was based on interleaved 250-MHz 8-bit flash analog-to-digital converters[24]. In addition to providing precise time and energy information for reconstructing the $\pi^+$ track, the TDs enabled us to observe the $\pi^+ \rightarrow \mu^+\nu$ decay at rest in the Range Stack stopping counter. The outputs are also recorded by ADCs. The resolution of the measurement on the range and energy in the Range Stack were estimated to be 3.88% and 3.82%, respectively[25].

For the $K^+ \rightarrow \pi^+\pi^0\gamma$ analysis, 1) ADC information is used for the measurement of $\pi^+$ kinetic energy, 2) the range measured in the Range Stack is used for $\pi^+$-track identification (described in Section 4.1.1) but is unused in the kinematic reconstruction of $K^+ \rightarrow \pi^+\pi^0\gamma$, and 3) the $\pi^+ \rightarrow \mu^+\nu$ signature in the stopping counter is unused in the $\pi^+$ identification in order not to reduce the signal acceptance.

2.5 Photon Detection

The photon detectors in E787 was primarily designed to detect photons from the $K^+ \rightarrow \pi^+\pi^0$ background in the $K^+ \rightarrow \pi^+\nu\bar{\nu}$ study. They surround the central region and cover the $4\pi$ solid angle. The Barrel part of the photon detectors is used to measure the photons in the $K^+ \rightarrow \pi^+\pi^0\gamma$ study. Other photon detectors including the EndCap calorimeters and additional photon detectors for filling minor openings along the beam direction, as well as any active parts of the detector (the Target, Range Stack, ...) not hit by the $\pi^+$ track, are used for detecting and vetoing extra particles including photons.

2.5.1 Barrel Photon Detector

The Barrel photon detector[25] covers the 2/3 of the solid angle and is located immediately outside the Range Stack. The counters of the Barrel detector are arranged in a cylindrical
way with 48 sectors in the azimuthal direction and 4 layers in the radial direction whose boundaries are tilted so that the inter-sector gaps do not project back to the Target. Each of the counters consists of alternating layers of 1 mm-thick lead and 5 mm-thick plastic scintillator sheets. The visible fraction of the energy of electromagnetic shower to the actual photon energy is about 30%. The total thickness of these 4 layers corresponds to 14.3 radiation lengths. Most of the photons from the stopped kaon to the Barrel region start showering in the Barrel photon detector (except for the case that a photon starts showering in the Range Stack; such events are rejected in the $K^+ \rightarrow \pi^+ \pi^0 \gamma$ analysis), and we can measure the energy of photons in a calorimetric method.

The inefficiency of the Barrel photon detector is about $10^{-3}$. The modules of the Barrel photon detector are 190-cm in length along the beam axis, and the scintillation light is read out from both ends of each counter. Geometrically, the fraction of scintillator light reaching photocathode varies from 30% and for the inner modules to 17% for the outer modules, and produces about 10 photoelectrons per MeV of visible energy loss in the scintillator. An ADC and a TDC measure the energy and time of the showers, respectively. The resolution of the energy measurement in the Barrel photon detector alone is $\delta E = \frac{1.5}{\sqrt{E[\text{MeV}]}}$ (Section 4.2.3).

### 2.5.2 EndCap

The two EndCap calorimeters cover the 1/3 of the solid angle along the beam direction and are used for vetoing extra particles including photons. They consist of 143 undoped-CsI crystals, which are read out by fine-mesh photo-tubes in the 1.0-Tesla field. The signals are sent into 500-MHz TDs based on charged coupled devices[26]. If an extra photon is detected in the EndCap calorimeters, the event is rejected.

### 2.6 Trigger Condition for Three Photons Events

Two stages of trigger, which are called the Level 0 and Level 1.1 in E787, work in a custom-built system of ECL logic boards. Level 0 is entirely composed of the combination of fast logic signals, and introduces about 40 ns of dead-time per trigger. Level 1.1 is for the online $\pi^+$ identification with the pulse shape analysis in the $\pi^+$ stopping counter in the Range Stack, which is unused for the $K^+ \rightarrow \pi^+ \pi^0 \gamma$ study.

Particles from $K^+ \rightarrow \pi^+ \pi^0 \gamma$ are the $\pi^+$, $\pi^0$ and photon, and the $\pi^0$ immediately decays to two photons. The events with one $\pi^+$ and three photons in the detector should be collected by the $K^+ \rightarrow \pi^+ \pi^0 \gamma$ trigger. It is requested in the trigger that three photons are detected in the Barrel photon detector, and $\pi^+$ comes to rest in the Range Stack. The trigger condition is defined by the symbols in Eq.2.1.

$$KB \cdot IC \cdot DC \cdot T \cdot 2 \cdot (3_{CT} + 4_{CT}) \cdot (7_{CT} + 8_{CT}) \cdot (9 + \cdots + 21) \cdot EC \cdot HEX \cdot NG3 \quad (2.1)$$
Figure 2.8: Three photons recognized by the on-line trigger(NG3). The two segments in the center are recognized as one photon cluster; the segments in the left and in the right is recognized as one photon cluster for each.

The meaning of each symbol is explained in Tab.2.1. $KB\cdot IC\cdot DC$ confirms that a kaon comes to rest in the Target. $T\cdot 2\cdot (3_{CT} + 4_{CT})$ confirms that a pion penetrates the 4th layer in the Range Stack and $(7_{CT} + 8_{CT})\cdot (9 + \cdots + 21)$ requires that a pion comes to rest in the layers up to the 6th. NG3 requires that there are three photons in the Barrel photon detector. $EC\cdot HEX$ requires that there is no extra photon in the EndCaps and in the Range Stack.

The algorithm of NG3: the on-line shower clustering is as follows. The output signals from the modules of two sectors (in the azimuthal direction) and four layers (in the radial direction) in the Barrel photon detector defined as a "segment" in each end are fed into a discriminator. If the signal corresponds to the shower energy more than 15 MeV, the on-line trigger recognizes that there is a shower in the segment. If there are no hits in the adjacent segments, it is regarded as an isolated cluster. An example of the on-line clustering is illustrated in Fig.2.8; the on-line trigger recognizes that there are three photon clusters.

For comparison, the trigger condition used in E787 1995 for the study of $K^+ \to \pi^+ \pi^0 \gamma$ is shown in Eq.2.2. The difference from the trigger in 1998 is a requirement of $\pi^+$ that comes to rest in the Range-Stack layers from the 6th to the 10th.

$$KB\cdot IC\cdot DC\cdot T\cdot 2\cdot (6_{CT} + 7_{CT})\cdot (11 + \cdots + 20)\cdot (19_{CT} + 20_{CT} + 21_{CT})\cdot EC\cdot HEX\cdot NG3$$

(2.2)

The new requirement in 1998 that $\pi^+$ comes to rest at the inner layers than in 1995
Table 2.1: Meaning of symbol in trigger equation.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>KB</td>
<td>A coincidence of signals from the kaon Cherenkov counter, spill gate from the AGS, the B4 counter and the sum of Target energy. KB requests that $K^+$ comes into the Target.</td>
</tr>
<tr>
<td>IC</td>
<td>The I-Counter has a signal.</td>
</tr>
<tr>
<td>DC</td>
<td>The timing of the I-Counter hit is at least 1.5 ns latter than the timing of the Cherenkov hit (delayed coincidence).</td>
</tr>
<tr>
<td>$T\cdot2$</td>
<td>A coincidence of the signals from the first two layers of Range Stack in the same sector.</td>
</tr>
<tr>
<td>$3_{CT} + 4_{CT}$</td>
<td>Signals from the 3rd or 4th layer of the Range Stack. CT means that these counters are in the same sector or two clockwise sectors of $T\cdot2$.</td>
</tr>
<tr>
<td>$(7_{CT} + 8_{CT})$</td>
<td>No signal in the 7th nor 8th layer in the Range stack.</td>
</tr>
<tr>
<td>$(9 + \cdots 21)$</td>
<td>No signal in any sector from the 9th to 21st layers in the Range Stack.</td>
</tr>
<tr>
<td>EC</td>
<td>No signal in the EndCap.</td>
</tr>
<tr>
<td>NG3</td>
<td>There are three or more photon clusters in the Barrel photon detector.</td>
</tr>
</tbody>
</table>

provides us with the acceptance in the lower region of $\pi^+$ energy. The number of events from the IB component increases as the kinetic energy gets larger, while the portion of events from the DE component gets larger for the lower region. We can therefore improve the ratio of DE to IB by requiring the lower energy region.

The trigger for $K^+ \rightarrow \pi^+\pi^0\gamma$ in 1998 was prescaled by 5 in order not to disturb the collection of the E787 main triggers for $K^+ \rightarrow \pi^+\nu\bar{\nu}$. The number of $K^+ \rightarrow \pi^+\pi^0\gamma$ events recorded to data tapes is 5 to 7 per beam spill.

BNL-E787 took data from September to December in the year of 1998 with $1.74 \times 10^{12}$ kaons entering into the Target.

2.7 Data Acquisition

The logic of the trigger system initiates the signal digitization in the data acquisition system. The data is stored at the memory of SSP (SLAC Scanning Processing board) in each Fastbus crate.

Between the beam spills, data are transferred from each crate over the Fast bus cable segments to the master SSP, and events are built by the SGI-Challenge computer and are recorded to magnetic tapes.

2.8 Monte Carlo Simulation in E787

The Monte Carlo simulation package in E787, called UMC, is based on the EGS4 electromagnetic shower simulation[27]. E787 collaborators have developed and installed the
algorithm for charged track transportation with $dE/dX$ losses in materials with the Bethe-Bloch formula and the $\pi^+$ nuclear interactions. The format of the UMC outputs is the same as the real data except that no pulse-form information is simulated. We can analyze both of the UMC and real data with the same analysis code, which provides us with a consistent analysis between the real and UMC data.

In the $K^+ \rightarrow \pi^+ \pi^0 \gamma$ analysis, the branching ratio is calculated by normalizing the distributions to those expected from the Monte Carlo simulation. The IB, DE and INT components are generated for spectrum analysis. Matrix elements for $K^+ \rightarrow \pi^+ \pi^0 \gamma$ are taken from the DAΦNE physics handbook[10], in which the theoretical expect on ChPT provides comprehensive reviews in radiative kaon decays as well as the Fortran codes for the matrix elements. Backgrounds from kaon decays are also generated for understanding their behaviors; however, the background estimations are made based on the real data.

Steps of UMC start at the decay point of kaons at rest. The kaon stopping positions are obtained by analyzing the $K\mu2$ events derived from real data. The energy deposit of a charged track is calculated at each step in the detector, and is converted to the output of scintillating light using the Birk’s formula[28].
Chapter 3

Event Reconstruction

There are three stages for the $K^+ \rightarrow \pi^+\pi^0\gamma$ analysis. At first, we need to reconstruct each event and calculate variables for further analysis. The second stage is to remove backgrounds and select a sample of $K^+ \rightarrow \pi^+\pi^0\gamma$ events with selection criteria (cuts). The third stage is to separate the measured spectrum into the IB, DE and INT components. How to reconstruct each event is explained in this chapter; background rejection is described in chapter 4, and the spectrum analysis is given in chapter 5 (Fig. 3.1).

The kinetic energy and momentum of $\pi^+$ and the energies and positions of photons in $K^+ \rightarrow \pi^+\pi^0\gamma$ are measured in the E787 detector. The energy of $\pi^+$ is measured by the Target, I-Counter and Range Stack. The momentum of $\pi^+$ is measured mainly by the UTC. The energy and position of each photon are measured by the Barrel photon detector.
Table 3.1: Variables for UTC.

<table>
<thead>
<tr>
<th>variable</th>
<th>definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{\pi^+}^{\text{UTC}}$</td>
<td>Momentum in UTC.</td>
</tr>
<tr>
<td>$\theta_{\pi^+}$</td>
<td>Dip angle measured by UTC.</td>
</tr>
<tr>
<td>$\phi_{\pi^+}$</td>
<td>Azimuthal angle measured by UTC.</td>
</tr>
<tr>
<td>$\varepsilon_{\text{DC}}$</td>
<td>Calculated dE/dX loss based on $P_{\pi^+}^{\text{UTC}}$.</td>
</tr>
<tr>
<td>$R_{\text{gas}}$</td>
<td>Amount of the material in the gas, equivalent to the range in scintillator.</td>
</tr>
<tr>
<td>$R_{\text{wall}}$</td>
<td>Amount of the material in inner wall, equivalent to the range in scintillator.</td>
</tr>
</tbody>
</table>

3.1 Reconstruction of Charged Track in the UTC

The momentum and the angle with respect to the beam axis (dip-angle) of charged tracks are measured by the UTC. The first step is a track finding in the $x-y$ plane. Hits on neighboring wires in each super-layer are combined into "vectors", which give a rough estimate of the track direction in each super-layer. These vectors are then linked to form a track that passed through all the three super-layers. Using the track found in this method, a $\chi^2$ minimization procedure is applied to find the final $x-y$ track parameters. In the next step for $z$-fit, the $x-y$ track parameters are used to convert the struck foil-strips into the $z$ positions. The $z$ positions are then fit to a straight line as a function of the turning angle of the track. From this fit, the dip-angle of the charged track ($\theta_{\pi^+}$) is calculated. The momentum of the charged track at the UTC ($P_{\pi^+}^{\text{UTC}}$) is determined from the $x-y$ components and $z$ components.

The momentum measured by the UTC is, exactly speaking, the momentum of the charged track at the midpoint of the UTC. The range in the UTC from the entering point to the midpoint is corrected for. Although the UTC has little material, charged particles deposit some energy in the wall and gas of the UTC. The energy deposit is estimated, assuming that the charged particle is $\pi^+$. The calculated energy deposits are converted to the range in a scintillator; the variables for the inner wall and gas are $R_{\text{wall}}$ and $R_{\text{gas}}$, respectively. The $dE/dX$ loss at the UTC ($\varepsilon_{\text{DC}}$) is calculated based on the Bethe-Bloch formula. The variables on the UTC are summarized in Tab.3.1.

3.2 Reconstruction of Charged Track in the Target

The decay $K^+ \rightarrow \pi^+ \pi^0 \gamma$ has a $\pi^+$ with an energy lower than $\pi^+$ in the $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ study, which results in two issues in the measurement: (1) the charged track deposits a larger energy per unit length, and (2) the relative error in the measurement of $\pi^+$ kinetic energy becomes larger. A small energy deposit therefore cannot be neglected in the $K^+ \rightarrow \pi^+ \pi^0 \gamma$ reconstruction. About 30% of the kinetic energy of $\pi^+$ is lost (and should be measured) in the Target; the Target plays an important role on the measurements in the $K^+ \rightarrow \pi^+ \pi^0 \gamma$ analysis.

The standard E787 reconstruction-method, developed for the $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ analysis, on the Target is described at first, and new developments for the $K^+ \rightarrow \pi^+ \pi^0 \gamma$ analysis are
explained in the next.

### 3.2.1 Standard Reconstruction in the Target

This section is a review of E787’s reconstruction of events in the Target.

The Target is composed of 412 fibers with a cross section (cell) of 5 mm square. There are other fibers called "edge fibers" in the Target. The edge fibers are located at the peripheral region of the Target. Edge fibers have not been used so far in the E787 standard reconstruction (except for the purpose of photon veto in the Target).

The first step of the Target reconstruction is an identification of charged tracks. The UTC track is extrapolated, assuming that the track in the \(x-y\) view is an arc in the Target. The cells located within the path of \(\pm 1\) cm along the track, called SWATH, are thought to be due to the charged tracks from kaon decay. The cells are then categorized as "kaon", "charged track" or "photon". If the energy deposit in a cell is more than 2 MeV and the cell is connected to SWATH, the cell is identified as a kaon cell. The cells which connected to SWATH and whose energy is less than 2 MeV are identified as pion cells. The cells which are isolated from SWATH is identified as photon cells. A schematic view of the clustering procedure with SWATH is shown in Fig.3.2. The energy deposit of charged track \(E_{\text{target}}^{\text{stn}}\) is defined as the sum of energy in the pion cells.

