Monte Carlo Approach to String Theory
—— An Overview

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Seminar at CQUEST, Sogang Univ. '08.4.28

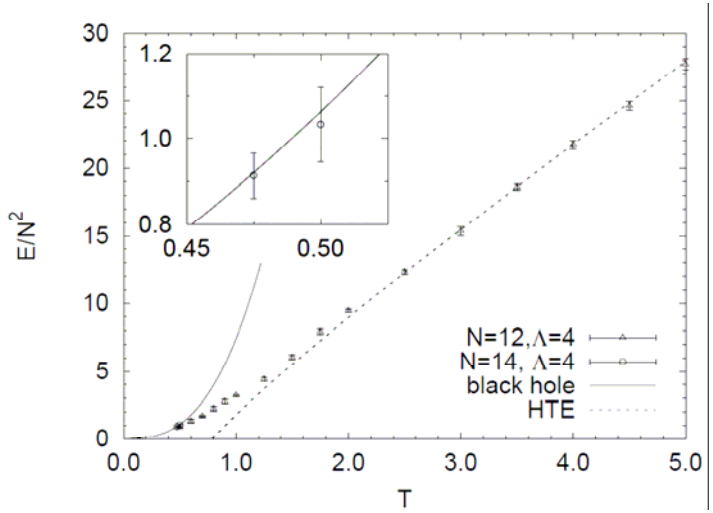
Why Monte Carlo simulations?

- Gauge/gravity duality:
  quantum field theories $\simeq$ classical gravity
  strong coupling, large N

- matrix models
  non-perturbative formulation of string theories
  play the role of lattice gauge theories for QCD

- I will explain the basic ideas and techniques
  necessary for matrix model simulations
  (in fact, simpler than field theories)

Testing gauge/gravity duality from first principles AND understanding microscopic description of black hole



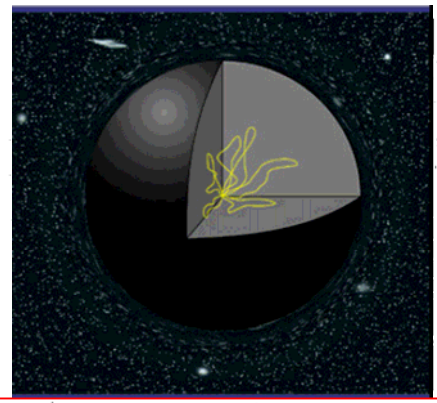
"black hole thermodynamics"

$$\frac{E}{N^2} = C\left(\frac{T}{\lambda}\right)^p$$

$$C = \frac{9}{14}\left(4^{13}15^2\left(\frac{\pi}{7}\right)^{14}\right)^{1/5} = 7.407\ldots$$

$$p = \frac{14}{5} = 2.8$$

Klebanov-Tseytlin '96



超弦理論が予測するブラックホール内部の様子。素粒子を表す多数の「弦」が、中心に端を持ちながら揺らいでいる＝高エネルギー加速器研究機構提供

ブラックホール内部 超弦理論で解明

高エネ研、スパコン使い再現

究極の理論とされる「超弦理論」に基づくブラックホール内部を、高エネルギー加速器研究機構（茨城県つくば市）などの研究チームがシミュレーションで再現し、英国のホーキング博士の理論に一致することを確かめた。謎の多いブラックホール研究を進展させると共に、極めて困難な超弦理論の証明にスーパーコンピューターのシミュレーションが有効であることを初めて示した点でも注目されそうだ。

ブラックホールは、質量が大きな星の進化の最終段階などで作られる天体。重力が極めて強いため、光さえも抜け出せないと考えられていた。これに対し、ホーキング博士は74年、真空中でも粒子と反粒子が対になって生成と消滅を繰り返す量子効果と、アインシュタインの一般相対性理論を組み合わせ、外から見るとブラックホールから熱的な放射が生じているようになるとする理論を提案。ブラックホールは最終的には「蒸発」すると主張し、大きく注目された。しかし、ブラックホールの中心付近は一般相対性理論を適用できず、どのような状態かは謎だった。

宇宙の起源解き明かせるか

超弦理論によると、ブラックホールの内部は強い重力のため、素粒子を表す「弦」が中心に凝縮。中心に両端を持つ多数の弦が揺らいだ状態になっていると予測される。

そこで研究チームは、同機構のスーパーコンピューターを使ってこの予測を検証した。「弦」の振動を周波数に応じて効率よく計算する新しい手法で計算した結果、ホーキング博士が示した理論と一致することが分かった。

研究員任者の西村淳同機構素粒子原子核研究所准教授（素粒子物理学）は「ブラックホールの性質が超弦理論で理解できたということは、ほとんど明らかでなかった超弦理論の実在を示す有力な証拠になる。今後もこの理論に基づく計算機シミュレーションによって、宇宙の起源や素粒子の性質などを解明したい」と話している。

米科学誌「フィジカル・レビュー・レターズ」（電子版）に15日掲載された。【河内敏康】

超弦理論

ことば　すべての素粒子は「粒」ではなく、一次元的な広がりを持った極めて微小な「弦」として考える理論。「超ひも理論」とも呼ばれる。弦の振動の仕方などによってさまざまな素粒子を表すことができる。

素粒子の間に働く基本的な相互作用には、電磁気力、弱い相互作用、強い相互作用、重力の4種類がある。重力以外の三つを記述する理論はあるが、重力まで含めた究極の理論はこれまでなかった。超弦理論では、重力を伝える素粒子「重力子」も含むため、四つの相互作用を統一して考えることが可能な理論として期待されている。しかし、弦の間に働く相互作用が強いため、具体的な計算が非常に難しく、実証された例はほとんどない。

Does 4d space-time emerge?

- IIB matrix model (Ishibashi-Kawai-Kitazawa-Tsuchiya '97)

  non-perturbative formulation of superstring theory

$$S = N\mathrm{tr}\left\{-\frac{1}{4}[X_\mu, X_\nu]^2 + \frac{1}{2}\psi_\alpha(\Gamma_\mu)_{\alpha\beta}[X_\mu, \psi_\beta]\right\}$$

- "moment of inertia" tensor

$$T_{\mu\nu} = \frac{1}{N}\mathrm{Tr}(X_\mu X_\nu)$$

10×10 real symmetric matrix

Eigenvalues : $\lambda_1 > \lambda_2 > \cdots > \lambda_{10}$

order parameter for the SSB of SO(10)

Does a phenomenon such as

$$\langle\lambda_1\rangle = \langle\lambda_2\rangle = \langle\lambda_3\rangle = \langle\lambda_4\rangle \gg \langle\lambda_5\rangle$$

occur in the $N \to \infty$ limit

SO(10)→SO(4)

The aim of these lectures

- Satisfy your curiosity

  Indeed MC studies of models with 16 SUSY is made possible only recently.