![Figure 3.2: Cells in the Target. The dashed line represents the SWATH from the UTC track extrapolation. The cells are categorized as kaon(blue), pion(red) or photon(yellow).](image)

The kaon stopping position, which is also the kaon decay point, is defined as following. SWATH is an extrapolation of the UTC track and the shape is an arc. The extreme clockwise and counter-clockwise kaon cells within SWATH are picked up, and the average of center positions of the two extreme cells is calculated. The point which is the closest to the averaged point on the track is defined to be the kaon stopping position. The schematic view of the determination is shown in Fig.3.3.
Figure 3.3: Determination of the stopping position in the Target. The dashed line is the SWATH. The small dots are the center positions of the two extreme cells in the kaon cells in SWATH. The large dots is the stopping position of kaon.

The path length of the track between the kaon decay point to the point where charged track exits from the Target is defined as the range in the Target \( R_{\text{target}} \). A summary on the Target variables is in Tab.3.2.

### 3.2.2 Monte Carlo Studies

For developing more precise Target reconstructions than in E787, the Monte Carlo \( K^+ \rightarrow \pi^+ \pi^0 \gamma \) events were generated by UMC to obtain the following information in each event.

1. "unsaturated" energy deposit by \( \pi^+ \) within the kaon cells \( (E_{\text{hid}}^{MC}) \),
2. "unsaturated" energy deposit by \( \pi^+ \) in the edge fibers \( (E_{\text{edge}}^{MC}) \),
3. "saturated" energy deposit by \( \pi^+ \) in the Target, except for the deposits in kaon cells and in the edge fibers \( (E_{\text{sat}}^{MC}) \),
4. "unsaturated" energy deposit by \( \pi^+ \) in the Target, except for the deposits in kaon cells and in the edge fibers \( (E_{\text{unsat}}^{MC}) \),

where the word "saturated" means that the energy is converted into the amount of scintillating light based on the Birk’s formula\[28\], and the word "unsaturated" means that the energy is just the energy deposit to the material.
Energy Corrections

**hidden energy** If a charged track deposits some energy within the kaon hit cells, the energy is hidden and not included to $E_{\text{target}}^{\text{stan}}$. The following three variables, which are obtainable from the event reconstruction, are turned out to be useful to estimate the hidden energy $E_{\text{hid}}$.

The first is an azimuthal angle between the cells of the kaon and the charged track in the Target ($\phi_{K\pi}$), as shown in Fig.3.4. If $\phi_{K\pi}$ is around 180 degrees, the charged track could be fully contained in the kaon hit cells. If the value of $\phi_{K\pi}$ is $\pm$90 degree, the charged track would not be contained to the kaon cells except for the cell including the vertex of the kaon decay.

![Figure 3.4: Definition of $\phi_{K\pi}$: the angle $\phi$ between pion cells (red) and kaon cells (blue) in the x-y view.](image)

The second is the number of kaon cells located within the SWATH ($N_{K\text{cell}}$). If the direction of the kaon to the Target is parallel in the beam direction, the kaon should hit only a few cells in the Target. If $N_{K\text{cell}}$ is small, the hidden energy is expected to be small.

The relation of $\phi_{K\pi}$ and $E_{\text{hid}}^{MC}$ (the hidden energy, available in the Monte Carlo events) for each value of $N_{K\text{cell}}$ is shown in Fig.3.5. In the plots, the events whose charged track propagates vertically to the beam axis is selected. The relation between the estimated hidden energy ($\varepsilon_{\text{hid}}$) and $\phi_{K\pi}$ for each $N_{K\text{cell}}$ is obtained from these plots.

The third is a dip angle of a charged track. If the charged track is nearly parallel to the beam axis (and the Target fibers), the charged track would deposit larger energy. The dip angle $\theta_{\pi^\pm}$, which has already been defined in Section 3.1, is used to estimate the hidden energy $E_{\text{hid}}$ from the $\varepsilon_{\text{hid}}$ in above as follows:

$$E_{\text{hid}} = \varepsilon_{\text{hid}} / \sqrt{1 - (\cos \theta_{\pi^\pm})^2}. \quad (3.1)$$

The estimated hidden energy ($E_{\text{hid}}$) is, in the end, added to the measured energy of the charged track.
Figure 3.5: Relations of the hidden energy ($E_{hid}^{MC}$) and the angle between the kaon and charged track ($\phi_{K\pi}$) for the number of kaon cells within the SWATH ($N_{K_{cell}}$). The solid curve in each plot is derived from the fitting of $E_{hid}^{MC}$ as a function of $\phi_{K\pi}$, and is regarded as the hidden energy $E_{hid}$ as a function of $\phi_{K\pi}$ for each $N_{K_{cell}}$. $E_{hid}$ is further corrected for with a dip angle of $\pi^+$ to obtain the estimated hidden energy $E_{hid}$. 
**edge fibers**  The energy deposit in edge fibers should be estimated. The energy deposit should depend on the path length of the track in the edge fibers. The distance between the outermost main fiber and the exiting point of the track from the Target ($R_{edge}$) is used. The definition of $R_{edge}$ is shown in Fig.3.6(left), and the relation between $R_{edge}$ and the energy deposit in the edge fibers available in the Monte Carlo events ($E_{MC}^{edge}$) is shown in Fig.3.6(right). The energy deposit in the edge fibers ($E_{edge}$) is estimated as a function of $R_{edge}$. The $E_{edge}$ is added to the measured energy of the charged track.

![Figure 3.6: Definition of $R_{edge}$: range in the edge fibers. $R_{edge}$ is the length of the red line in the left figure. The energy deposit in the edge fibers ($E_{MC}^{edge}$) versus the range is shown in the right figure. The solid curve is the relation between the estimated energy deposit in the edge fibers ($E_{edge}$) and $R_{edge}$ from the fitting of $E_{MC}^{edge}$ as a function of $R_{edge}$.](image)

**saturation effect**  The amount of scintillating-light output from a fiber is not exactly proportional to the energy deposit in the fiber. E787’s standard Target reconstruction assumes that the amount of light is proportional to the deposit energy, although UMC has simulated the saturation effect. We need to correct for this mismatch to reconstruct the correct Target energy in the UMC events exclusively. The relation between the measured energy ($E_{sat}$) and the mismatch due to the saturation effect ($E_{MC}^{sat}/A_0E_{MC}^{unsat}$) is shown in Fig.3.7; the relation is fitted as a function of the measured energy in Fig.3.7. For UMC events, the value of the energy correction at the measured target energy ($E_{sat}$) is added to the measured energy in the Target.

**energy deposit in the dead material**  The energy deposit in the dead material in the Target ($E_{dead}$) should be estimated. The energy deposit depends on the range in the Target ($R_{target}$) and the dip angle of $\pi^+ (\theta_{\pi^+})$. The deposit as a function of $R_{target}$ and $\theta_{\pi^+}$ is obtained by measuring the monochromatic $\pi^+$ energy from the $K\pi2$ decay. Since the UMC does not simulate the dead material in the Target, this correction ($E_{dead}$) is made only for real data.
Figure 3.7: Mismatch due to the saturation effect in the Monte Carlo events \((E^{MC}_{sat} - E^{MC}_{unsat})\) as a function of the saturated energy \(E_{sat}\) in the Target. The red line is the relation between the energy correction for the UMC events and \(E_{sat}\) from the fitting of \((E^{MC}_{sat} - E^{MC}_{unsat})\) as a function of \(E_{sat}\).

UTC track extrapolation

E787’s standard routine extrapolates the UTC track to the Target, assuming that the track in the Target is an arc with a constant curvature in the x-y view. However, the charged particle deposits energy in the Target in each step and the curvature changes accordingly. We need corrections for the changes of the arc with respect to the \(\pi^+\) energy, in the Target analysis, to take care of lower-momentum charged tracks correctly.

We have calculated the momentum of the charged track in the UTC. The point where the charged track passed through in the I-Counter is calculated by assuming that the dE/dX is constant. By adding the energy deposit in the I-Counter, the energy of the charged track at the outermost point in the Target is obtained from the UTC momentum. If the dE/dX is assumed to be constant for each small step in the target scintillator, the energy deposit in each step of the track propagation can be estimated by the Bethe-Bloch formula. This step is set to 0.01 cm in the new analysis code. The step-by-step calculation continues up to close to the measured stopping position of kaon. If the calculated point is closest to the measured kaon decay point, the propagation is stopped.

This method provides more precise reconstruction of \(\pi^+\) azimuthal angle at the kaon decay point. The correlation between the range in the Target and the error of the \(\pi^+\) azimuthal angle with UMC data is shown in Fig.3.8; the new reconstruction provides us with better resolutions in the azimuthal angle.

The Target energy \((E^{cal}_{target})\) in Tab.3.2, expected from the UTC momentum and the range in the Target, is calculated to check that the charged track is consistent with \(\pi^+\); \(E^{cal}_{target}\) is used for an event selection and not for a correction to the target energy. If there is a discrepancy between the calculated energy and the measured energy, it means that the charged track is not \(\pi^+\) \((E^{cal}_{target} \neq E_{target})\).
Figure 3.8: Plots on the error of the $\pi^+$ azimuthal angle versus the range in the Target. The left figure is derived from the E787 standard algorithm; the right figure is derived from the new analysis code with the target tracking based on the Bethe-Bloch formula.

<table>
<thead>
<tr>
<th>variable</th>
<th>definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{\text{stn target}}$</td>
<td>Energy measured in the Target by the E787 standard reconstruction. (Sec 3.2.1)</td>
</tr>
<tr>
<td>$E_{\text{target}}$</td>
<td>Energy measured in the Target by a new reconstruction. (Sec 3.2.2)</td>
</tr>
<tr>
<td>$R_{\text{target}}$</td>
<td>Range measured in the Target.</td>
</tr>
<tr>
<td>$X_{\text{stop}}$</td>
<td>$x$ coordinate of the $K^+$ stopping position.</td>
</tr>
<tr>
<td>$Y_{\text{stop}}$</td>
<td>$y$ coordinate of the $K^+$ stopping position.</td>
</tr>
<tr>
<td>$Z_{\text{stop}}$</td>
<td>$z$ coordinate of the $K^+$ stopping position.</td>
</tr>
<tr>
<td>$E_{\text{cal target}}$</td>
<td>Expected energy in the Target from the UTC momentum and the Target range, assuming that the charged track is $\pi^+$.</td>
</tr>
</tbody>
</table>

3.2.3 Summary of New Corrections

Five variables after the new corrections to the Target reconstruction are listed in Tab.3.3. We succeed in obtaining more precise energy deposit in the Target:

For UMC data,

$$E_{\text{target}} = E_{\text{stn target}} + E_{\text{hid}} + E_{\text{edge}} + E_{\text{sat}}$$

and for real data,

$$E_{\text{target}} = E_{\text{stn target}} + E_{\text{hid}} + E_{\text{edge}} + E_{\text{dead}}.$$  \hspace{1cm} (3.3)

The UMC data is used for checking the improvements in the reconstruction. The distributions of the difference between the true and measured values of energy deposit in the Target is shown in Fig.3.9. The peak of the distribution after the correction is located at zero; the resolution is also improved.
Figure 3.9: Distribution of the difference between the true and measured energy deposits in the Target. The unhatched histogram and the hatched histogram are the distributions before and after the corrections, respectively.

<table>
<thead>
<tr>
<th>variable</th>
<th>definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{hid}$</td>
<td>Estimated hidden energy.</td>
</tr>
<tr>
<td>$E_{edge}$</td>
<td>Estimated energy deposit in the edge fibers.</td>
</tr>
<tr>
<td>$E_{sat}$</td>
<td>Estimated saturation effect, only for UMC data.</td>
</tr>
<tr>
<td>$E_{dead}$</td>
<td>Estimated energy deposit in the dead material in the Target, only for real data</td>
</tr>
<tr>
<td>$E_{target}$</td>
<td>Energy deposit in the Target after including all the corrections.</td>
</tr>
</tbody>
</table>

### 3.3 I-Counter

The energy deposits in the I-Counter are measured by ADCs. If the hits in adjacent counters are found, the energies are summed up. The energy deposit in the I-Counter ($E_{IC}$) is used for the $\pi^+$ energy reconstruction. The path length of the extrapolated UTC track in the I-Counter is regarded as the range in the I-Counter ($R_{IC}$). A summary of the I-Counter variables is in Tab.3.4.

<table>
<thead>
<tr>
<th>variable</th>
<th>definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{IC}$</td>
<td>Measured energy in the I-Counter.</td>
</tr>
<tr>
<td>$R_{IC}$</td>
<td>Measured range in the I-Counter.</td>
</tr>
</tbody>
</table>
Table 3.5: Variables for Range Stack

<table>
<thead>
<tr>
<th>variable</th>
<th>definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{RS}$</td>
<td>Energy measured in the Range Stack.</td>
</tr>
<tr>
<td>$R_{RS}$</td>
<td>Range measured in the Range Stack.</td>
</tr>
<tr>
<td>$t_{RS}$</td>
<td>Track time measured in the Range Stack.</td>
</tr>
</tbody>
</table>

3.4 Range Stack

We need to find the Range-Stack counters hit by a charged track. The counters whose timing is the closest in time to the on-line $T \cdot 2$ timing (detector strobe) are selected. The pattern-recognition program finds the outer layer in the same sector at first. If a hit is not found, it looks for the counter in the clockwise adjacent sector in the same layer. If a hit is still not found, it will find the counter in the outer layer and in the adjacent sector. The above procedure continues until it cannot find an appropriate counter any more. The schematic view of the order of finding the next Range Stack counter is illustrated in Fig.3.10.

![Figure 3.10: The order of finding the next step in the Range Stack track finding, which is started from the left bottom counters.](image)

The last counter in the Range-Stack track is called the stopping counter. We need to take special care of the stopping counter for summing up energies because additional energy due to the muon from the $\pi^+ \rightarrow \mu^+ \nu$ decay at rest is expected in the counter and is not negligible; the muon energy should be subtracted from the energy in the stopping counter. The range in the stopping counter is calculated with the Bethe-Bloch formula, assuming that the charged track is $\pi^+$. The range measured in the Range Stack is stored as $R_{RS}$. The average of the timings in the counters with the weight of the energy in each counter is stored as the track time $t_{RS}$. The variables on the Range Stack are summarized in Tab.3.5.
3.5 Barrel Photon Detector

3.5.1 Visible Fraction

The Barrel photon detector is a sandwich calorimeter; in order to calculate the actual photon energy, we must obtain and use a correction factor named as a visible fraction. The visible fraction is obtained from the two photons from the $K^+ \rightarrow \pi^+ \pi^0$ decay in a special trigger for calibration. The $\pi^+$ from $K\pi\pi$ has monochromatic momentum and energy. If appropriate cuts are imposed on the kinematics of the charged track, pure $K\pi\pi$ sample can be selected. If the events are identified as $K\pi\pi$, two photon clusters in the Barrel photon detector are reconstructed and are assumed to be from $\pi^0$ in $K\pi\pi$. Sum of the energies of two photons from $\pi^0$ should also be monochromatic(245 MeV). Comparing the observed energy and the energy expected to $\pi^0$ from $K\pi\pi$, the visible fraction in the Barrel photon detector is obtained to be 0.29.

In order to check the consistency between the real and UMC data, plots of the reconstructed $\pi^0$ mass from $K^+ \rightarrow \pi^+ \pi^0 \gamma$ is shown at Fig.3.11. There are three combination to select two photons from $\pi^0$; in this plot, the combination which gives the invariant mass closest to the $\pi^0$ mass is selected. The real data is well reproduced by the UMC.