- Convince you that "MC sim." in general
  is a powerful and (in fact) rather easy method

  Once you know how to use it, you can add it in your "tool box"!
  Not necessarily mean that you have to do it yourself.
  You can also ask some students or experts to do it for you!

- Enable you to read papers based on MC with a lot of technical terms
  some good ones must be useful for you to gain new insights
  into the physics you are interested in.

I believe, for further developments of string theory,
Monte Carlo approach should really play an important role
(Like lattice simulation in QCD!)

I hope these lectures help you participate in such a development.

References

- H.J. Rothe (2005):
  Lattice Gauge Theories —— An Introduction

- T. DeGrand, C. De Tar (2006):
  Lattice Methods For Quantum Chromodynamics

Advertisement :

I am planning to write a review article on MC simulations focusing on matrix models based on the present lecture.

Any requests or suggestions would be very helpful!

Plan of this lecture

4/28

1. The principle of Monte Carlo simulation

   "Simulating" Gaussian matrix model

2. How to generate an ensemble for a general model

   ——— algorithm

   detailed balance AND ergodicity

3. Basic algorithms for simulating bosonic models

   heat-bath algorithm, Metropolis algorithm

   examples: bosonic IKKT model, one-matrix model

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4. Basic algorithms for fermions

   hybrid Monte Carlo algorithms

   example: 4d version of IKKT model (supersymmetric)

5. An overview of previous works and future prospects

1. The principle of Monte Carlo simulation

1. The principle of Monte Carlo simulation

- Generate configurations $\{C\}$ with the probability $\propto \mathrm{e}^{-S[C]}$

ensemble $\mathcal{E}$

real positive

$$\langle \mathcal{O} \rangle \equiv \frac{\int d\phi\, \mathcal{O}\, \mathrm{e}^{-S[\phi]}}{\int d\phi\, \mathrm{e}^{-S[\phi]}} \simeq \frac{1}{n} \sum_{C \in \mathcal{E}} \mathcal{O}[C]$$

no. of configs. in the ensemble $\mathcal{E}$

- By increasing *n*, one can obtain VEVs as accurate as one wishes.

Possible error due to finite *n* (finite statistics)
= statistical error
can be estimated from the fluctuations of $\mathcal{O}[C]$

"Simulating" Gaussian matrix model

——— the simplest possible example

ϕ : $N \times N$ hermitian matrix

$$Z = \int d\phi \, \mathrm{e}^{-S} \ , \quad S = \frac{1}{2} N \, \mathrm{tr} \phi^2$$

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int d\phi \, \mathcal{O} \, \mathrm{e}^{-S}$$

Vacuum **E**xpectation **V**alue

Exact results : $\left\langle \frac{1}{N} \mathrm{tr} \phi^2 \right\rangle = 1 \ , \quad \left\langle \frac{1}{N} \mathrm{tr} \phi^4 \right\rangle = 2 + \frac{1}{N^2} \ , \quad$ etc.

$$\phi_{ii} = \frac{A_i}{\sqrt{N}} \qquad (1 \le i \le N)$$

$$\phi_{ij} = \frac{X_{ij} + i Y_{ij}}{\sqrt{2N}} \qquad (1 \le i < j \le N)$$

$$S = \frac{1}{2} \left[\sum_{i=1}^{N} (A_i)^2 + \sum_{i<j} (X_{ij})^2 + \sum_{i<j} (Y_{ij})^2 \right]$$

The ensemble can be obtained by just generating A_i , X_{ij} , Y_{ij} as Gaussian variables.

$$\rho(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}x^2}$$

Generating Gaussian variables

- Random number generator

  ——— the heart of MC sim.

  generates a sequence of random numbers within the range [0,1) with uniform probability

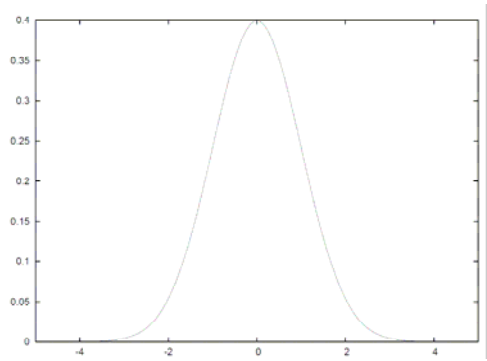
Suppose x , $y \in [0,1)$ are 2 such numbers.

- Generating Gaussian variables

$$r = \sqrt{-2\ln x} \qquad \xi = r\cos\theta$$
$$\theta = 2\pi y \qquad \eta = r\sin\theta$$

$$dx\,dy = \frac{1}{2\pi}\exp\left(-\frac{r^2}{2}\right) r\,dr\,d\theta$$
$$= \frac{1}{2\pi}\exp\left(-\frac{\xi^2+\eta^2}{2}\right) d\xi\,d\eta$$

$$\rho(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$$



ξ , η are independent Gaussian variables

2. How to generate an ensemble for a general model ——algorithm

How to generate an ensemble for a general model? —— "algorithm"

- Define the transition probability $P[C \to C']$
- Given an initial config. C_0

  one can generate (probabilistically)

$$C_0 \to C_1 \to \cdots \to C_{n-1} \to C_n \to C_{n+1} \to \cdots$$

$w_n[C]$: probability of obtaining C at n-th step

$$w_{n+1}[C] = \sum_{C'} w_n[C']\, P[C' \to C] \ , \quad w_0[C] = \delta_{C,C_0}$$

We want to choose $P[C \to C']$ in such a way that

$$\lim_{n\to\infty} w_n[C] \propto \mathrm{e}^{-S[C]}$$

How to generate an ensemble for a general model? —— "algorithm" (cont'd)

$$w_{n+1}[C] = \sum_{C'} w_n[C'] P[C' \to C]$$

Taking the $n \to \infty$ limit,

$$e^{-S[C]} = \sum_{C'} e^{-S[C']} P[C' \to C]$$

necessary condition

summing over C'

detailed balance :

$$e^{-S[C]} P[C \to C'] = e^{-S[C']} P[C' \to C]$$

a stronger condition

the flow of probability is balanced between arbitrary pairs of configurations

How to generate an ensemble for a general model? —— "algorithm" (cont'd)

Choose $P[C \to C']$ to satisfy also :

Ergodicity

For arbitrary pairs of configs. C , C'
there is a finite probability of moving from C to C'
within finite steps unless $e^{-S[C']} = 0$

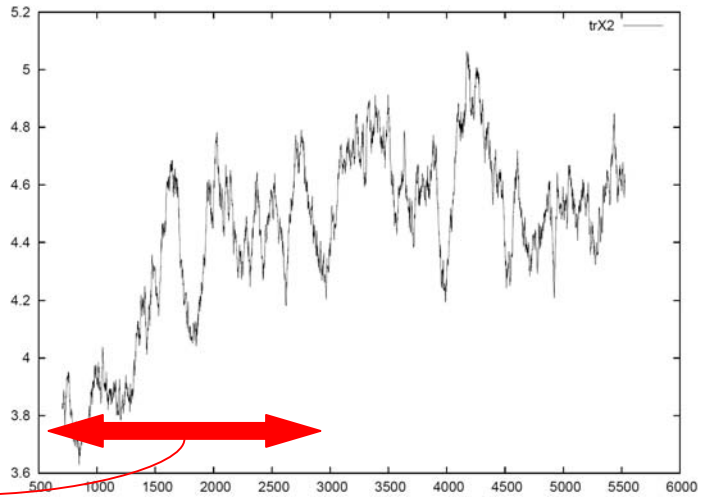
Theorem:

If { the detailed balance / the ergodicity } are satisfied, $\lim_{n\to\infty} w_n[C] \propto e^{-S[C]}$

Various choices for $P[C \to C']$ lead to various algorithms.

Thermalization and auto-correlation

$\mathcal{O}[C_n]$



history of an observable

$$\langle \mathcal{O} \rangle \simeq \frac{1}{n} \sum_{C \in \mathcal{E}} \mathcal{O}[C]$$

- thermalization

$$\lim_{n \to \infty} w_n[C] \propto \mathrm{e}^{-S[C]}$$

One has to discard sufficiently many configs.
generated before "thermal equilibrium" is achieved

- auto-correlation

should be taken into account in estimating statistical errors
jack-knife method

Efficiency = CPU time required to produce an independent config.

3. The basic algorithms for simulating bosonic models

Heat bath algorithm

- Divide the whole system into sub-systems

$$C = \{C^{(1)}, C^{(2)}, \cdots, C^{(k)}\}$$

- Update $C^{(1)}$ fixing the rest

$$C' = \{\tilde{C}^{(1)}, C^{(2)}, \cdots, C^{(k)}\}$$

with the probability $P_1[C \to C'] \propto \mathrm{e}^{-S[C']}$,

which satisfies the detailed balance

$$\mathrm{e}^{-S[C]} P_1[C \to C'] = \mathrm{e}^{-S[C']} P_1[C' \to C]$$

- Repeat this for each sub-system ➡ One sweep

bosonic IKKT model

(Hotta-J.N.-Tsuchiya NPB 545 ('99)

$$A_\mu \quad (\mu = 1, \cdots, D) \quad N \times N \text{ traceless hermitian}$$

$$\begin{aligned} S &= -\frac{1}{4} N \,\mathrm{tr}\,[A_\mu, A_\nu]^2 \\ &= N \sum_{\mu<\nu} \left[-\frac{1}{2} \mathrm{tr}(G_{\mu\nu})^2 + 2\,\mathrm{tr}(A_\mu^2 A_\nu^2) \right] \end{aligned}$$

$$G_{\mu\nu} \equiv \{A_\mu, A_\nu\} \ : \ \text{hermitian}$$

introduce auxiliary variables:

$$Q_{\mu\nu} \ (\mu < \nu) \ : \quad N \times N \text{ hermitian}$$

$$\tilde{S} = N \sum_{\mu<\nu} \left[\frac{1}{2} \mathrm{tr}(Q_{\mu\nu})^2 - \mathrm{tr}(Q_{\mu\nu} G_{\mu\nu}) + 2\,\mathrm{tr}(A_\mu^2 A_\nu^2) \right]$$

bosonic IKKT model (cont'd)

- Updating $Q_{\mu\nu}$

$$\tilde{S} = N \sum_{\mu<\nu} \left[\frac{1}{2} \mathrm{tr}(Q_{\mu\nu})^2 - \mathrm{tr}(Q_{\mu\nu} G_{\mu\nu}) + 2\,\mathrm{tr}(A_\mu^2 A_\nu^2) \right]$$

$$= \frac{1}{2} N \sum_{\mu<\nu} \mathrm{tr}(Q_{\mu\nu} - G_{\mu\nu})^2 + (Q\text{-indep.})$$

Just the same as in the Gaussian matrix model !

Generate normal Gaussian variables

$$(Q_{\mu\nu})_{ii} = \frac{A_{\mu\nu,i}}{\sqrt{N}} + (G_{\mu\nu})_{ii} \qquad (1 \le i \le N)$$

$$(Q_{\mu\nu})_{ij} = \frac{X_{\mu\nu,ij} + iY_{\mu\nu,ij}}{\sqrt{2N}} + (G_{\mu\nu})_{ij} \qquad (1 \le i < j \le N)$$

bosonic IKKT model (cont'd)

- Updating A_μ

$$\begin{aligned}\tilde{S} &= N \sum_{\mu<\nu} \left[\frac{1}{2} \mathrm{tr}(Q_{\mu\nu})^2 - \mathrm{tr}(Q_{\mu\nu} G_{\mu\nu}) + 2\,\mathrm{tr}(A_\mu^2 A_\nu^2) \right] \\ &= N \left[2\,\mathrm{tr}(S_\mu A_\mu^2) - \mathrm{tr}(T_\mu A_\mu) \right] + (A_\mu\text{-indep.})\end{aligned}$$

$$S_\mu = \sum_{\nu \neq \mu} A_\nu^2$$

$$T_\mu = \sum_{\nu \neq \mu} (A_\nu Q_{\mu\nu} + Q_{\mu\nu} A_\nu)$$

$O(N^3)$ calculations

$(A_\mu)_{11}, \cdots, (A_\mu)_{NN}$ can be updated simultaneously.