![Figure 3.11: Reconstructed $\pi^0$ mass from $K^+ \rightarrow \pi^+ \pi^0 \gamma$: real data(dots) and the IB component in UMC(dashed histogram).](image)

3.5.2 $z$ Position of Photons

The scintillating light is recorded in both sides of each Barrel counter by photo-tubes. We can measure the $z$ position with the measured energies by ADCs in both ends ($A_1$ and $A_2$, respectively), the attenuation length in each counter($\lambda$), and an offset($f$) as

$$z_{adc} = \frac{\lambda}{2} \log \frac{A_2}{A_1} + f.$$  (3.4)
We can also measure the \( z \) position with the timings recorded in both ends (\( T_1 \) and \( T_2 \), respectively), a calibration constant (\( c \)) which is equivalent to the speed of light in the counter, and an offset (\( d \)) as:

\[
z_{tdc} = \frac{c}{2}(T_1 - T_2) + d
\]  

(3.5)

The \( z \) measurement in a single Barrel counter uses both ADC-based and TDC-based measurements with a proper weight. The best \( z \) measurement is obtained by combining the two measurements as [29]

\[
z_{com} = \frac{z_{tdc} + z_{adc}}{1 + \sqrt{E/10}}
\]  

(3.6)

where \( E \) is the counter energy in MeV.

### 3.5.3 Shower Clustering

The electromagnetic shower in the Barrel photon detector is distributed in more than one Barrel module. If hits are found in a module and an adjacent module, these two modules are regarded to be in the same shower. An example is illustrated in Fig.3.12; these hits are recognized as two clusters.

![Figure 3.12: Shower clustering of the hits in the Barrel photon detector. These hits are recognized as two clusters.](image)

The number of clusters (\( N_\gamma \)) is used for further analysis. The energy of each module consisting in a cluster is summed up and considered to be the photon energy (\( E_{\gamma(i)}, i = 1, 2, \ldots, N_\gamma \)). The position of each cluster is obtained by averaging the center position of each module with the weight of the energy. The point of each cluster in x-y view is stored as \( X_{\gamma(i)}, Y_{\gamma(i)}, Z_{\gamma(i)} \) (i=1,2,\ldots,N_\gamma). The five variables on the Barrel photon detector are summarized in Tab.3.6. The clusters are arranged so that \( E_{\gamma(1)} > E_{\gamma(2)} > \cdots \).
Table 3.6: Variables for Barrel photon detector.

<table>
<thead>
<tr>
<th>variable</th>
<th>definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_\gamma$</td>
<td>Number of photon clusters.</td>
</tr>
<tr>
<td>$X_{\gamma(i)}$</td>
<td>$x$ position of the $i$-th energetic cluster.</td>
</tr>
<tr>
<td>$Y_{\gamma(i)}$</td>
<td>$y$ position of the $i$-th energetic cluster.</td>
</tr>
<tr>
<td>$Z_{\gamma(i)}$</td>
<td>$z$ position of the $i$-th energetic cluster.</td>
</tr>
<tr>
<td>$E_{\gamma(i)}$</td>
<td>Energy of the $i$-th energetic cluster.</td>
</tr>
</tbody>
</table>

Table 3.7: Reconstructed variables and the detectors used for reconstruction.

<table>
<thead>
<tr>
<th>variable</th>
<th>detectors for reconstruction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Charged track energy ($T_{\pi^+}^{\text{recon}}$)</td>
<td>Target, I-Counter, UTC, Range Stack</td>
</tr>
<tr>
<td>Charged track momentum ($P_{\pi^+}^{\text{recon}}$)</td>
<td>Target, I-Counter, UTC</td>
</tr>
<tr>
<td>Charged track azimuthal angle ($\phi_{\pi^+}$)</td>
<td>Target, I-Counter, UTC</td>
</tr>
<tr>
<td>Charged track dip angle ($\theta_{\pi^+}$)</td>
<td>Target, UTC</td>
</tr>
<tr>
<td>Photon energy ($E_{\gamma(i)}^{\text{recon}}$)</td>
<td>Barrel photon detector</td>
</tr>
<tr>
<td>Photon momentum ($P_{\gamma(i)}^{\text{recon}}$)</td>
<td>Target, Barrel photon detector</td>
</tr>
</tbody>
</table>

3.6 Kinematic Values of Charged Track

We obtained the azimuthal angle and the dip angle of a charged track using the UTC information. There are two kinematic variables for charged track: momentum and kinetic energy. This redundant measurement allows us to identify the charged track as $\pi^+$. We can obtain the total kinetic energy of the charged track $T_{\pi^+}^{\text{recon}}$ by the following way.

$$T_{\pi^+}^{\text{recon}} = E_{RS} + E_{\text{target}} + E_{IC} + \epsilon_{DC} \times (R_{\text{wall}} + 0.5 \times R_{\text{GAS}})$$ (3.7)

We need to calculate the total momentum of the charged track ($P_{\pi^+}^{\text{recon}}$). The charged track loses energy in the Target and I-Counter before entering the UTC. The path length of the charged track in the Target and the I-Counter is calculated. The path length to the midpoint in the UTC is evaluated to the equivalent length in a scintillator. The path length in the three detectors are summed up. In the next, $P_{\pi^+}^{\text{UTC}}$ is converted to the range in a scintillator. We can add these ranges, and the total "range" is re-converted to the total momentum at the decay vertex, assuming that the charged particle is $\pi^+$. The momentum of the charged track is stored as $P_{\pi^+}^{\text{recon}}$. The kinematic variables are summarized at Tab.3.7.
Chapter 4

Background Rejection

In the earlier part of this chapter, the selection criteria (cuts) to reject backgrounds against the $K^+ \rightarrow \pi^+ \pi^0 \gamma$ signal are described. The definitions of the cuts are summarized in the middle of this chapter (Tab. 4.3). In the E787 detector, we can measure all the kinematic variables of the decay products from the stopped kaon decay: one $\pi^+$ and three photons. Imposing the kinematic constraints is a key to reject backgrounds, and the most important cuts in this analysis are based on the constraints with the procedure called the 'kinematic fit'. The confidence level on the agreement of kinematic variables to the kinematic constraints provides us with a good separation between the signal and the backgrounds. The kinematic fit is also valuable in improving the accuracy of each kinematic variable.

The way to estimate the background level of each source is discussed in the later part of this chapter. A method of background estimation with real data is employed. The background level is found out to be negligible in this analysis.

4.1 Cuts for Background Rejection

4.1.1 Consistency of the Charged Track Between the UTC and the Range Stack

The consistency of the charged track between the UTC and the Range Stack is checked (Cut 1). The difference between the expected range, assuming that the charged track is a $\pi^+$, from the momentum measured by the UTC ($R_{\text{expected}}$) and the measured range in the Range Stack ($R_{\text{measured}}$) is used. The difference $R_{\text{diff}} \equiv R_{\text{expected}} - R_{\text{measured}}$ should be zero for $\pi^+$; if the charged track is not a $\pi^+$, there should be a difference between the expected range and the measured range. The distributions of $R_{\text{diff}}$ in the real data and in the UMC events are shown in Fig. 4.1. There are two peaks in the real data: one peak is at $R_{\text{diff}} = 0.0$ from $\pi^+$ and another peak is at $R_{\text{diff}} = 3.0$ from $\mu^+$. $\pi^+$ tracks are selected by requiring that $R_{\text{diff}}$ is close to zero ($-2.2 < R_{\text{diff}} < 1.6$) for real data.
4.1.2 Consistency of the Charged Track in the Target

The cut on $R_{diff}$ is not powerful to remove the events involving the $e^+$ track; however, $\pi^+$ can be separated from $e^+$ in the Target. The difference between the expected energy deposit in the Target from the momentum by the UTC ($E_{\text{cal}}^{\text{target}}$) and the energy measured in the Target ($E_{\text{target}}$) is used for identification of the charged track (Cut 2). The plots of the difference $E_{\text{diff}} \equiv E_{\text{cal}}^{\text{target}} - E_{\text{target}}$ after the $\pi^+$ selection in the Range Stack by Cut 1 are shown in Fig. 4.2. There is a tail from positrons in the real data, and the events with $|E_{\text{diff}}| \geq 6$ MeV are removed by Cut 2.

4.1.3 Photon to the Direction of $\pi^+$

These two cuts (Cut 1 and Cut 2) still do not have enough rejection to the events involving the $e^+$ track. A new cut is introduced in order to improve the rejection, by considering the angle between the charged track and each photon in the final state. Since a positron tends to
radiate a photon in the forward direction and mimic a photon in the $K^+ \rightarrow \pi^+ \pi^0 \gamma$ decay, a cut (Cut 3) is imposed to remove the events with a small angle ($\theta_{\pi^+ \gamma}$) between the charged track and one of the three photons. There are two detector-subsystems where the $e^+$ track could radiate photons: the Target and the Range Stack.

The distributions of the cosine of the angle between the charged track and the photons whose direction is the closest to the track at the outermost point of the Target are shown in Fig.4.3; the distributions for the angle between the track and the photon whose direction is the closest to the track at the entrance to the Range Stack are shown in Fig.4.4.

There are a lot of positron events in the real data, compared to the UMC data. The events with $\cos \theta_{\pi^+ \gamma} \geq 0.98$ are removed for both the real and UMC data.

Figure 4.3: Distributions of the cosine of the angle between the charged track and the photon whose direction is the closest to the charged track at the outermost point of the Target in the real data(left), and in the UMC data for the IB component(center) and the DE component(right) of $K^+ \rightarrow \pi^+ \pi^0 \gamma$.

Figure 4.4: Distributions of the cosine of the angle between the charged track and the photon whose direction is the closest to the charged track at the entrance of the Range Stack in the real data(left), and in the UMC data for the IB component(center) and the DE component(right) of $K^+ \rightarrow \pi^+ \pi^0 \gamma$. 

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4.1.4 Missing Momentum

The events of $K^+ \rightarrow \pi^+ \pi^0 \gamma$ have three photon-clusters in the final state. It is required that the number of clusters must be three (Cut 4), and the events with extra photons in the Range Stack, EndCap, V-Counter, I-Counter and Target must be vetoed (Cut 5). The plots of the number of clusters are shown in Fig.4.5. The missing momentum ($P_{\text{miss}}$) is calculated from the sum of four-momentum vectors of the $\pi^+$ and three photons. The plots of the squared missing momentum in the $\pi^+ \pi^0 \gamma$ final state are shown in Fig.4.6. There is a long tail in the large missing-momentum region in the real data. Since the $K^+ \rightarrow \pi^+ \pi^0 \gamma$ events should not have larger missing momenta than that expected due to the resolution effects, the events with a large missing momentum: $P_{\text{miss}}^2 > 10000$ (MeV/c)$^2$ are removed (Cut 6).

Figure 4.5: Distributions of the number of photon-clusters reconstructed by the Barrel photon detector in the real data(left), and in the UMC data for the IB component(center) and the DE component(right) of $K^+ \rightarrow \pi^+ \pi^0 \gamma$.

Figure 4.6: Distributions of the squared missing momentum in the real data(left), and in the UMC data for the IB component(center) and the DE component(right) of $K^+ \rightarrow \pi^+ \pi^0 \gamma$. 
4.1.5 Photon-Shower Split

If a photon shower splits into two clusters in the Barrel photon detector in the reconstruction, the number of photon-clusters is screwed up. In order to confirm that the number of clusters is correct, it is required that the distance between the photon clusters should be larger than 70 cm (Cut 7). The distributions of the distance are shown in Fig.4.7. There is a small peak at around 40 cm in the real data; these events are considered to be from the photon-shower split.

![Figure 4.7: Distributions of the distance between the photon clusters (the closest one) in the real data(left), and in the UMC data for the IB component(center) and the DE component(right) of $K^+ \rightarrow \pi^+\pi^0\gamma$.](image)

4.1.6 Invariant-mass of Charged Tracks

The invariant mass of the charged track ($m_{\pi^+}$) from the measured momentum and the measured kinetic-energy is used for identification of the $\pi^+$ track. The distributions of $m_{\pi^+}$ are shown in Fig.4.8. It is required that the mass should be from 100 MeV/c^2 to 200 MeV/c^2 (Cut 8). The resolutions are different between the real data and the UMC data. The cut positions are, however, loose enough; tighter rejection would be made by the kinematic fitting.

4.1.7 Photon Timing

Events with a photon cluster due to accidental hits in the Barrel photon detector should be removed. The distributions of the photon time with respect to the charged-track time are shown in Fig.4.9. The timing of each photon-cluster is required to be within ±2 ns (Cut 9).

4.1.8 Dip Angle of Photons

If a photon hits a boundary region between the module and the light guide of the Barrel photon detector, energy and position of the photon would be mis-measured. In order to
Figure 4.8: Distributions of the invariant mass of the charged track in the real data (left), and in the UMC data for the IB component (center) and the DE component (right) of $K^+ \rightarrow \pi^+ \pi^0 \gamma$.

Figure 4.9: Distributions of the timing of photons in the real data (left), and in the UMC data for the IB component (center) and the DE component (right) of $K^+ \rightarrow \pi^+ \pi^0 \gamma$.

confirm that photons are measured correctly in the fiducial volume of the Barrel photon detector, the cosine of the dip angle of each photon ($\cos \theta_{\gamma(i)} (i = 1, 2, 3)$) is required to be within $\pm 0.6$ (Cut 10). The distributions of the cosine of the dip angle are shown in Fig.4.10.

### 4.1.9 Trigger Threshold of Photons

It is difficult to estimate the energy threshold of the on-line photon trigger (NG3) precisely in the Barrel photon detector and to reproduce it in the Monte Carlo simulation. The light yield of the Barrel photon detector corresponding to the energy loss in the detector is attenuated while the light propagates in the scintillator. The offline energy threshold for each photon is determined (Cut 11) with the averaged attenuation length of the scintillators (87.3 cm), the half length of the module of the Barrel photon detector (95 cm), the $z$ position (ZG) of the photon and the energy ($E_\gamma$) of photon as:

$$E_{\text{trig}} = E_\gamma \times \exp\left(-\frac{95.0 - |\text{ZG}|}{87.3}\right) > 22.0.$$  

(4.1)
Figure 4.10: Distributions on the cosine of the dip angle of photons in the real data (left), and in the UMC data for the IB component (center) and the DE component (right) of $K^+ \rightarrow \pi^+ \pi^0 \gamma$.

The plots of the energy versus the $z$ position of a photon are plotted in Fig.4.11. Events in the upper side of the solid line in the plots are accepted.

In order to confirm that the offline energy threshold is higher than the on-line threshold, the events with small $E_\gamma$ have to be removed. The $W$ defined as Eq.1.7, with the values obtained after the kinematic fit which is described later, is required to be larger than 0.1 (Cut 12).

4.2 Kinematic Fit

4.2.1 method of kinematic fit

By imposing the kinematic constraints on all of the detected particles, we can obtain more accurate values for the kinematic variables as well as reject the backgrounds which do not satisfy the constraints of $K^+ \rightarrow \pi^+ \pi^0 \gamma$.

A list of the variables used as the inputs of the kinematic fit is shown in Tab.4.1.