$(A_\mu)_{i_1 i_2}, \cdots, (A_\mu)_{i_{n-1} i_n}$, where $i_1, \cdots i_n$ are all different, can be updated simultaneously*.

Both by generating Gaussian variables.

(* Repeat until all the off-diagonal elements get updated.)

bosonic IKKT model (cont'd)

dynamical variables	no. of arithmetic operations for updating
$Q_{\mu\nu}$	$O(N^2)$
A_μ diagonal	$O(N^2)$
A_μ off-diagonal	$O(N^3)$

one sweep

dominant part

exact result: $\left\langle \frac{1}{N} \mathrm{tr}(F_{\mu\nu})^2 \right\rangle = D\left(1 - \frac{1}{N^2}\right)$ $\qquad F_{\mu\nu} \equiv i\,[A_\mu, A_\nu]$

Exercise 1) Write a code and plot the history of $\frac{1}{N}\mathrm{tr}(F_{\mu\nu})^2$
2) Check that the ensemble average agrees with the exact result within the statistical error

See Hotta-J.N.-Tsuchiya ('99) for $\left\langle \frac{1}{N}\mathrm{tr}(A_\mu)^2 \right\rangle$ etc.

Metropolis algorithm

Less efficient than heat bath algorithm,
but applicable to any model.
Important idea for including fermions.

- Generate a trial config. C'
  from the previous config. C
  with the probability $f(C \to C')$
  obeying $f(C \to C') = f(C' \to C)$ reversibility

- Accept C' with the probability $\min(1, \mathrm{e}^{-\Delta S})$
  $$\Delta S \equiv S[C'] - S[C]$$
  otherwise stay with C

- One can again divide the system into
  sub-systems, and visit each of them sequentially.

One-matrix model

Kawahara-J.N.-Yamaguchi, JHEP 0706 ('07)

Solvable in the large N limit.

$$S = \frac{N}{g}\left(-\mathrm{tr}\,\phi^2 + \frac{1}{4}\mathrm{tr}\,\phi^4\right)$$

$$\phi = U\Lambda U^\dagger\ , \quad \Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$$

Integrating over U, one obtains

$$\tilde{S} = \frac{N}{g}\sum_i\left(-\lambda_i^2 + \frac{1}{4}\lambda_i^4\right) - \sum_{i<j}\log|\lambda_i - \lambda_j|^2$$

Eigenval. Dist.

$$\rho(x) \equiv \frac{1}{N}\sum_{i=1}^{N}\delta(x - \lambda_i)$$

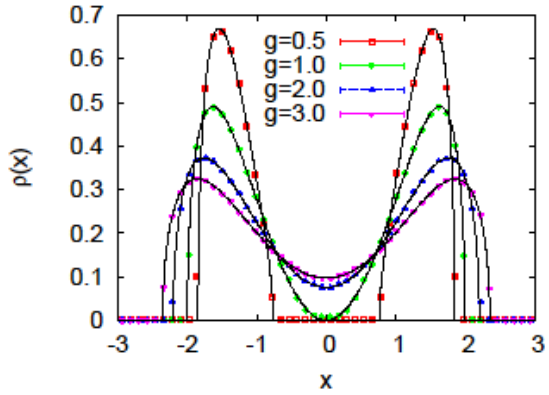


Figure 1: The eigenvalue distribution $\rho(x)$ is plotted for $g = 0.5, 1.0, 2.0, 3.0$ with $N = 32$. The curves represent the exact results (3.2), (3.3) obtained in the planar large-N limit.

Update λ_1

Choose the trial value for λ_1'
randomly within a fixed interval $[-X, X]$.
Calculate ΔS and perform the Metropolis reject/accept.

Repeat this for $\lambda_2\ , \cdots, \lambda_N$

$O(N^2)$ calculations

Summary of the first part

- Monte Carlo simulation
  Euclidean path integral formalism
  Calculation of VEVs, correlation fns, etc.
  by generating configs. and taking average.

- heat bath algorithm : efficient, but not general
  Metropolis algorithm : less efficient, but general
  important idea for including fermions

- Simulating bosonic models is very easy.
  Do try : Gaussian matrix model (< a few hours)
  bosonic IKKT model, one-matrix model

Pedagogical Lectures on Monte Carlo Simulations and Its Application to Matrix Models

Jun Nishimura

KEK & Graduate University for Advanced Studies
(SOKENDAI)

Seminar at CQUEST, Sogang Univ. '08.4.30

A brief review of the first part (Ch.1,2,3)

- Generate configurations $\{C\}$ with the probability $\propto \mathrm{e}^{-S[C]}$

ensemble $\mathcal{E}$

real positive

$$\langle \mathcal{O} \rangle \equiv \frac{\int d\phi\, \mathcal{O}\, \mathrm{e}^{-S[\phi]}}{\int d\phi\, \mathrm{e}^{-S[\phi]}} \simeq \frac{1}{n} \sum_{C \in \mathcal{E}} \mathcal{O}[C]$$

n: no. of configs. in the ensemble $\mathcal{E}$

- Define the transition probability $P[C \to C']$
- Given an initial config. C_0

  one can generate (probabilistically)

$$C_0 \to C_1 \to \cdots \to C_{n-1} \to C_n \to C_{n+1} \to \cdots$$

detailed balance :

$$\mathrm{e}^{-S[C]} P[C \to C'] = \mathrm{e}^{-S[C']} P[C' \to C]$$

Various "algorithm" —— the choice of $P[C \to C']$

- Heat bath algorithm

$$C = \{C^{(1)}, C^{(2)}, \cdots, C^{(k)}\}$$

$$C' = \{\tilde{C}^{(1)}, C^{(2)}, \cdots, C^{(k)}\}$$

with the probability $P_1[C \to C'] \propto \mathrm{e}^{-S[C']}$

- Metropolis algorithm

propose a trial config. C' with probability $f(C \to C')$

such that $f(C \to C') = f(C' \to C)$ reversibility

Accept C' with the probability $\min(1, \mathrm{e}^{-\Delta S})$

$$\Delta S \equiv S[C'] - S[C]$$

otherwise stay with C

Plan of this lecture

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1. The principle of Monte Carlo simulation

   "Simulating" Gaussian matrix model

2. How to generate an ensemble for a general model

   ——— algorithm

   detailed balance AND ergodicity

3. Basic algorithms for simulating bosonic models

   heat-bath algorithm, Metropolis algorithm

   examples: bosonic IKKT model, one-matrix model

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4. Basic algorithms for fermions

   hybrid Monte Carlo algorithms

   example: 4d version of IKKT model (supersymmetric)

5. An overview of previous works and future prospects

4. The basic algorithms for fermions

How to treat fermions in simulations

$$S_{\mathrm{f}} = -\bar{\psi}_i \mathcal{M}_{ij}[\phi] \psi_j$$

There is **no way** to put Grassmann variables, as they are, on a computer!

$$\begin{aligned} Z_{\mathrm{f}} &= \int d\psi \, d\bar{\psi} \, \mathrm{e}^{-S_{\mathrm{f}}[\psi,\bar{\psi},\phi]} \\ &= \det \mathcal{M}[\phi] \end{aligned}$$

fermion determinant

One has to simulate : size : $\mathcal{N} \propto$ no. of d.o.f.

$$S_{\mathrm{eff}}[\phi] = S_{\mathrm{b}}[\phi] - \ln \det \mathcal{M}[\phi]$$

requires $O(\mathcal{N}^3)$ arithmetic operations

- matrix models : $\mathcal{N} \propto N^2$
- field theories : $\mathcal{N} \propto V \equiv L^D$

One needs clever techniques to deal with them efficiently.

Hybrid Monte Carlo (HMC) algorithm

e.g.) bosonic IKKT model

$$S[A] = -\frac{1}{4} N \, \mathrm{tr}[A_\mu, A_\nu]^2$$

introduce auxiliary variables

$$X_\mu \quad (\mu = 1, \cdots, D) \ : \ N \times N \text{ hermitian}$$

$$\tilde{S}[A, X] = \frac{1}{2} \mathrm{tr}(X_\mu)^2 + S[A]$$

As a particular way of proposing a trial config.
in Metropolis algorithm, we consider
an auxiliary classical dynamics regarding

$$X_\mu \ : \text{ the conjugate momentum of } A_\mu$$
$$\tilde{S}[A, X] \ : \text{ the Hamiltonian}$$

HMC algorithm (cont'd)

$$\tilde{S}[A,X] = \frac{1}{2}\,\mathrm{tr}(X_\mu)^2 + S[A]$$

Hamiltonian eq.

Molecular Dynamics

$$\begin{cases} \dfrac{dA_\mu}{d\tau} = \dfrac{\partial \tilde{S}[A,X]}{\partial X_\mu} = X^*_\mu \\ \dfrac{dX_\mu}{d\tau} = -\dfrac{\partial \tilde{S}[A,X]}{\partial A_\mu} = -\dfrac{\partial S[A]}{\partial A_\mu} \end{cases}$$

"force term"
main part of
The calculation

Solve it for a fixed "time" interval T

$$(A,X) \to (A',X')$$

old config. trial config.

One trajectory

reversibility OK

$\tilde{S}[A,X] = \tilde{S}[A',X']$ ⇒ always accepted

HMC algorithm (cont'd)

- Can one keep on generating new configs.
  by the Molecular Dynamics alone?

NO! Ergodicity problem

e.g.) Configs. with different $\tilde{S}$ cannot be reached.

➡ refresh momenta *X* after each trajectory

$$\tilde{S}[A, X] = \frac{1}{2}\,\mathrm{tr}(X_\mu)^2 + S[A]$$

One can update *X* by generating Gaussian variables.

Hence the name : "hybrid" Monte Carlo

HMC algorithm (cont'd)

- In practice, Hamilton eq. should be discretized.

reversibility → should be respected

$\tilde{S}[A, X] = \tilde{S}[A', X']$ → will be sacrificed

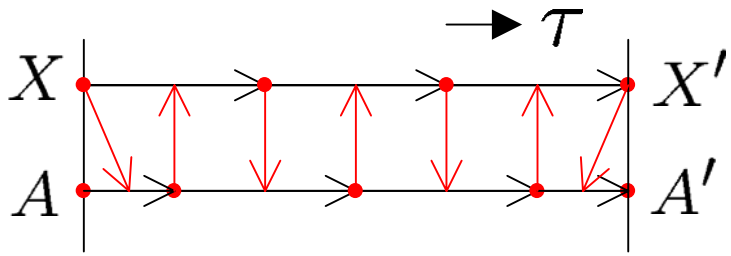
Metropolis procedure :

accept (*A'*,*X'*) with the probability $\min(1, \mathrm{e}^{-\Delta\tilde{S}})$

$$\Delta\tilde{S} = \tilde{S}[A', X'] - \tilde{S}[A, X]$$

Then the detailed balance will still be satisfied.

leap-frog discretization



4d version of IKKT model

Ambjorn-Anagnostopoulos-Bietenholz-Hotta-J.N. JHEP07,013 ('00)

$A_\mu \quad (\mu = 1, \cdots, 4)$: $N \times N$ traceless hermitian

$\psi_\alpha \ , \bar{\psi}_\alpha \quad (\alpha = 1, 2)$: $N \times N$ traceless matrices with Grassmann entries

$$
\begin{aligned}
S_{\mathrm{b}}[A] &= -\frac{1}{4} N \, \mathrm{tr}[A_\mu, A_\nu]^2 \\
S_{\mathrm{f}}[A] &= -(\Gamma_\mu)_{\alpha\beta} \, \mathrm{tr}(\bar{\Psi}_\alpha [A_\mu, \Psi_\beta]) \\
&= -\bar{\psi}_{a\alpha} \mathcal{M}_{a\alpha b\beta} \psi_{b\beta}
\end{aligned}
$$

$$
\mathcal{M}_{a\alpha,b\beta} \equiv (\Gamma_\mu)_{\alpha\beta} \mathrm{tr}(t_a [A_\mu, t_b])
$$