The conservation of energy and momentum in the $K^+ \rightarrow \pi^+ \pi^0 \gamma$ decay is represented as follows.

\[
P_\pi \cos(\phi_\pi) \sin(\theta_\pi) + P_{\gamma_1} \cos(\phi_{\gamma_1}) \sin(\theta_{\gamma_1}) + P_{\gamma_2} \cos(\phi_{\gamma_2}) \sin(\theta_{\gamma_2}) + P_{\gamma_3} \cos(\phi_{\gamma_3}) \sin(\theta_{\gamma_3}) = 0 \tag{4.2}
\]

\[
P_\pi \sin(\phi_\pi) \sin(\theta_\pi) + P_{\gamma_1} \sin(\phi_{\gamma_1}) \sin(\theta_{\gamma_1}) + P_{\gamma_2} \sin(\phi_{\gamma_2}) \sin(\theta_{\gamma_2}) + P_{\gamma_3} \sin(\phi_{\gamma_3}) \sin(\theta_{\gamma_3}) = 0 \tag{4.3}
\]

\[
P_\pi \cos(\theta_\pi) + P_{\gamma_1} \cos(\theta_{\gamma_1}) + P_{\gamma_2} \cos(\theta_{\gamma_2}) + P_{\gamma_3} \cos(\theta_{\gamma_3}) = 0 \tag{4.4}
\]

\[
T_\pi + m_\pi + P_{\gamma_1} + P_{\gamma_2} + P_{\gamma_3} = m_k, \tag{4.5}
\]

where $P_\pi$ is the $\pi^+$ momentum, $T_\pi$ is the $\pi^+$ kinetic energy, and $P_{\gamma_1}$, $P_{\gamma_2}$, $P_{\gamma_3}$ are the momenta of three photons. $\theta_\pi$, $\theta_{\gamma_1}$, $\theta_{\gamma_2}$, $\theta_{\gamma_3}$ are the polar angles of $\pi^+$ and three photons, and $\phi_\pi$, $\phi_{\gamma_1}$,
Figure 4.11: Plots of the trigger effects of photons observed in the real data (Top), and in the UMC data for the IB component (bottom left) and the DE component (bottom right) of $K^+ \rightarrow \pi^+ \pi^0 \gamma$. The horizontal axis is the $z$ position of a photon cluster, and the vertical axis is the energy of the cluster. The solid curves show the cut at $E_{\text{trig}} = 22$.

$\phi_{\gamma 2}, \phi_{\gamma 3}$ are their azimuthal angles. The constraint on the $\pi^+$ mass is

$$\sqrt{p_{\pi}^2 + m_{\pi}^2} = T_{\pi} + m_{\pi}. \quad (4.6)$$

Two of the three photons in the final state should be from a $\pi^0$ decay. The constraint on the $\pi^0$ mass is imposed as:

$$m_{\pi^0} = \sqrt{2E_{\gamma 1}E_{\gamma 2}(1 - \cos \theta_{\gamma 1\gamma 2})}. \quad (4.7)$$

The shower-clustering routine provides us with the X, Y and Z positions of each photon in the Barrel photon detector. The directions of photons are defined as $(X_{\gamma(i)} - X_{\text{stop}}, Y_{\gamma(i)} - Y_{\text{stop}}, Z_{\gamma(i)} - Z_{\text{stop}})$ with the kaon stopping position $(X_{\text{stop}}, Y_{\text{stop}}, Z_{\text{stop}})$.

The number of observed variables is thirteen. We use the column vector $\mathbf{y}$ with thirteen
Table 4.1: Inputs for the kinematic fit. $y$ is the column vector representing the inputs to the fitting, and $y_i$ ($i = 1 - 13$) are the components.

<table>
<thead>
<tr>
<th>$y$</th>
<th>variable</th>
<th>definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>$\phi_\pi$</td>
<td>azimuthal angle of $\pi^+$.</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$\phi_{\gamma 1}$</td>
<td>azimuthal angle of one photon from $\pi^0$.</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$\phi_{\gamma 2}$</td>
<td>azimuthal angle of another photon from $\pi^0$.</td>
</tr>
<tr>
<td>$y_4$</td>
<td>$\phi_{\gamma 3}$</td>
<td>azimuthal angle of the radiative photon, not from $\pi^0$.</td>
</tr>
<tr>
<td>$y_5$</td>
<td>$\theta_\pi$</td>
<td>dip angle of $\pi^+$.</td>
</tr>
<tr>
<td>$y_6$</td>
<td>$\theta_{\gamma 1}$</td>
<td>dip angle of one photon from $\pi^0$.</td>
</tr>
<tr>
<td>$y_7$</td>
<td>$\theta_{\gamma 2}$</td>
<td>dip angle of another photon from $\pi^0$.</td>
</tr>
<tr>
<td>$y_8$</td>
<td>$\theta_{\gamma 3}$</td>
<td>dip angle of the radiative photon, not from $\pi^0$.</td>
</tr>
<tr>
<td>$y_9$</td>
<td>$P_\pi$</td>
<td>momentum of $\pi^+$.</td>
</tr>
<tr>
<td>$y_{10}$</td>
<td>$P_{\gamma 1}$</td>
<td>energy of one photon from $\pi^0$.</td>
</tr>
<tr>
<td>$y_{11}$</td>
<td>$P_{\gamma 2}$</td>
<td>energy of another photon from $\pi^0$.</td>
</tr>
<tr>
<td>$y_{12}$</td>
<td>$P_{\gamma 3}$</td>
<td>energy of the radiative photon, not from $\pi^0$.</td>
</tr>
<tr>
<td>$y_{13}$</td>
<td>$T_\pi$</td>
<td>kinetic energy of $\pi^+$.</td>
</tr>
</tbody>
</table>

Each variable is assigned to a component of $y$ as in Tab.4.1. The number of constraints on these thirteen variables is six from Eq.4.2 to Eq.4.7. The set of variables which minimize the chi-square defined as

$$\chi^2 = (y - \eta)^T V^{-1}(y - \eta),$$

is obtained as a column vector $\eta$ under the condition that these variables satisfy the six constraints exactly. $V$ in Eq.4.8 is a covariant matrix, and the diagonal terms in $V^{-1}$ are calculated as the reciprocal of the square of the resolutions of the thirteen variables ($\sigma_i^{\text{meas}} (i = 1, 2, \ldots, 13)$). The off-diagonal terms in $V$ are neglected in this analysis. The resolutions are obtained by the monochromatic peak of energy and momentum of photons and charged track from the $K^+ \rightarrow \pi^+ \pi^0$ decay, and are tuned with the “stretch functions” described later.

There are several ways to obtain the numerical solution to the problem of minimization with constraints. In this case, the method of Lagrangian multipliers is employed. We introduce six additional parameters $\lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$. The problem is to minimize the following new $\chi^2$, where $f(\eta)$ is the functions expressing the six constraints(from Eq.4.2 to 4.7).

$$\chi^2 = (y - \eta)^T V^{-1}(y - \eta) + 2\lambda^T f(\eta)$$

(4.9)

Derivatives of $\chi^2$ to the parameters in $\eta$ and $\lambda$ give

$$\nabla_\eta \chi^2 = -2V^{-1}(y - \eta) + 2F^T_\eta \lambda = 0 \quad (13 \text{ equations})$$

(4.10)

1 Italic bold face means a vector.
\[ \nabla \lambda = 2f(\eta) = 0 \quad (6 \text{ equations}) \quad (4.11) \]

where the matrix \( F_\eta \) is defined as \( (F_\eta)_{ki} \equiv \frac{\partial f_k}{\partial \eta_i} \). The equations imply

\[ -V^{-1}(y - \eta) + F_\eta^T \lambda = 0 \quad (4.12) \]

\[ f(\eta) = 0. \quad (4.13) \]

If we can solve the above 19 equations, the most appropriate values are obtained and the constraints (Eq.4.13) are automatically satisfied.

There is an iteration method for estimating the most appropriate values. Suppose the \( \nu \)-th iteration has been performed and the \( \nu \)-th values for \( \eta \) and \( \lambda \) is represented as \( \eta^\nu \) and \( \lambda^\nu \), respectively. We perform the Taylor expansion of the equations for constraints (Eq.4.13) around the values in \( \eta \):

\[ f_k^\nu + \sum_{i=1}^{N} \left( \frac{\partial f_k}{\partial \eta_i} \right)^\nu (\eta_i^{\nu+1} - \eta_i^\nu) + \cdots = 0, \quad k = 1, 2, 3, 4, 5, 6. \quad (4.14) \]

Suppose that the terms in the second and higher orders are neglected, this can be written

\[ f^\nu + F_\eta^\nu (\eta^{\nu+1} - \eta^\nu) = 0. \quad (4.15) \]

For the \( (\nu + 1) \)-th iteration, \( \eta \) and \( y \) should follow the equation of:

\[ V^{-1}(\eta^{\nu+1} - y) + (F_\eta^T)^\nu \lambda^{\nu+1} = 0 \quad (4.16) \]

We can solve these equations as

\[ f^\nu + F_\eta^\nu (y - V(F_\eta^T)^\nu \lambda^{\nu+1} - \eta^\nu) = 0 \quad (4.17) \]

\[ r = S\lambda^{\nu+1} \quad (4.18) \]

where

\[ r \equiv f^\nu + F_\eta^\nu (y - \eta^\nu) \quad (4.19) \]

\[ S \equiv F_\eta^\nu V(F_\eta^T)^\nu. \quad (4.20) \]

We can obtain the \( \nu + 1 \)-th values as

\[ \lambda^{\nu+1} = S^{-1}r \quad (4.21) \]

\[ \eta^{\nu+1} = y - VF_\eta^T \lambda^{\nu+1}. \quad (4.22) \]
Table 4.2: Output of kinematic fit.

<table>
<thead>
<tr>
<th>variable</th>
<th>definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{\pi^+}$</td>
<td>fitted momentum of charged track in the most kinematically probable case</td>
</tr>
<tr>
<td>$P_{x \pi^+}, P_{y \pi^+}, P_{z \pi^+}$</td>
<td>fitted $x,y,z$ component of momentum of charged track in the most kinematically probable case</td>
</tr>
<tr>
<td>$T_{\pi^+}$</td>
<td>fitted kinetic energy in the most kinematically probable case</td>
</tr>
<tr>
<td>$E_{\gamma(i)}$</td>
<td>fitted photon energy in the most kinematically probable case. The photon with $i = 1,2$ are photons from $\pi^0$. The photon with $i = 3$ is the radiated photon from kaon.</td>
</tr>
<tr>
<td>$P_{x \gamma(i)}, P_{y \gamma(i)}, P_{z \gamma(i)}$</td>
<td>fitted $x,y,z$ component of direction of photon in the most kinematically probable case.</td>
</tr>
<tr>
<td>$W$</td>
<td>$W$ spectrum in the most kinematically probable case</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>probability of fitting chi-square</td>
</tr>
</tbody>
</table>

Comparing the $(\chi^2)^{\nu+1}$ with $(\chi^2)^\nu$ and, if the difference is small enough, the values are supposed to be the results. For the $K^+ \rightarrow \pi^+ \pi^0 \gamma$ analysis, the difference should be less than $10^{-3}$. If the number of the iterations reaches forty, the flag indicating the failure of the kinematic fit is activated and such events are rejected.

The distributions of the differences between the fitted values ($X_{fit}^i$) and the originally measured values ($X_{meas}^i$), divided by a factor below, must be a normal Gaussian with the mean and sigma of 0.0 and 1.0, respectively. If the “stretch functions” defined as such are not consistent with the distribution of a normal Gaussian, we need more corrections to the parameters in the covariant matrix $V$ in the kinematic fit.

The standard deviation for each variable is determined so that the stretch functions defined as

$$s_i \equiv \frac{X_{meas}^i - X_{fit}^i}{\sqrt{\sigma_{meas}^{i^2} - \sigma_{fit}^{i^2}}} \quad (4.23)$$

have a distribution of a normal Gaussian. The standard deviation $\sigma_{fit}^i$ is calculated from that of the measured one ($\sigma_{meas}^i$, which was described in Page 52), based on an error propagation formula. In order to set the mean of the stretch functions to be zero, the momentum and energy of $\pi^+$ have to be shifted by introducing offsets. $P_{\pi^+}^{recon}$ and $T_{\pi^+}^{recon}$ are the original values of $\pi^+$ momentum and energy in Chapter 3, and $P_{\pi^+}^{kinf}$ and $T_{\pi^+}^{kinf}$ are the $\pi^+$ momentum and energy after the corrections with the offsets. The distributions of the stretch functions as well as the chi-square probability of the kinematic fit for real data are shown in Fig.4.12. We need the offsets as:

$$P_{\pi^+}^{kinf} = P_{\pi^+}^{recon} - 0.67, \quad (4.24)$$
$$T_{\pi^+}^{kinf} = T_{\pi^+}^{recon} + 0.5.$$
Figure 4.12: Distributions of the stretch functions and the chi-square probability of the kinematic fit for real data.
The distributions of the stretch functions as well as the chi-square probability of the kinematic fit for the IB and DE components of the $K^+ \rightarrow \pi^+\pi^0\gamma$ from the UMC are shown in Fig.4.13 and Fig.4.14, respectively. The events whose chi-square probability is less than 0.1 are rejected (Cut 13). The variables of the outputs from the kinematic fit are stored in the variables in Tab.4.2.
Figure 4.13: Distributions of the stretch functions and the chi-square probability of the kinematic fit for the IB component of $K^+ \rightarrow \pi^+ \pi^0 \gamma$ in the UMC.
Figure 4.14: Distributions of the stretch functions and the chi-square probability of the kinematic fit for the DE component of $K^+ \rightarrow \pi^+ \pi^0 \gamma$ in the UMC.
4.2.2 Photons from $\pi^0$

The two photons which come from the $\pi^0$ have to be correctly selected among three photons in the final state. The chi-square probability in the kinematic fit times the square of the amount of the IB matrix element, calculated with the fitted kinematic values, is calculated as a likelihood\(^2\), and the combination which maximizes the likelihood is selected as the correct combination. In order to make the probability of correct combinations large, the events whose best combination is not much different from the worst combination in the likelihood values are removed (Cut 14). The probabilities of the correct pairing are estimated using the UMC data. The probabilities of the correct pairing for IB and DE in this scheme are 97.2% and 83.4%, respectively. The $W$ spectrum and the spectrum excluding the events with incorrect pairing are shown in Fig.4.15.

![Figure 4.15](image)

Figure 4.15: $W$ spectrum of the IB component(left) and DE component(right) in the UMC. The solid line is the spectrum including both the correct and incorrect pairings of the photons from $\pi^0$. The spectrum excluding the incorrect pairing is indicated by the dashed line.

4.2.3 Power of Kinematic Fit

This section illustrates the power of the kinematic fit and the chi-square probability cut.

Background Rejection

Fig.4.16(left) shows the distributions of the charged-track before and after the kinematic fit is performed in the real data. Two peaks are observed in the distribution before the kinematic fit; the events in the left peak is from the $K^+ \rightarrow \pi^+ \pi^0 \pi^0$ decay, and the events in the right peak

\(^2\)The square of the IB matrix element, whose formula is available exactly from QED, is considered to be proportional to the likelihood that the combination is from the $\pi^0$ in the IB component of $K^+ \rightarrow \pi^+ \pi^0 \gamma$. We did NOT take the DE matrix element for the selection, because the matrix element was not powerful to discriminate the IB and DE components in the selection.
peak is from $K^+ \rightarrow \pi^+ \pi^0 \gamma$. The $K^+ \rightarrow \pi^+ \pi^0 \pi^0$ events are reduced by the kinematic fit. The $\pi^+$ momentum region of $140\text{MeV/c} < P_{\pi^+} < 180\text{MeV/c}$ is chosen for the spectrum fitting (Cut 15). Another example is the distribution of $R_{\text{diff}}$ (Fig. 4.16 right). There are two peaks before the kinematic fit and, if a charged track is a $\pi^+$, $R_{\text{diff}}$ must be 0.0. Most of the $\pi^+$ events remain after the kinematic fit.

Figure 4.16: Distributions of the charged track momentum (left) and $R_{\text{diff}}$ (right). The unhatched and hatched histograms show the distributions before and after the kinematic fit, respectively.
Improvements in resolutions

The improvement in resolutions are represented by UMC. The photon energy before and after the kinematic fit is shown in Fig.4.17. The resolution is much improved by the kinematic fit.