$2(N^2-1) \times 2(N^2-1)$ matrix

$$
\begin{aligned}
\Psi_\alpha &= \sum_a \psi_{a\alpha} t^a \\
\bar{\Psi}_\alpha &= \sum_a \bar{\psi}_{a\alpha} t^a
\end{aligned}
$$

generators of SU(N)

$$
Z = \int dA \, \mathrm{e}^{-S_{\mathrm{b}}[A]} \det \mathcal{M}[A]
$$

real positive

4d version of IKKT model (cont'd)

- apply HMC

$$S[A] = S_{\rm b}[A] - \ln \det \mathcal{M}[A]$$

$$\frac{dX_\mu}{d\tau} = -\frac{\partial S}{\partial A_\mu} \qquad \text{"force term"}$$

$$= -\frac{\partial S_{\rm b}}{\partial A_\mu} - \mathrm{tr}\left(\mathcal{M}^{-1}\frac{\partial \mathcal{M}}{\partial A_\mu}\right)$$

needs to be calculated
at each Molecular Dynamics step

At the end of each trajectory,

$\det \mathcal{M}$ needs to be calculated

		comp. effort	c.f.)bosonic models
matrix models :	$\mathcal{N} \propto N^2$	$O(N^6)$	$O(N^3)$
field theories :	$\mathcal{N} \propto V \equiv L^D$	$O(V^2)$	$O(V)$

pseudo-fermions

a crucial trick for further efficiency

- Represent fermion determinant by integration over auxiliary bosonic variables

$$\det \mathcal{M}[A] \propto \int dF dF^* \mathrm{e}^{-F^* \mathcal{K}[A] F}$$

real positive

$\mathcal{D} = \mathcal{M}^\dagger \mathcal{M}$: all the eigenvalues > 0

$$\mathcal{K}[A] = \mathcal{D}[A]^{-1/2}$$

rational approx.

$$x^{-1/2} \simeq a_0 + \sum_{i=1}^{Q} \frac{a_i}{x + b_i}$$

a_i , b_i : real positive

can be optimized for $\epsilon \leq x \leq 1$

$$\mathcal{K} = \mathcal{D}^{-1/2} \simeq a_0 + \sum_{i=1}^{Q} a_i (\mathcal{D} + b_i)^{-1}$$

pseudo-fermions (cont'd)

- Apply HMC to the whole system

$$S_{\mathrm{PF}}[A,F,F^*] = S_{\mathrm{b}}[A] - a_0 F^* F - \sum_{i=1}^{Q} a_i \, F^* (\mathcal{D}[A] + b_i)^{-1} F \equiv G_i$$

the main task boils down to solving linear eq.

$$(\mathcal{D}[A] + b_i)\, G_i = F$$ for a given *F*

instead of calculating $\det \mathcal{M}$, $\mathcal{M}^{-1}$

Conjugate gradient method

➡ iterative multiplication of $(\mathcal{D}[A] + b_i)$

{ matrix models : $O(N^3)$
field theories : $O(V)$ arithmetic operations

comparable to bosonic models!

Multi-mass CG solver

- Actually, one does not have to solve

$$(\mathcal{D}[A] + b_i)\, G_i = F$$

for each of b_i $(i = 1, \cdots, Q)$ separately.

Solve it for the smallest b_i with the CG method

➡ The solution for larger b_i can be obtained as a by-product of the CG procedure.

Jegerlehner, hep-lat/9612014

Thus, one can save the factor of Q.

c.f.) typically $Q \sim 10$

Hybrid Monte Carlo
rational approximation
multi-mass CG solver

Clark-Kennedy-Sroczynski('05)

Rational Hybrid Monte Carlo

The standard algorithm for QCD, SUSY theories

5. An overview of previous works and future prospects

An overview of previous works and future prospects

- Bosonic models can be studied very easily.
- applications:

  including Myers terms to bosonic IKKT model

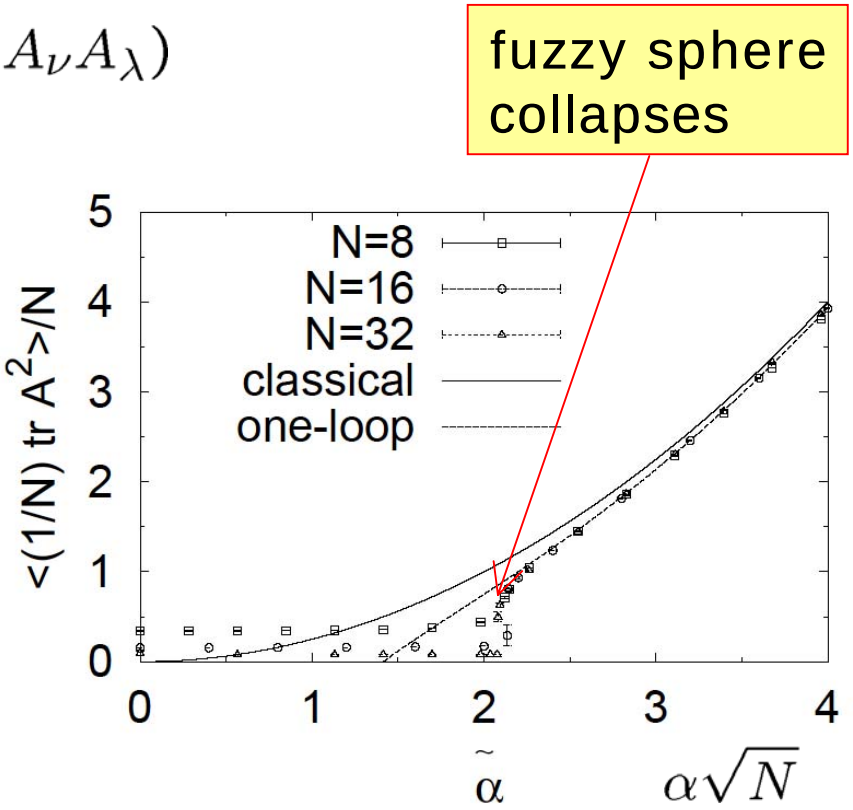
$$S_3 = \frac{2}{3} i \alpha N \epsilon_{\mu\nu\lambda} \mathrm{tr}(A_\mu A_\nu A_\lambda)$$

gauge theories on a fuzzy sphere

Iso-Kimura-Tanaka-Wakatsuki,
Nucl.Phys.B604,121 ('01)

fuzzy sphere becomes unstable as the coupling becomes strong.