We use the value of W to determine the branching ratio of the DE component of $K^+ \rightarrow \pi^+ \pi^0 \gamma$. The distributions of the accuracy of W (measured value of W subtracted by the W calculated by the true kinematic values) for the IB and DE components in the UMC are shown in Fig.4.18 and Fig.4.19, respectively. The accuracy of W after the kinematic fit is 2.7 times better than that before the kinematic fit.

![Figure 4.17: Plots of the photon energy before(left) and after(right) the kinematic fit is performed in the UMC. The horizontal axis is the true photon energy, and the vertical axis is the measured photon energy.](image1)

![Figure 4.18: Distributions on the improvement of W for the IB data in the UMC. The left and right figures are the distributions before and after the kinematic fit is performed, respectively.](image2)
Figure 4.19: Distributions on the improvement of $W$ for the DE data in the UMC. The left and right figures are the distributions before and after the kinematic fit is performed, respectively.

### 4.3 Summary of Cuts and Data Reduction

A complete list of the cuts is shown in Tab.4.3, and a data reduction table is shown in Tab.4.4. Details on the cut conditions are presented in Appendix A. When all cuts are imposed to the real data, 20571 events remain. The number of events with $W/BQ < 0.5$ is 735.

### 4.4 Rejection of Cuts for Events by the UMC

Each component of the IB, DE and INT components of $K^+ \rightarrow \pi^+ \pi^0 \gamma$ is generated by the UMC. The numbers of the generation kaon decays are summarized in Tab.4.5. The kinetic region for those generation is $10 \text{ MeV} < T_{\pi^+} < 95 \text{ MeV}$. The events with $55 \text{ MeV} < T_{\pi} < 90 \text{ MeV}$ are used for the spectrum fitting. The momentum region is determined by Cut 15: $140 \text{MeV}/c < P_{\pi^+} < 180 \text{MeV}/c$ is tighter than the region $55 \text{ MeV} < T_{\pi^+} < 90 \text{ MeV}$ for the UMC generations.

The data reduction table for IB and DE is shown in Tab.4.6 and Tab.4.7, respectively. The cut 0 is a cut set to reduce the amount of the data; it is a looser cut condition than that of cuts for selecting signals. The acceptance of the IB and DE components($55\text{MeV} < T_{\pi^+} < 90\text{MeV}$) are $6.84 \times 10^{-4}$ and $8.86 \times 10^{-4}$, respectively.
Table 4.3: Summary of cuts.

<table>
<thead>
<tr>
<th>cut number</th>
<th>description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\pi^+$ identification based on the consistency of the range: $-2.2 &lt; R_{\text{diff}} &lt; 1.6$ for real data; $-1.9 &lt; R_{\text{diff}} &lt; 1.9$ for UMC data.</td>
</tr>
<tr>
<td>2</td>
<td>The energy deposit in Target, calculated from the range with pion assumption, is consistent with the measured energy. $</td>
</tr>
<tr>
<td>3</td>
<td>Photons do not exist in the direction of the charged track in the Target and the Range Stack: $\cos \theta_{\pi^+ \gamma} &lt; 0.98$.</td>
</tr>
<tr>
<td>4</td>
<td>Events with three photon clusters ($N_{\gamma} = 3$).</td>
</tr>
<tr>
<td>5</td>
<td>Photon veto in the Range Stack, Endcap, V-counter, I-Counter and Target.</td>
</tr>
<tr>
<td>6</td>
<td>Squared missing momentum is required to be less than $10000 \text{ (MeV/c)}^2$.</td>
</tr>
<tr>
<td>7</td>
<td>The distances among the photon clusters are larger than 70cm.</td>
</tr>
<tr>
<td>8</td>
<td>It is required that the invariant mass of the charged track is close to the pion mass: $100 &lt; m_{\pi^+} &lt; 200$.</td>
</tr>
<tr>
<td>9</td>
<td>The time of each photon is coincide with the charged track time ($\pm 2 \text{ ns}$).</td>
</tr>
<tr>
<td>10</td>
<td>All the photon clusters are located in the fiducial region: $</td>
</tr>
<tr>
<td>11</td>
<td>Confirm that the energy of each photon satisfies the condition of the trigger: $E_{\text{trig}} &gt; 22$.</td>
</tr>
<tr>
<td>12</td>
<td>Events with a small $W$ is rejected. $W$ must be $&gt; 0.1$.</td>
</tr>
<tr>
<td>13</td>
<td>Cut on the chi-square of the kinematic fit. It is required that the chi-square probability($\mathcal{P}$) is greater than 0.1.</td>
</tr>
<tr>
<td>14</td>
<td>There are three combination to the two photons from $\pi^0$. The selected combination is compared with the combination whose selection likelihood was the worst. Unless there is large difference between these combinations, the events are rejected.</td>
</tr>
<tr>
<td>15</td>
<td>The $\pi^+$ momentum after the kinematic fit is from 140 MeV/c to 180 MeV/c is accepted.</td>
</tr>
</tbody>
</table>
Table 4.4: Summary of the reduction of Real data.

<table>
<thead>
<tr>
<th>Cut number</th>
<th>Examined</th>
<th>Passed</th>
<th>Rejection</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>799385</td>
<td>376427</td>
<td>2.12</td>
</tr>
<tr>
<td>2</td>
<td>376427</td>
<td>319144</td>
<td>1.18</td>
</tr>
<tr>
<td>3</td>
<td>319144</td>
<td>256723</td>
<td>1.24</td>
</tr>
<tr>
<td>4</td>
<td>256723</td>
<td>180452</td>
<td>1.42</td>
</tr>
<tr>
<td>5</td>
<td>180452</td>
<td>122474</td>
<td>1.47</td>
</tr>
<tr>
<td>6</td>
<td>122474</td>
<td>106559</td>
<td>1.15</td>
</tr>
<tr>
<td>7</td>
<td>106559</td>
<td>97516</td>
<td>1.09</td>
</tr>
<tr>
<td>8</td>
<td>97516</td>
<td>88050</td>
<td>1.11</td>
</tr>
<tr>
<td>9</td>
<td>88050</td>
<td>76871</td>
<td>1.15</td>
</tr>
<tr>
<td>10</td>
<td>76871</td>
<td>75679</td>
<td>1.02</td>
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<tr>
<td>11</td>
<td>75679</td>
<td>46681</td>
<td>1.62</td>
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<tr>
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<td>45720</td>
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<td>45720</td>
<td>27834</td>
<td>1.64</td>
</tr>
<tr>
<td>14</td>
<td>27834</td>
<td>26605</td>
<td>1.05</td>
</tr>
<tr>
<td>15</td>
<td>26605</td>
<td>20571</td>
<td>1.29</td>
</tr>
</tbody>
</table>

Table 4.5: Generation of UMC data.

<table>
<thead>
<tr>
<th>Term</th>
<th>Number of K decays in the region of $10 \text{ MeV} &lt; T_\pi &lt; 95 \text{ MeV}$ ($55 \text{ MeV} &lt; T_\pi &lt; 90 \text{ MeV}$)</th>
<th>Trigger Passed</th>
</tr>
</thead>
<tbody>
<tr>
<td>IB</td>
<td>$6.00 \times 10^8 \ (3.77 \times 10^8)$</td>
<td>938351</td>
</tr>
<tr>
<td>DE</td>
<td>$1.50 \times 10^8 \ (7.26 \times 10^7)$</td>
<td>237220</td>
</tr>
<tr>
<td>INT</td>
<td>$1.50 \times 10^8 \ (8.76 \times 10^7)$</td>
<td>218374</td>
</tr>
</tbody>
</table>
Table 4.6: Summary of off-line analysis of UMC IB.

<table>
<thead>
<tr>
<th>Cut number</th>
<th>Examined</th>
<th>Passed</th>
<th>Rejection</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>938351</td>
<td>852354</td>
<td>1.10</td>
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<td>852354</td>
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<td>1.11</td>
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<td>3</td>
<td>753742</td>
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<td>1.02</td>
</tr>
<tr>
<td>4</td>
<td>739243</td>
<td>724215</td>
<td>1.02</td>
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<tr>
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<td>1.01</td>
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<td>660323</td>
<td>657410</td>
<td>1.00</td>
</tr>
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</tr>
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<td>644361</td>
<td>1.01</td>
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<td>11</td>
<td>644361</td>
<td>426747</td>
<td>1.51</td>
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<td>12</td>
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</tr>
<tr>
<td>13</td>
<td>414960</td>
<td>334054</td>
<td>1.24</td>
</tr>
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<td>14</td>
<td>334054</td>
<td>318262</td>
<td>1.05</td>
</tr>
<tr>
<td>15</td>
<td>318262</td>
<td>258380</td>
<td>1.23</td>
</tr>
</tbody>
</table>

\[ A_{UMC}^{IB} = 6.84 \times 10^{-4} \text{ (55 MeV < } T_π < 90 \text{ MeV)} \]

Table 4.7: Summary of off-line analysis of UMC DE.

<table>
<thead>
<tr>
<th>Cut number</th>
<th>Examined</th>
<th>Passed</th>
<th>Rejection</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>237220</td>
<td>210082</td>
<td>1.13</td>
</tr>
<tr>
<td>1</td>
<td>210082</td>
<td>194760</td>
<td>1.08</td>
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<td>2</td>
<td>194760</td>
<td>191232</td>
<td>1.02</td>
</tr>
<tr>
<td>3</td>
<td>191232</td>
<td>188088</td>
<td>1.02</td>
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<tr>
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<td>188088</td>
<td>183840</td>
<td>1.02</td>
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<td>183840</td>
<td>172359</td>
<td>1.07</td>
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<td>8</td>
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<td>160237</td>
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<td>159147</td>
<td>1.01</td>
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<td>159147</td>
<td>157411</td>
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<tr>
<td>11</td>
<td>157411</td>
<td>125572</td>
<td>1.25</td>
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<tr>
<td>15</td>
<td>91555</td>
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</tbody>
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\[ A_{UMC}^{DE} = 8.86 \times 10^{-4} \text{ (55 MeV < } T_π < 90 \text{ MeV)} \]
Table 4.8: Summary of off-line analysis of UMC INT.

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<th>Passed</th>
<th>Rejection</th>
</tr>
</thead>
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<tr>
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<td>261629</td>
<td>234050</td>
<td>1.12</td>
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<tr>
<td>1</td>
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</tr>
<tr>
<td>2</td>
<td>214985</td>
<td>210867</td>
<td>1.02</td>
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<tr>
<td>3</td>
<td>210867</td>
<td>207765</td>
<td>1.01</td>
</tr>
<tr>
<td>4</td>
<td>207765</td>
<td>203363</td>
<td>1.02</td>
</tr>
<tr>
<td>5</td>
<td>203363</td>
<td>190451</td>
<td>1.07</td>
</tr>
<tr>
<td>6</td>
<td>190451</td>
<td>188832</td>
<td>1.01</td>
</tr>
<tr>
<td>7</td>
<td>188832</td>
<td>182235</td>
<td>1.04</td>
</tr>
<tr>
<td>8</td>
<td>182235</td>
<td>181473</td>
<td>1.00</td>
</tr>
<tr>
<td>9</td>
<td>181473</td>
<td>180183</td>
<td>1.01</td>
</tr>
<tr>
<td>10</td>
<td>180183</td>
<td>177839</td>
<td>1.01</td>
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<td>177839</td>
<td>135005</td>
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<td>135005</td>
<td>134394</td>
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<td>134394</td>
<td>106464</td>
<td>1.26</td>
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<td>106464</td>
<td>102145</td>
<td>1.04</td>
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<td>15</td>
<td>102145</td>
<td>76053</td>
<td>1.34</td>
</tr>
</tbody>
</table>

$A_{INT}^{UCMC} = 8.68 \times 10^{-4}$ (55 MeV < $T_\pi$ < 90 MeV)
Table 4.9: Processes of producing background to $\pi^+\pi^0\gamma$.

<table>
<thead>
<tr>
<th>charged track</th>
<th>photons</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^+\pi^0$</td>
<td>$\pi^+$ momentum is mis-measured.</td>
</tr>
<tr>
<td>$\pi^+\pi^+\pi^0$</td>
<td>$\pi^+$ momentum is mis-measured.</td>
</tr>
<tr>
<td>$\mu^+\pi^0\nu$</td>
<td>$\mu^+$ is mis-identified as $\pi^+$.</td>
</tr>
<tr>
<td>$e^+\pi^0\nu$</td>
<td>$e^+$ is mis-identified as $\pi^+$.</td>
</tr>
<tr>
<td>$\text{One of the two photons from } \pi^0 \text{ is split into two clusters}$ OR $\text{&quot;there is an accidental hit in Barrel photon detector&quot;}$.</td>
<td></td>
</tr>
<tr>
<td>$\text{One of the four photons from } \pi^0\pi^0 \text{ is missed.}$</td>
<td></td>
</tr>
<tr>
<td>$\text{&quot;One of the two photons from } \pi^0 \text{ is split into two clusters&quot; OR &quot;there is an accidental hit in the Barrel photon detector&quot;}$.</td>
<td></td>
</tr>
<tr>
<td>$\text{&quot;One of the two photons from } \pi^0 \text{ is split into two clusters&quot; OR &quot;there is an accidental hit in the Barrel photon detector&quot;}$.</td>
<td></td>
</tr>
</tbody>
</table>

4.5 Background Estimation

The background levels to the $K^+ \rightarrow \pi^+\pi^0\gamma$ sample after imposing all the cuts in Tab.4.3 should be estimated. $K^+$ decays to $\mu^+\pi^0\nu$, $\pi^+\pi^0$, $\pi^+\pi^0\pi^0$ and $e^+\pi^0\nu$ could be backgrounds due to some unfortunate processes. Other background sources are assumed to be negligible. The processes in which these $K^+$ decays could contribute to the $K^+ \rightarrow \pi^+\pi^0\gamma$ sample by mistakes are summarized in Tab.4.9.

These backgrounds are categorized into five types, according to the source of photons (Tab.4.10). The background level for each type is estimated with real data by a "bifurcation method", which has been used in the analysis of the BNL-E787 experiment. At first, cuts for rejecting background events except for the targeting background source are imposed (stage 1). In the next, a cut to tag a background sample is imposed to the data (stage 2), and other cuts which have a large rejection to the background are imposed to estimate the rejection power (stage 3). We need to estimate the efficiency of the tagging; another sample of the background is selected by a different tagging cut (stage 4), which has to be independent of the tagging cut in question, and the efficiency of the tagging is estimated. A summary of the methods for estimating the background level is also shown in Tab.4.10.

The background sources not listed in Table 4.9, if exists, are not taken into account in the background estimation based on E787’s bifurcation method; they are however considered to be negligible in the $K^+ \rightarrow \pi^+\pi^0\gamma$ analysis because of the powerful rejection of the kinematic fit cuts.

4.5.1 Backgrounds due to Accidental Hits

In this section, the background level from $K^+ \rightarrow \mu^+\pi^0\nu$, $K^+ \rightarrow \pi^+\pi^0$, $K^+ \rightarrow \pi^+\pi^0\pi^0$ and $K^+ \rightarrow e^+\pi^0\mu$ with accidental hits in the Barrel photon detector is estimated.

We need an indicator of accidental hits. The relative timing between the $\pi^+$ and the
Table 4.10: Categorization of the types of background, and methods for estimating the background levels.

<table>
<thead>
<tr>
<th>photon source</th>
<th>mode</th>
<th>tag</th>
<th>section</th>
</tr>
</thead>
<tbody>
<tr>
<td>extra photon to $\pi^0$</td>
<td>accidental</td>
<td>$\pi^+\pi^0$ $\mu^+\pi^0\nu$ $e^+\pi^0\nu$</td>
<td>timing of photon</td>
</tr>
<tr>
<td></td>
<td>split</td>
<td>$\pi^+\pi^0$</td>
<td>momentum of charged track</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\mu^+\pi^0\nu$ $e^+\pi^0\nu$</td>
<td>identification of the charged track</td>
</tr>
<tr>
<td></td>
<td>bremsstrahlung</td>
<td>$e^+\pi^0\nu$</td>
<td>direction of photon with respect to the $e^+$ track</td>
</tr>
<tr>
<td>missing photon from $\pi^0\pi^0$</td>
<td>missing</td>
<td>$\pi^+\pi^0\pi^0$</td>
<td>momentum of the charged track</td>
</tr>
</tbody>
</table>
Table 4.11: Cuts used for estimating the accidental-hits background for each stage in Fig.4.20.

<table>
<thead>
<tr>
<th>stage</th>
<th>cuts</th>
<th>remained events</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3, 4, 5, 7, 10, 11</td>
<td>132215</td>
</tr>
<tr>
<td>2</td>
<td>offtime photon</td>
<td>3626</td>
</tr>
<tr>
<td>3</td>
<td>1, 2, 6, 8, 12, 13, 14, 15</td>
<td>17</td>
</tr>
<tr>
<td>4</td>
<td>$W &gt; 0.5$ at stage 3</td>
<td>1</td>
</tr>
</tbody>
</table>

A photon cluster should be random for the accidental photon hits. The photon from $K^+$ decay should coincide with the $\pi^+$ track. When the photon time is apart from the $\pi^+$ time (from -6 ns to -4 ns and from +4 ns to +6 ns), the event is tagged as a background event due to accidental hits. All the cuts except for the photon timing cut should have been imposed to the events.

We assume that the time distribution of the photons due to accidental hits should be flat in all over the time. The acceptable time width ($\pm 2.0$ ns) of the signal is the same as that for tagging (-6 ns to -4 ns and +4 ns to +6 ns, 4 ns in total). We can therefore assume that the number of the expected background events is the same as the number of tagged events. The number of tagged events after all the cuts except for photon timing cut are imposed is 17; the number of background is therefore estimated to be 17. We have 20571 events after all the cuts are imposed; the background level from the photon accidental hits is 0.08% in the $K^+ \to \pi^+ \pi^0 \gamma$ sample. The chart for the background estimation is shown in Fig.4.20, and the timing distribution of photons is shown in Fig.4.21. The cuts are listed in Tab.4.11.

The number of background events in the region of large $W$ is crucial to the spectrum analysis. The $W$ spectrum for the tagged events is shown in Fig.4.22. There are 735 events after all the cuts, and only one event is expected due to the background in the region of $W > 0.5$. The background with accidental hits is therefore negligible.
Figure 4.20: Chart for estimating the backgrounds from the accidental hit.

Figure 4.21: Timing distributions of the 3rd energetic photon at the stage 2 in Fig.4.20. The right figure is the distribution after all the cuts except for photon timing cut are imposed. The numbers of the tagged events in the left and right figure are $N_2 (= 3626)$ and $N_3 (= 17)$ in Fig.4.20, respectively.
Figure 4.22: W spectrum of the remained background events due to the accidental hits.
4.5.2 Backgrounds from $\pi^+\pi^0$ with Split Photon

The decay $K^+ \rightarrow \pi^+\pi^0$ with a photon split into two clusters is a possible background source. The bifurcation for the $\pi^+\pi^0$ background needs to tag the $\pi^+$ from $K^+ \rightarrow \pi^+\pi^0$ with the momentum of 205 MeV/c. Although the events with higher momentum are removed at first in the other background studies, a special data set which keeps the events with higher momentum events is used in this section.

The $K^+ \rightarrow \pi^+\pi^0$ events are easily tagged by requiring the $\pi^+$ momentum to be high. However, there should be a contamination from $K^+ \rightarrow \pi^+\pi^0\gamma$ in such a sample, and the background level would be overestimated due to the contamination. The estimated background level in this section should therefore be regarded an upper limit.

At first, the cuts to reject other background sources (called the “setup cuts”) are imposed. The cut to select $K^+ \rightarrow \pi^+\pi^0$, 200.0 MeV/c $< P_{\pi^+}^{\text{recon}} < 210.0$ MeV/c, is imposed in the next. In order to reject the backgrounds further, the cuts on the kinematic fit are imposed. The number of events at this stage ($N_3$) is 0. The $N_3$ is set to be 2.3 in stead of 0 in order to obtain the 90% confidence-level upper limit. In order to check the tagging efficiency, and the distribution of the $\pi^+$ momentum is shown in Fig.4.24. The cuts of kinematic fit reject the $K\pi\pi$ events, while it keeps the signal of $K^+ \rightarrow \pi^+\pi^0\gamma$.

The efficiency of the tagging should be estimated, and the distance between the closest photon-clusters is used. If the distance is smaller than 55.0 cm, it is assumed that a photon cluster in this event has been split into two clusters. The distributions of the closest distance in several stages are shown in Fig.4.25. We can see that the events with a split photon are enhanced if a cut $200.0 < P_{\pi^+}^{\text{kin}} < 210.0$ to select $\pi^+\pi^0$ is imposed, while the events with a split photon are suppressed if the momentum region for $K^+ \rightarrow \pi^+\pi^0\gamma$ is selected. The tagging efficiency ($\varepsilon_{\text{tag}}$) is estimated to be $N_5/N_4$, where $N_4$ and $N_5$ are the numbers of event before and after imposing the momentum cut selecting $\pi^+\pi^0$. A rejection of the $\pi^+$ momentum cut has to be estimated. The rejection ($R_{\text{cut}}$) of the $\pi^+$ momentum cut is $N_4/N_6$ where $N_6$ is the number of events selected as $\pi^+\pi^0\gamma$ by the momentum cut. A chart for the background estimation is shown in Fig.4.23. The cuts in each stage are listed in Tab.4.12.

The background level $N_{\pi^+\pi^0}^{\text{BKG}}$ is estimated to be as:

$$N_{\pi^+\pi^0}^{\text{BKG}} = N_3 \times \frac{1}{\varepsilon_{\text{tag}}} \times \frac{1}{R_{\text{cut}}} = N_3 \times \frac{N_5}{N_4} \times \frac{N_6}{N_5} = N_3 \times \frac{N_6}{N_5} = 2.3 \times \frac{163}{138} = 2.7.$$ (4.26)

The estimated background level (2.7 events, a 90% confidence-level upper limit), corresponds to 0.01% in the $K^+ \rightarrow \pi^+\pi^0\gamma$ events. This background source is negligible.
Table 4.12: Cuts used for estimating split-photon background with $\pi^+\pi^0$ for each stage in Fig.4.23.

<table>
<thead>
<tr>
<th>stage</th>
<th>cuts</th>
<th>remained events</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>1, 2, 3, 4, 5, 6, 8, 9, 10, 11</td>
<td>49930</td>
</tr>
<tr>
<td>2</td>
<td>$200.0 &lt; P_{\pi_+}^{\text{recon}} &lt; 210.0$</td>
<td>549</td>
</tr>
<tr>
<td>3</td>
<td>7, 12, 13, 14</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>distance among the nearest clusters is less than 55.0 cm</td>
<td>996</td>
</tr>
<tr>
<td>5</td>
<td>$200.0 &lt; P_{\pi_+}^{\text{recon}} &lt; 210.0$</td>
<td>138</td>
</tr>
<tr>
<td>6</td>
<td>15</td>
<td>163</td>
</tr>
<tr>
<td>7</td>
<td>$W &gt; 0.5$ at stage 3</td>
<td>0</td>
</tr>
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</table>

![Figure 4.23](image.png)

Figure 4.23: Chart for estimating the background level from $\pi^+\pi^0$ with split photon.
Figure 4.24: Distributions of the $\pi^+$ momentum $P_{\pi^+}^{\text{recon}}$. The left figure is in the first stage after imposing the setup cuts, and the right figure is at the stage after imposing the cuts on the kinematic fit. The numbers of the tagged events in the left and right figures are $N_2 (=549)$ and $N_3 (=0)$ in Fig.4.23, respectively.

Figure 4.25: Distributions of the distance between the closest photon-clusters. The left figure is the distribution at the setup. The number of the tagged events in the left figure is $N_4 (=996)$. The center figure is at the stage to select $\pi^+\pi^0(200 < P_{\pi^+}^{\text{recon}} < 210)$. The number of the tagged events is $N_5 (=138)$. The right figure is at the stage to select $K^+ \rightarrow \pi^+\pi^0\gamma(140 < P_{\pi^+} < 180)$. The number of the tagged events is $N_6 (=163)$. 

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Table 4.13: Cuts used for estimating the split-photon background with $e^+\pi^0\nu$ and $\mu^+\pi^0\nu$ for each stage in Fig.4.26.

<table>
<thead>
<tr>
<th>stage</th>
<th>cuts</th>
<th>remained events</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>2, 3, 4, 5, 9, 10, 11, 15</td>
<td>47372</td>
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<td>2</td>
<td>$R_{\text{diff}} &gt; 2.5$</td>
<td>8444</td>
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<tr>
<td>3</td>
<td>6, 7, 8, 12, 13, 14</td>
<td>206</td>
</tr>
<tr>
<td>4</td>
<td>distance among photons is less than 55 cm</td>
<td>1113</td>
</tr>
<tr>
<td>5</td>
<td>$R_{\text{diff}} &gt; 2.5$</td>
<td>670</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>216</td>
</tr>
<tr>
<td>7</td>
<td>$W &gt; 0.5$ at stage 3</td>
<td>8</td>
</tr>
</tbody>
</table>

4.5.3 Backgrounds from $e^+\pi^0\nu$ and $\mu^+\pi^0\nu$ with Split Photon

There are two methods for tagging the backgrounds. The first method uses a cut for an identification of charged particles. The particles with $R_{\text{diff}} > 2.5$ are tagged as muon or positron. The distributions of $R_{\text{diff}}$ are shown in Fig.4.27; the tagged events are suppressed by imposing the kinematic fit.

The second method uses the minimum distance between the photon clusters. If the distance is smaller than 55.0 cm, the events are assumed to be due to a split photon. The distributions of the distance between the closest photon-clusters are shown in Fig.4.28. The events with a split photon-cluster are enhanced if the cuts to select the background are imposed; on the other hand, the events with a split photon are suppressed if the cuts to select $K^+ \rightarrow \pi^+\pi^0\gamma$ are imposed. These confirm that the tagging with the minimum distance works.

A chart for the background estimation is shown in Fig.4.26. The cuts in each stage are listed in Tab.4.13.

The background level $N_{\text{split}}^{\text{BKG}}$ is estimated to be:

$$N_{\text{split}}^{\text{BKG}} = N_3 \times \frac{1}{\varepsilon_{\text{tag}}} \times \frac{1}{R_{\text{cut}}} = N_3 \times \frac{N_6}{N_5} = 206 \times \frac{216}{670} = 66.4.$$  \hspace{1cm} (4.27)

We have 20571 events after all the cuts imposed; the background from $e^+\pi^0\nu$ and $\mu^+\pi^0\nu$ with split photon is 0.3% against $K^+ \rightarrow \pi^+\pi^0\gamma$ sample. The $W$ spectrum for the tagged events is shown in Fig.4.29. Eight events with $W > 0.5$ are observed, and the expected number of background events with $W > 0.5$ is 2.6. We have 735 events in the region of $W > 0.5$ after all the cuts. The expected number of events corresponds to 0.4% in the region of $W > 0.5$. This background source is negligible.
Figure 4.26: Chart for estimating the background level from $e^+\pi^0\nu$ and $\mu^+\pi^0\nu$ with split photon.

Figure 4.27: Distributions of $R_{diff}$ at the setup (left) and at the stage after imposing the cuts on the kinematic fit (right). The numbers of the tagged events in the left and right figure are $N_2(= 8444)$ and $N_3(= 206)$, respectively.
Figure 4.28: Distributions of the distance between the closest photon clusters. The left figure is at the setup. The number of the tagged events is \( N_4 (= 1113) \). The center figure is at the stage to select the background; the number of the tagged events is \( N_5 (= 670) \). The right figure is at the stage to select \( K^+ \rightarrow \pi^+ \pi^0 \gamma \); the number of the tagged events is \( N_6 (= 216) \).

Figure 4.29: \( W \) spectrum of the tagged events after imposing the cuts on the kinematic fit for the split photon study.
Table 4.14: Cuts used for estimating the radiated-photon background for each stage in Fig. 4.30.

<table>
<thead>
<tr>
<th>stage</th>
<th>cuts</th>
<th>remained events</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4, 5, 7, 9, 10, 11, 15</td>
<td>84626</td>
</tr>
<tr>
<td>2</td>
<td>( R_{\text{diff}} &gt; 5 )</td>
<td>7597</td>
</tr>
<tr>
<td>3</td>
<td>2, 3, 6, 8, 12, 13, 14</td>
<td>34</td>
</tr>
<tr>
<td>4</td>
<td>angle between photons and charged track in the Target or in the Range Stack</td>
<td>28904</td>
</tr>
<tr>
<td>5</td>
<td>( R_{\text{diff}} &gt; 5 )</td>
<td>3254</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>13973</td>
</tr>
<tr>
<td>7</td>
<td>( W &gt; 0.5 ) at stage 3</td>
<td>1</td>
</tr>
</tbody>
</table>

### 4.5.4 Backgrounds from \( e^+ \pi^0 \nu \) with Radiated Photon

A tagging method is to use a cut for an identification of charged particles. The distributions of \( R_{\text{diff}} \) are shown in Fig. 4.31. Background events are much suppressed after imposing the cuts on kinematic fit.

An \( e^+ \) track tends to radiate photons in passing through the material. There are two detector-subsystems where the \( e^+ \) has bremsstrahlung: the Target and the Range Stack. The \( e^+ \) radiates photons in the same direction as the \( e^+ \) transportation. The radiated photon can be tagged by the position of the photon cluster. The distributions of the angle between the charged track at the entrance of the Range Stack and the photon in the Barrel photon detector are shown in Fig. 4.32; backgrounds can be tagged correctly.

A chart of this bifurcation is shown in Fig. 4.30. The cuts in each stage are listed in Tab. 4.14.

The background level \( N^{BKG}_{e^+ \pi^0 \nu} \) is estimated to be:

\[
N^{BKG}_{e^+ \pi^0 \nu} = N_3 \times \frac{1}{\epsilon_{\text{tag}}} \times \frac{1}{R_{\text{cut}}} = N_3 \times \frac{N_6}{N_5} = 34 \times \frac{13973}{3254} = 146.0. \tag{4.28}
\]

The \( W \) spectrum of the tagged events is shown in Fig. 4.33; the number of events with \( W > 0.5 \) is one. Assuming the shape of \( W \) spectrum in the tagging branch with the particle identification is the same as that in the tagging branch with extra photon in the Barrel photon detector, the expected number of background events with \( W > 0.5 \) is \( 1 \times \frac{13973}{3254} = 4.3 \), corresponding to 0.6%. This background source is negligible.
Figure 4.30: Chart for estimating the background level from $e^+\pi^0\nu$ with radiated photon.

Figure 4.31: Distributions of $R_{diff}$ at the setup (left) and at the stage after imposing the cuts on the kinematic fit. The numbers of the tagged events in the left and right figures are $N_2 (= 7597)$ and $N_3 (= 34)$, respectively.
Figure 4.32: Distributions of the angle between the charged track at the entrance of the Range Stack and the photon. The left figure is at the stage tagged by the extra photon in the Barrel photon detector. The center figure is the $e^\pm$ events with $R_{diff} > 5.0$ cm. The right figure is the events after $\pi^+$ identification. The number of the tagged events in the center and right figures are $N_3 (=3254)$ and $N_6 (=13973)$, respectively.