Azuma-Bal-Nagao-J.N.,
JHEP 05,005 ('04)



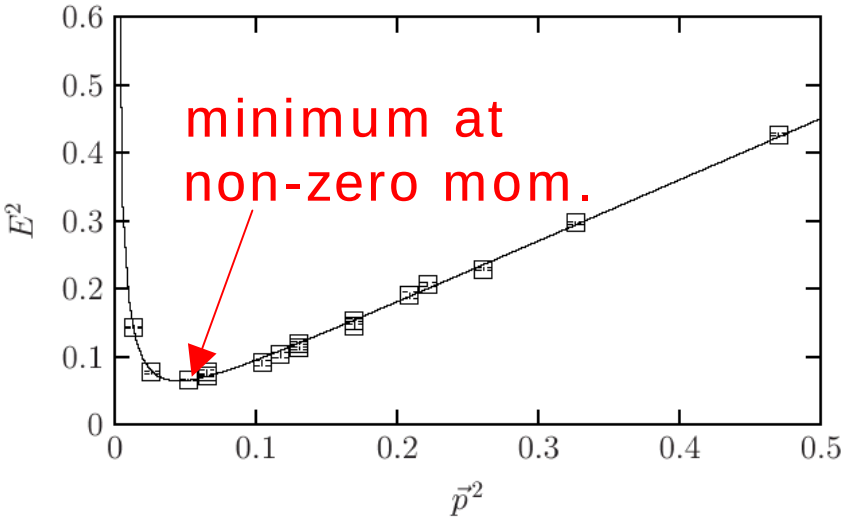
An overview of previous works and future prospects (cont'd)

Field theories on a non-commutative torus
can be formulated non-perturbatively using matrix models

Ambjorn-Makeenko-J.N.-Szabo, JHEP 05,023 ('00)

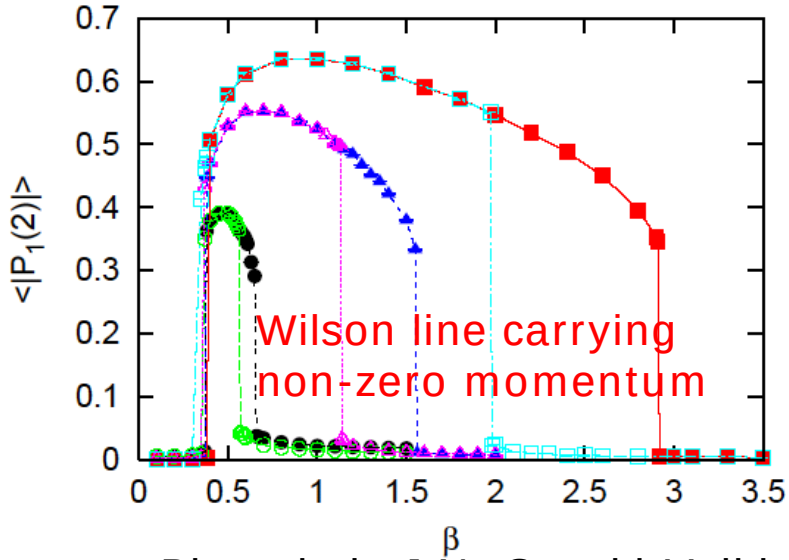
Spontaneous breakdown of translational symmetry
due to UV/IR mixing effect

(2+1)d scalar field theory



Bietenholz-Hofheinz-J.N.,
JHEP 06,042 ('04)

(2+2)d U(1) gauge theory



Bietenholz-J.N.-Susaki-Volkholz,
JHEP 10,042 ('06)

An overview of previous works and future prospects (cont'd)

- 1d SUSY gauge theories at finite temperature

  Anagnostopoulos-Hanada-J.N.-Takeuchi, Phys.Rev.Lett.100,021601('08)

  Fourier mode simulation using RHMC algorithm

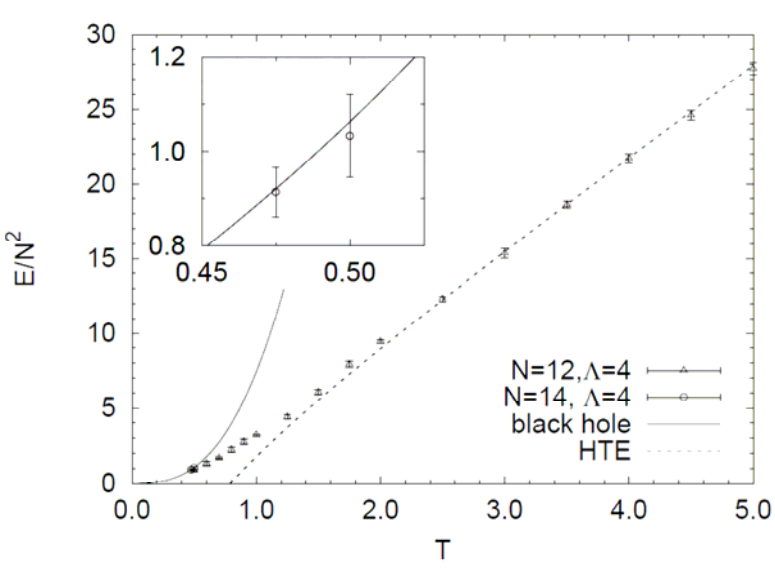
  ⇒ The first non-perturbative studies of a system with 16 supercharges

  Gauge/gravity correspondence
  We can "study" gravity from gauge theory !

Testing the correspondence
- at the operator level
- in the case with less SUSY

Extension to higher dimensions
various proposals
using lattice approach

Kaplan, Sugino, Catterall,...