Figure 4.33: W spectrum of the tagged events after imposing the cuts on the kinematic fit cuts for the $e^+ \pi^0 \nu$ study.
Table 4.15: Cuts used for estimating the $\pi^+\pi^0\pi^0$ background for each stage in Fig. 4.34.

<table>
<thead>
<tr>
<th>stage</th>
<th>cuts</th>
<th>remained events</th>
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<tbody>
<tr>
<td>1</td>
<td>1, 2, 3, 5, 7, 8, 9, 10, 11</td>
<td>72690</td>
</tr>
<tr>
<td>2</td>
<td>$N_\gamma = 4, E_{\gamma 4} &gt; 20$</td>
<td>18486</td>
</tr>
<tr>
<td>3</td>
<td>6, 12, 13, 14, 15</td>
<td>13</td>
</tr>
<tr>
<td>4</td>
<td>$100 &lt; P_{\text{recon}}^{\pi^+} &lt; 130$</td>
<td>33183</td>
</tr>
<tr>
<td>5</td>
<td>$N_\gamma = 4, E_{\gamma 4} &gt; 20$</td>
<td>15710</td>
</tr>
<tr>
<td>6</td>
<td>$N_\gamma = 3$</td>
<td>12642</td>
</tr>
<tr>
<td>7</td>
<td>$W &gt; 0.5$ at stage 3</td>
<td>2</td>
</tr>
</tbody>
</table>

4.5.5 $K^+ \rightarrow \pi^+\pi^0\pi^0$ with a missed photon

If $K^+ \rightarrow \pi^+\pi^0\pi^0$ is a background for $K^+ \rightarrow \pi^+\pi^0\gamma$, one photon should have been missed and the momentum of the charged track should have been measured incorrectly. The $\pi^+\pi^0\pi^0$ background is estimated by a bifurcation method. There are two branches with different tagging methods. $\pi^+$ from $K^+ \rightarrow \pi^+\pi^0\pi^0$ is tagged by requiring that the number of photon in the Barrel photon detector is equal to four. The distributions on the number of photon clusters are shown in Fig. 4.35. The cuts on the kinematic fit remove the backgrounds with four clusters and, after imposing the momentum cuts, 13 events remain.

We need to measure the tagging efficiency. $K^+ \rightarrow \pi^+\pi^0\pi^0$ is tagged by the $\pi^+$ momentum region from 100 MeV/c to 130 MeV/c. The tagged events also contain $K^+ \rightarrow \pi^+\pi^0\gamma$ events because the $K^+ \rightarrow \pi^+\pi^0\gamma$ decay occurs in the lower-momentum region; the background level would be overestimated. The distributions of $P_{\text{recon}}^{\pi^+}$ are shown in Fig. 4.36. Since the cut on the $\pi^+$ momentum rejects the backgrounds, we can see the tagging works correctly.

A chart of this bifurcation is shown in Fig. 4.34. The detailed cut list is shown in Tab. 4.15.

$$N_{\pi^+\pi^0\pi^0}^{BKG} = N_3 \times \frac{1}{\epsilon_{\text{tag}}} \times \frac{1}{R_{\text{cut}}} = N_3 \times \frac{N_6}{N_5} = 13 \times \frac{12642}{15710} = 10.5.$$ (4.29)

The 10.5 events corresponds to the 0.05% contribution to the $K^+ \rightarrow \pi^+\pi^0\gamma$ sample. The $W$ spectrum of the tagged events is shown in Fig. 4.37. The estimated number of events at $W > 0.5$ is 1.6 and is 0.2%. This background source is negligible.
Figure 4.34: Chart for estimating the background level from the $\pi^+ \pi^0\pi^0$ background.

Figure 4.35: Distributions of the number of photon-clusters. The numbers of the tagged events in the left and right figure is $N_2(=18486)$ and $N_3(=13)$, respectively.
Figure 4.36: Distributions of the $\pi^+$ momentum. The left figure is at the setup. The number of the tagged events in the left figure is $N_4 (= 33183)$. The center figure is at the stage tagged by the number of photons ($N_\gamma = 4$). The number of the tagged events is $N_5 (= 15710)$. The right figure is at the stage tagged by the number of photons ($N_\gamma = 3$) for selecting the $K^+ \rightarrow \pi^+ \pi^0 \gamma$ decay. The number of the tagged events is $N_6 (= 12642)$.

Figure 4.37: $W$ spectrum of the tagged events after imposing the cuts on the kinematic fit and the momentum cut for $\pi^+ \pi^0 \pi^0$. 

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Table 4.16: Summary of the background studies.

<table>
<thead>
<tr>
<th>description</th>
<th>number</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^+ \pi^0$, $\mu^+ \pi^0 \nu$ with accidental hits</td>
<td>17</td>
<td>0.1%</td>
</tr>
<tr>
<td>$\pi^+ \pi^0$ with a split photon</td>
<td>2.7</td>
<td>0.0%</td>
</tr>
<tr>
<td>$\mu^+ \pi^0 \nu$ and $e^+ \pi^0 \nu$ with a split photon</td>
<td>66</td>
<td>0.3%</td>
</tr>
<tr>
<td>$e^+ \pi^0 \nu$ with a radiated photon</td>
<td>146</td>
<td>0.7%</td>
</tr>
<tr>
<td>$\pi^+ \pi^0 \pi^0$ with a missed photon</td>
<td>11</td>
<td>0.0%</td>
</tr>
<tr>
<td>number of events after all cuts</td>
<td>20571</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.17: Summary of background studies for the events with $W > 0.5$.

<table>
<thead>
<tr>
<th>description</th>
<th>number</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^+ \pi^0$, $\mu^+ \pi^0 \nu$ with accidental hits</td>
<td>1</td>
<td>0.1%</td>
</tr>
<tr>
<td>$\pi^+ \pi^0$ with a split photon</td>
<td>0</td>
<td>0.0%</td>
</tr>
<tr>
<td>$\mu^+ \pi^0 \nu$ and $e^+ \pi^0 \nu$ with a split photon</td>
<td>2.6</td>
<td>0.4%</td>
</tr>
<tr>
<td>$e^+ \pi^0 \nu$ with a radiated photon</td>
<td>4.3</td>
<td>0.6%</td>
</tr>
<tr>
<td>$\pi^+ \pi^0 \pi^0$ with a missed photon</td>
<td>1.6</td>
<td>0.2%</td>
</tr>
<tr>
<td>number of events after all cuts and $W &gt; 0.5$</td>
<td>735</td>
<td></td>
</tr>
</tbody>
</table>

4.5.6 Summary of the Background Studies

The numbers of the estimated background levels are summarized in Tab.4.16. For the spectrum analysis, the region with large $W$ is important; the numbers of the estimated background levels at $W > 0.5$ are summarized in Tab.4.17. All background sources are negligible in the $K^+ \rightarrow \pi^+ \pi^0 \gamma$ analysis.
Chapter 5

Spectrum Analysis

We have succeeded in suppressing the backgrounds to be negligible levels. Since the contribution from the Inner Bremsstrahlung component is expected to be dominant in the $\pi^+\pi^0\gamma$ sample, we have to develop a method to determine how much the DE component and the INT component contribute to the decay. The IB component is calculated reliably by QED; it might be assumed that the difference between the number of observed events and the expected number of the IB-component events is originated from the DE component. However, this method requires the knowledge of the $K^+$ flux, reconstruction efficiencies and numerous factors to calculate the absolute number of the IB events, and the DE and INT components cannot be separated.

The analysis method employed here is to choose a kinematic variable in which the IB, DE and INT components have significantly different distributions. The UMC simulation is performed to reproduce the distributions of the variables for each component. The relative amount of the components which gives the best fit to the observed distributions is determined, and yields a ratio of the DE and INT components to the IB component.

If this spectrum-fitting method is taken to separate the DE and INT components from the IB component, it should have been confirmed that the UMC simulation reproduces the real data sufficiently.

In this chapter, it is confirmed at first that the UMC reproduces the real data; the fitting of spectra to the IB, DE and INT components of $K^+ \rightarrow \pi^+\pi^0\gamma$ are subsequently described. The results on the measurement in this analysis and the systematic uncertainties are presented at the end of this chapter.

5.1 Spectra of the Kinematic Variables

The kinematic variable $W$, which has already been defined and discussed in Chapter 1, consists of the $\pi^+$ momentum ($P_{\pi^+}$ in Fig.5.1) and kinetic-energy ($E_{\pi^+}$ in Fig.5.2), the energy of the radiated photon ($E_{\gamma}$ in Fig.5.3), and the cosine of the angle between the $\pi^+$ and the radiated photon ($\cos \theta_{\pi^+\gamma}$ in Fig.5.4). Significant differences in the spectra of the IB and DE
samples in the UMC are useful to separate the IB and DE components. The DE component is expected to be a few percent in the real data, and the spectra of variables observed in the real data should basically similar to the spectra of the IB component in the UMC.

Figure 5.1: Spectra of the $\pi^+$ momentum in real data(left), and in the IB(center) and DE(right) components of the UMC data.

Figure 5.2: Spectra of the $\pi^+$ kinetic-energy in real data(left), and in the IB(center) and DE(right) components of the UMC data.
Figure 5.3: Spectra of the photon energy in real data (left), and in the IB (center) and DE (right) components of the UMC data.

Figure 5.4: Spectra of the cosine of the angle between $\pi^+$ and the radiated photon in real data (left), and in the IB (center) and DE (right) components of the UMC data.
The spectra of the IB and DE components are most significantly different in the W spectrum (Fig. 5.5). The W is adopted to be used in this analysis for separating the IB and DE components.

![Graphs showing spectra of W in real data (left), and in the IB (center) and DE (right) components of the UMC data.]

Figure 5.5: Spectra of W in real data (left), and in the IB (center) and DE (right) components of the UMC data.

5.2 Consistency between the Real Data and UMC

An additional cut of $W < 0.4$ is imposed in order to select a pure sample of the IB component in real data. The distributions of the real data and the IB component in the UMC are compared; the $\pi^+$ momentum, the $\pi^+$ energy, the energy of the radiated photon and the angle between $\pi^+$ and the radiated photon are shown in Fig. 5.6. There are good agreements between the real data and the UMC data.
Figure 5.6: Distribution of the $\pi^+$ momentum (top left), $\pi^+$ kinetic-energy (top right), photon energy (bottom left), and the angle between $\pi^+$ and the radiated photon (bottom right). The dots are from real data, and the dashed line is from the IB component of the UMC.
5.2.1 Method of Spectrum Fitting

In order to verify the consistency of the real data (with $W < 0.4$) and the IB component in the UMC, a fitting of the UMC to the real is performed.

Chi-square

The Maximum Likelihood method is employed to fit the UMC data to the real data. If a likelihood function is expressed as $\mathcal{L}$, it is known that the minimum of $-2\ln \mathcal{L}$ follows a $\chi^2$ distribution[4]. Bins of a histogram contain numbers of events as integers. The distribution of the errors obeys a Poisson distribution. Let $n_i$ be the number of the real data in the $i$-th bin, and $y_i^{IB}$ be the corresponding number of events from the IB component in the UMC data. $AMP$ is the coefficient to be determined by this fitting procedure. The chi-square is defined as[4]:

$$\chi^2 = 2\sum [y_i - n_i + n_i \cdot \ln(n_i/y_i)].$$  \hspace{1cm} (5.1)

$$y_i \equiv AMP \times y_i^{IB}$$ \hspace{1cm} (5.2)

The error in $AMP$ for this fitting is estimated by finding the variables which add one to the chi-square value.

Finding the Best Variables.

For finding the best variables, all possible values of $AMP$ are substituted for Eq.5.1. The variable is determined if the variable provides the minimum chi-square value. Although this method takes a lot of time for calculation, this method is reliable.

The $W$ spectrum from the real data is fitted by the UMC IB data and the minimum chi-square is 11.2 with number of degree of freedom(n.d.f) is 14. The result of this fitting is shown in Fig.5.7. The UMC IB is consistent with real data with $W < 0.4$.

5.3 DE Component in the $W$ Spectrum

Since it has been confirmed that the UMC reproduces the spectra of the real data, we proceed to extract the DE component by the spectrum fitting.

5.3.1 Method

Chi-square

The chi-square is defined in the same way as having been described in Section 5.2.1. Here the spectrum in the real data in $0.1 < W < 0.9$ is fitted to the IB spectrum and the DE spectrum from the UMC. Let $n_i$ be the number of events in real data in the $i$-th bin, and $y_i^{IB}$ and $y_i^{DE}$ be
the corresponding numbers of events from the IB spectrum and the DE spectrum of UMC, respectively. \(AMP\) and \(a^{DE}\) are the coefficients to be determined by this fitting procedure to scale the IB spectrum and the DE spectrum to real data, respectively. The chi-square is defined as:

\[
\chi^2 = 2\Sigma[y_i - n_i + n_i \cdot \ln(n_i/y_i)]
\]  

(5.3)

where

\[y_i \equiv AMP \times (y_i^{IB} + a^{DE}y_i^{DE}).\]  

(5.4)

\(a^{DE}\) represents the ratio of the DE component to the IB component.

The W spectrum from the real data is fitted with the spectra of the IB and DE components from the UMC. Since the W spectra of backgrounds events are similar to the spectrum of IB component (Section 4.5), and the number of expected background events are small comparing with the statistical fluctuation of the data, the contribution of the background events to the spectrum fitting for extracting the DE component is negligible. The result of the fitting is shown in Fig.5.8. The chi-square is 7.6 with the n.d.f of 6. \(AMP\) and \(a^{DE}\) are determined to be 0.07825 and 0.07016 ± 0.01116 in the assumption that the IB and DE components in the UMC data, without the INT component, explain the shape of the spectrum in the real data.

As a cross check, the W spectrum is fitted only to the spectrum of the IB component. The chi-square is 49.0 with the n.d.f of 7. The sole IB component does not explain the shape of the spectrum in the real data.
5.4 Systematic Uncertainties

Systematic uncertainties on $a_{DE}^{DE} = 0.07016 \pm 0.01116$ should be estimated, although many sources of systematic uncertainties have been removed by taking a ratio to the IB spectrum. Incomplete detector simulation in the UMC could be an origin of systematic uncertainties, which are estimated by varying various parameters and assessing the effects on $a_{DE}^{DE}$. This estimation method is based on a BNL-E787 technical note for the analysis of 1995 data set [31], which is a conservative estimation.

It is assumed that the total systematic error on $a_{DE}^{DE}$ is composed of six systematic uncertainties explained in the following sections, in which the method for estimating each uncertainty and the origin of the systematic uncertainty is described.

5.4.1 Fitting Variable

There are several variables (and spectra) of candidate to be used for fitting the real data with the UMC. The spectrum of $W$ is employed to obtain the results in this analysis. Ideally speaking, the UMC should be able to reproduce any spectrum from the real data. The spectrum of the cosine of the angle between $\pi^+$ and the radiated photon is used for calculating the branching ratio of the direct emission for a cross check. There are two branching-ratios from fitting of $W$ and the angle, respectively, and the difference is employed as the systematic uncertainty from the fitting method. The $a_{DE}^{DE}$ based on the spectrum of the angle is obtained
to be $0.075 \pm 0.0117$. The fitted spectra are shown in Fig.5.9. The estimated systematic uncertainty derived from the method is $\pm 6.9\%$.

Figure 5.9: Spectrum of the cosine of the angle between $\pi^+$ and the radiated photon (left). The dots are from the real data, and the solid line is the result by fitting with the IB spectrum and DE spectrum obtained from the UMC. The right figure is the spectrum normalized to the IB component.

### 5.4.2 $\pi^+$ Momentum Shift

In order to estimate the error due to the uncertainty in the absolute scale of the $\pi^+$ momentum, various momentum shifts are tested. The amount of variation are studied with the $\pi^+\pi^0$ data. Fig.5.10 represents the effect of the variation by using the monochromatic $\pi^+$ momentum in $K\pi2$ at 205 MeV/c. Distributions of the real and UMC data are consistent with the optimized parameters. If a positive or negative variation is imposed by hand, distributions of the real and UMC data would be significantly different.

$\Delta a^{DE}$ with a positive $\pi^+$-momentum shift of 0.3 MeV/c gives $0.07212 \pm 0.01122$, and $\Delta a^{DE}$ with a negative shift of 0.3 MeV/c gives $0.0685 \pm 0.0111$. Since we cannot distinguish the optimized distribution from those with the shift of $\pm 0.3$ MeV/c, it should be assigned as a systematic uncertainty. The uncertainty due to the pion momentum shift is $+2.8\%$ and $-2.4\%$.

### 5.4.3 Photon $z$ Position Smearing

If the $z$ positions of photons in the Barrel photon detector are shifted, it would introduce differences between the UMC and real data in the BR(DE) measurement because the photon directions change systematically. The $z$ position provides in particular the angle between
Figure 5.10: Distributions of the $\pi^+$ momentum in the $K^+ \rightarrow \pi^+ \pi^0$ data. The dots are from the real data, and the dashed line is from the UMC. The left figure is based on the optimized parameters, and the center and right figures are with the momentum shifts of -0.3 MeV/c and +0.3 MeV/c, respectively.

Pion and radiated photon. Different parameters of $z$ smearing are tested. The shift of $\pm 1.5$ cm is added to the parameter of photon $z$-position smearing. Fig.5.11 shows that the distributions of $z$ momentum valance with the optimum parameters and with the shifts. $a^{DE}$ is obtained to be $0.06402 \pm 0.01112$ and $0.07222 \pm 0.01122$ for the positive and negative changes, respectively. The systematic uncertainty is estimated to be -8.8% and 2.9%.

In the previous analysis, photon $z$ position smearing were determined so that the RMS’s of stretch functions for photon dip angles would become unity. The RMS’s for real data distributions are 3% smaller than those of the UMC distributions. If the $z$ position smearing parameter is changed by 1.5 cm, the RMS’s of the UMC distributions became 1% smaller than the real data values. The uncertainty from the smearing parameters of photon $z$ position was estimated to be +3%.

5.4.4 Attenuation Length and Speed of Light in the Barrel photon detector

Each module in the Barrel photon detector has a different attenuation length and speed of light for $z$ measurements. The attenuation length and the speed of light used in UMC are the average values of the real data. This assumption for the UMC introduces possible sources of a systematic uncertainty. The speed of light in the Barrel photon detector in the UMC is shifted by hand. Fig.5.12 shows the distribution of $z$ position with the optimum value(left), a negative shift by 5% (center) and a positive shift by 5% (right). $a^{DE}$ is obtained to be $0.06229 \pm 0.01111$ and $0.07889 \pm 0.1159$ for the positive and negative shifts, respectively. The systematic uncertainty is estimated to be -17.0% and 9.8%.
Figure 5.11: Distributions of the z momentum valance in $K^+ \to \pi^+ \pi^0$. The dots are from the real data, and the dashed line is from the UMC. The left figure is based on the optimized parameters, and the center and right figures are with the smearing-parameter shifts of -1.5 cm and +1.5cm, respectively.

5.4.5 Photon Energy

Each module in the Barrel photon detector might have a different visible fraction. The UMC assumes that all the modules have the same visible fraction, which comes from the average of those in the real data. This assumption could be a source of a systematic uncertainty. The visible fraction is varied up to the amount of 2%. Fig.5.13 shows the distribution of $\pi^0$ mass with the optimum parameter (left), a positive shift by 2% (center) and a negative shift by 2% (right). $a^{DE}$ is obtained to be $0.06229 \pm 0.01111$ and $0.07889 \pm 0.01159$ for the positive and negative shifts, respectively. The systematic uncertainty is estimated to be 11.2% and 12.4%.

5.4.6 UMC Statistics

Since the amount of UMC data is finite, the errors derived from the UMC statistics are estimated. UMC data is sub-divided into three parts, and the fitting is performed with each sample. The largest deviation from the obtained BR(DE) is systematic uncertainty from the UMC statistics. The $a^{DE}$ is 0.02349 ± 0.00381.

5.4.7 Summary of Systematic Uncertainties

All the possible systematic errors on $a^{DE}$ are obtained and are summarized in Tab.5.1. Total systematic error is estimated by adding them in quadrature. We have obtained the systematic error of +18% and −23% to $a^{DE}$ at present. The largest errors come from the photon measurements in the Barrel photon detector.
5.5 Branching Ratio for the DE Component

In the end, the branching ratio for the Direct Emission component (BE(DE)) is obtained as follows. The fitting result on the ratio of DE to IB is $a^{DE} = 0.07016 \pm 0.01116$. We have to normalize the UMC spectra by the number of events consisting each of the spectra: $N_{IB} = 258380$ and $N_{DE} = 64306$ for IB and DE (as described in Table 4.6 and Table 4.7), respectively. The acceptance factors of the IB and DE components: $A_{UMC}^{IB} = 6.84 \times 10^{-4}$ and $A_{UMC}^{DE} = 8.86 \times 10^{-4}$ from the UMC simulation (as described in Section 4.4), respectively, are used. BR(DE) is obtained by taking a ratio with the branching ratio for the IB component.
Table 5.1: Systematic uncertainties.

<table>
<thead>
<tr>
<th>error source</th>
<th>variation</th>
<th>$a^{DE}$ error</th>
</tr>
</thead>
<tbody>
<tr>
<td>pion momentum</td>
<td>+0.3MeV/c</td>
<td>2.8%</td>
</tr>
<tr>
<td></td>
<td>-0.3MeV/c</td>
<td>-2.4%</td>
</tr>
<tr>
<td>photon position</td>
<td>+5%</td>
<td>-17.0%</td>
</tr>
<tr>
<td></td>
<td>-5%</td>
<td>9.8%</td>
</tr>
<tr>
<td>photon position resolution</td>
<td>+1.5cm</td>
<td>-8.8%</td>
</tr>
<tr>
<td></td>
<td>-1.5cm</td>
<td>2.9%</td>
</tr>
<tr>
<td>visible fraction</td>
<td>+2%</td>
<td>-11.2%</td>
</tr>
<tr>
<td></td>
<td>-2%</td>
<td>12.4%</td>
</tr>
<tr>
<td>fitting method</td>
<td>fitting with the angle</td>
<td>±6.9%</td>
</tr>
<tr>
<td>UMC statistics</td>
<td>smaller sample</td>
<td>±2.1%</td>
</tr>
<tr>
<td>combined</td>
<td></td>
<td>+18%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-23%</td>
</tr>
</tbody>
</table>

predicted by QED: $BR(K_{\pi^2\gamma}(QED)) = 2.61 \times 10^{-4}$ as

$$BR(DE) = a^{DE} \times \frac{N_{DE}}{N_{IB}} \times \frac{A_{UMC}^{IB}}{A_{UMC}^{DE}} \times BR(K_{\pi^2\gamma}(QED))$$  \hspace{1cm} (5.5)

$$= (0.07016 \pm 0.01116) \times \frac{64306}{258380} \times \frac{6.84 \times 10^{-4}}{8.86 \times 10^{-4}} \times 2.61 \times 10^{-4}. \hspace{1cm} (5.6)$$

The final result for the Direct Emission component of $K^+ \rightarrow \pi^+ \pi^0 \gamma$ is

$$BR(DE, 55 < T_{\pi^+} < 90\text{MeV}) = (3.5 \pm 0.6(stat)^{+0.6}_{-0.8}(sys)) \times 10^{-6}, \hspace{1cm} (5.7)$$

where the statistical uncertainty(stat) is from the errors $\pm 0.01116$ in $a^{DE}$ in the spectrum fitting, and the systematic uncertainty(sys) is from the systematic errors +18% and -23% in $a^{DE}$. The result of the previous analysis by BNL-E787 in the 1995 data set was

$$BR(DE, 55 < T_{\pi^+} < 90\text{MeV}) = (4.7 \pm 0.8(stat) \pm 0.3(sys)) \times 10^{-6}, \hspace{1cm} (5.8)$$

and the result from the 1998 data is consistent with it. These two results are combined with their central values and statistical errors. The larger systematic error in 1998 is employed as the systematic error of the combined result. The BR(DE) in E787 is

$$BR(DE, 55 < T_{\pi^+} < 90\text{MeV}) = (3.9 \pm 0.5(stat)^{+0.6}_{-0.8}(sys)) \times 10^{-6}. \hspace{1cm} (5.9)$$

Assuming that there is only the magnetic transition for the DE amplitude (and the electric transition is negligible), the result for $BR(DE)$ gives $|M| = (1.9 \pm 0.2) \times 10^{-7}$ to the dimensionless magnetic amplitude in Eq.1.5, based on the formula of Eq.1.3. The absolute value of the reducible anomalous amplitude $M_{reducible}$ has been theoretically predicted to be

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}\[M_{\text{reducible}}\mid = 1.8 \times 10^{-7}\] with the standard \(O(p^2)\) ChPT coupling constants \(f = 93\ \text{MeV}\) and \(|G_8| = 9 \times 10^{-6}\ \text{GeV}^{-2}\) to the Eq.1.1. The result \(|M| = (1.9 \pm 0.2) \times 10^{-7}\) from E787 supports the hypothesis that the dominant contribution to the DE component is the reducible anomalous amplitude and other magnetic contributions are negligibly small or canceled out.

### 5.6 INT Components

The chi-square is defined with the IB, DE, and INT components. Let \(n_i\) be the number of the real data and \(y_i^{IB}, y_i^{DE}, y_i^{INT}\) be the corresponding number of events from the IB, DE and INT components of the UMC data. \(AMP, a^{DE}\) and \(b^{INT}\) are the coefficients to be determined by this fitting procedures.

\[
\chi^2 = 2\Sigma [y_i - n_i + n_i \cdot \ln(n_i/y_i)]
\]

where \(y_i\) is

\[
y_i = AMP \times (y_i^{IB} + a^{DE}y_i^{DE} + b^{INT}y_i^{INT})
\]

The results of fitting are \(AMP = 0.0812, a_i^{DE} = 0.06 \pm 0.01, a_i^{INT} = 0.004 \pm 0.015\); the INT component is turned out to be consistent with zero in this measurement.

### 5.7 Measurement of the Radiative \(K\pi_2\) Branching Ratio

A calculation of an absolute value of the \(K^+ \rightarrow \pi^+\pi^0\gamma\) Branching Ratio is performed with the following parameters obtained from the \(K^+ \rightarrow \mu^+\pi^0\nu\gamma\) analysis, which is currently in progress with the same trigger as for the \(K^+ \rightarrow \pi^+\pi^0\gamma\) study in the E787-1998 data set[32].

- \(K^+\) stopping fraction: \(f_s = 0.66\)
- On-line PV(with accidental loss): \(A_{PV} = 0.606\)

The acceptance of the on-line delayed coincidence is determined to be \(A_{delco} = 0.89\) with the \(K\pi_2\) data in the trigger of km21.

We have obtained the following parameters.

- UMC acceptance for IB: \(A_{UMC} = 258380/(3.77 \times 10^8) = 8.83 \times 10^{-4}\)
- The prescale factor\(N_{PS}\) of the trigger: 5.
- The number of the selected events: \(N_{ev} = 20571\). The number of the expected background events is estimated to be 242.7 at most. \(N_{ev}\) should have an error of 1.2% to the lower side.
- The total number of kaon: \(KB = 1.74 \times 10^{12}\)
By combining these numbers, we get:

\[
BR = \frac{N_{es} \cdot N_{PS}}{KB \cdot A_{UMC} \cdot F_s \cdot A_{PV} \cdot A_{delco}} \\
= \frac{20571 \times 5}{1.74 \times 10^{12} \cdot 6.85 \times 10^{-4} \cdot 0.66 \cdot 0.606 \cdot 0.89} \\
= 2.42 \times 10^{-4}
\]  

(5.12)

The measured value is 7.3% less than the value expected from QED calculation \(2.61 \times 10^{-4}\) and is within the systematic uncertainty achieved in the \(K^+ \rightarrow \pi^+ \pi^0 \gamma\) study.
Chapter 6

Conclusion

In the BNL-E787 experiment at the BNL-AGS, the events for studying the $K^+ \to \pi^+\pi^0\gamma$ decay at rest were collected by the trigger requiring a low-energy $\pi^+$ and three photons in the final state. The data whose amount is equivalent to $1.74 \times 10^{12}$ $K^+$ decays were taken in the year of 1998 and were analyzed. The most important selection-criteria for the $K^+ \to \pi^+\pi^0\gamma$ events were on the 'kinematic fit', which takes care of the kinematic constraints on the measured values. It was confirmed that the background level against $K^+ \to \pi^+\pi^0\gamma$ was negligible in the sample. The data from the Monte Carlo simulation (UMC) were generated to reproduce the spectra of the IB and DE components of the $K^+ \to \pi^+\pi^0\gamma$ decay. The spectrum of the kinematic variable $W$ in the real data, 20571 events in total, was reproduced by the IB and DE components by the fitting with a chi-square method. The branching ratio for the DE component was obtained to be

$$BR(\text{DE}, 55 < T_{\pi^+} < 90 \text{ MeV}) = (3.5 \pm 0.6 \text{ (stat)} \pm 0.6 \text{ (sys)}) \times 10^{-6}, \quad (6.1)$$

from the 1998 data. By combining it with the previous E787 result from the 1995 data set, the branching ratio is

$$BR(\text{DE}, 55 < T_{\pi^+} < 90 \text{ MeV}) = (3.9 \pm 0.5 \text{ (stat)} \pm 0.6 \text{ (sys)}) \times 10^{-6}, \quad (6.2)$$

which results in

$$|M| = (1.9 \pm 0.2) \times 10^{-7}, \quad (6.3)$$

to the amplitude of magnetic transition $M$, assuming that there is only the magnetic transition for the DE amplitude. The result is consistent with the hypothesis that the reducible anomalous amplitude with standard coupling constants is the dominant contribution to the DE component.
Appendix A

Cut condition

In this appendix, the cut positions are indicated in the distributions of the variables used for the cuts. The left, center, and right distributions are from the real data, the UMC data for the IB component and the DE component of $K^+ \rightarrow \pi^+ \pi^0\gamma$, respectively.

![Figure A.1: Cut 1: the distributions of $R_{diff}$.](image)

![Figure A.2: Cut 2: the distribution of the charged track identification in the Target.](image)
Figure A.3: Cut 3: the distributions of the photon angle with respect to the charged track.

Figure A.4: Cut 4: the distributions of $N_\gamma$. 
Figure A.5: Cut 5: visible energy in the Range Stack.

Figure A.6: Cut 5: visible energy in the EndCap A.
Figure A.7: Cut 5: visible energy in the EndCap B.

Figure A.8: Cut 5: visible energy in the V-Counter.
Figure A.9: Cut 5: visible energy in the I-Counter.

Figure A.10: Cut 5: visible energy in the Target.

Figure A.11: Cut 6: the distributions of $P_{\text{miss}}$.

Figure A.12: Cut 7: the distribution of the least distance between the photon clusters
Figure A.13: Cut 8: the distributions of $m_{\pi^+}$.

Figure A.14: Cut 9: the distributions of photon timing.

Figure A.15: Cut 10: the distributions of $\cos \theta_{\gamma}$. 
Figure A.16: Cut 11: the distributions of $E_\gamma$ versus $Z$ positon of the photons.

Figure A.17: Cut 12: the distributions of $W$. 

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Figure A.18: Cut 13: the distributions of $\mathcal{P}$.

Figure A.19: Cut 14: cut for improving combination.
Figure A.20: Cut 15: the distributions of the $\pi^+$ momentum after kinematic fit ($P_{\pi^+}$).
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