An overview of previous works and future prospects (cont'd)

- "Partial use" of Monte Carlo simulation

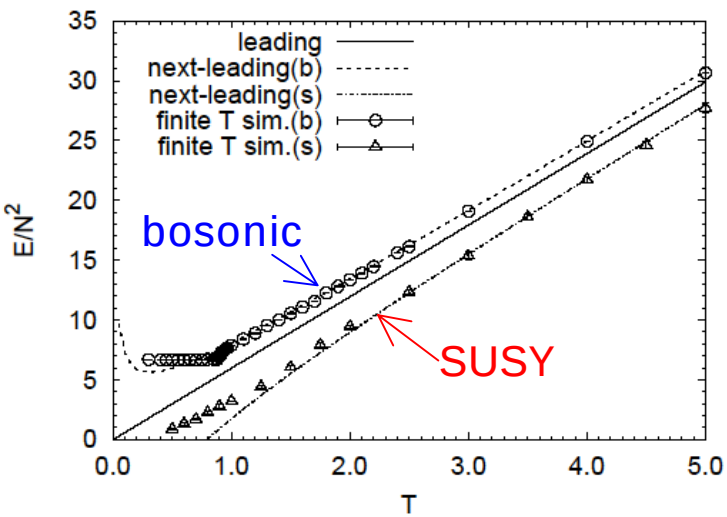
High temperature expansion of 1d SUSY gauge theory

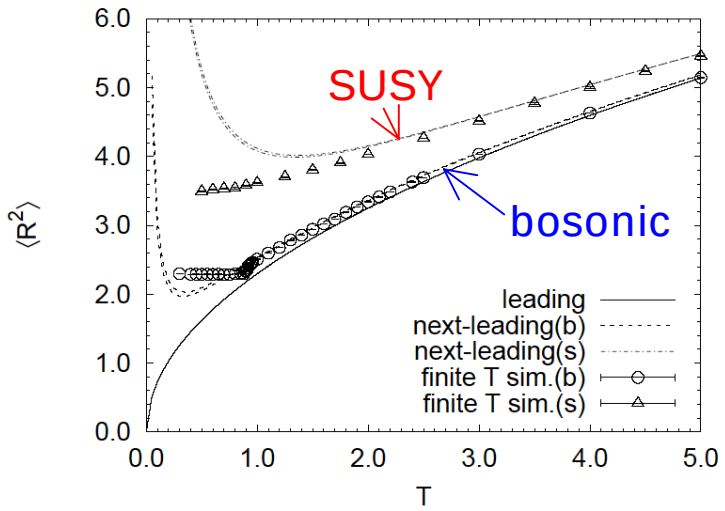
Kawahara-J.N.-Takeuchi, JHEP 12,103 ('07)

$$\frac{\partial}{\partial t} \to \begin{cases} 2\pi i n T & (n = 0, \pm 1, \pm 2, \cdots) \quad \text{bosons} \\ 2\pi i r T & (r = \pm\frac{1}{2}, \pm\frac{3}{2}, \cdots) \quad \text{fermions} \end{cases}$$

At high T, one can integrate out all the modes except the bosonic zero mode

➡ bosonic IKKT model





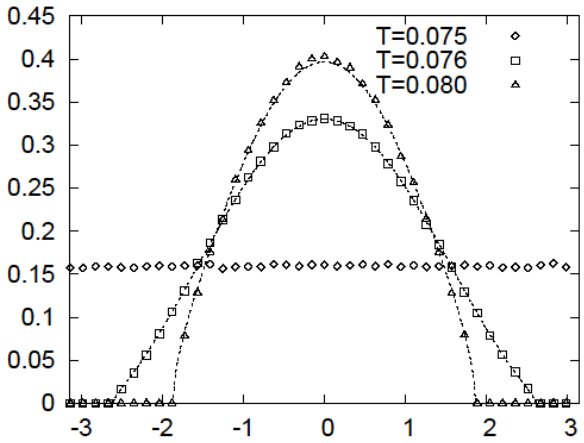
An overview of previous works and future prospects (cont'd)

plane wave matrix model μ : mass parameter

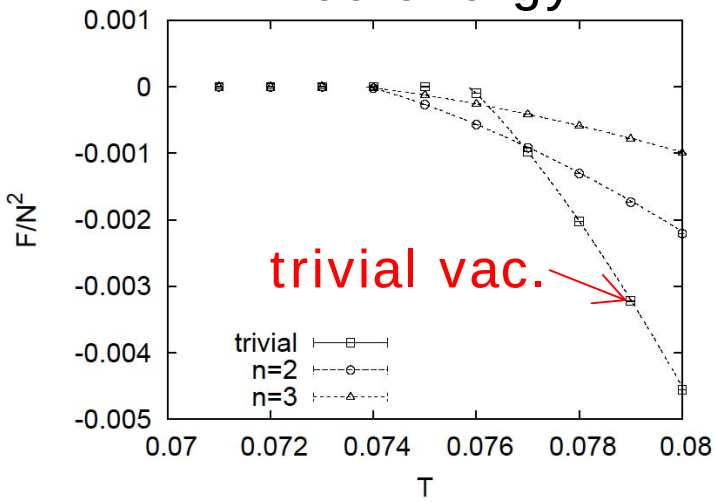
At large μ, one can integrate out
all the modes except the gauge field moduli.

only $O(N)$ d.o.f.

distribution of the gauge field moduli (trivial vac.)



free energy



Kawahara-J.N.-Yoshida JHEP 06,052 ('06)

An overview of previous works and future prospects (cont'd)

- IKKT model, BFSS model

  Fermion determinant (Pfaffian) becomes complex, but

$$Z_0 = \int dA \, \mathrm{e}^{-S_\mathrm{b}[A]} |\det \mathcal{M}[A]|$$

  can be simulated in the same way (e.g., RHMC).

The effect of the phase can be included by the reweighting method, but

sign problem

$$\langle \mathcal{O} \rangle = \frac{\langle O \, \mathrm{e}^{i\Gamma} \rangle_0}{\langle \mathrm{e}^{i\Gamma} \rangle_0}$$

both $\langle O \, \mathrm{e}^{i\Gamma} \rangle_0$ and $\langle \mathrm{e}^{i\Gamma} \rangle_0$ becomes exponentially small as N increases

An idea to sample efficiently the region in the config. space, where the fluctuation of Γ is not so violent.

Anagnostopoulos-J.N. Phys.Rev.D66,106008 ('02)

An overview of previous works and future prospects (cont'd)

- Monte Carlo simulation

a powerful method to study strongly coupled systems
from first principles
As such, one should be able to find many more places
for applications in string theory.

If you think you've found one,
you can try it out yourself
(I gave you all the basics. More details in text books)
or you can contact us.
(We can tell you whether it is feasible or not.
We may also collaborate if you wish.)

I hope MC sim has the potential to revolutionize
the research of string theory (as in the case of QCD) !