Making quantum cosmology well defined by simplices and thimbles

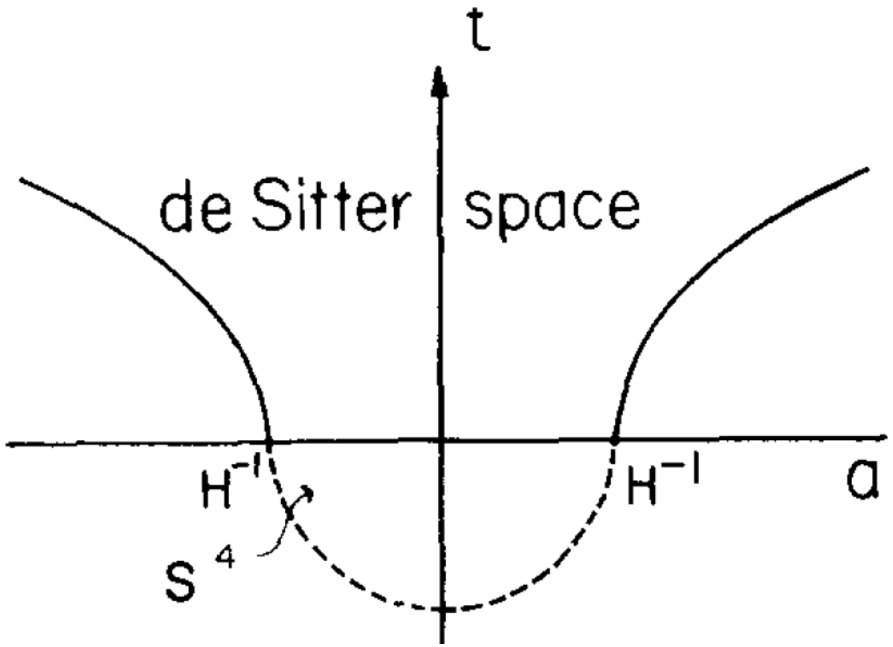
Jun Nishimura (KEK, Sokendai)

Journal club at KEK, December 3, 2021

Ref.) Ding Jia, arXiv 2110.05953[gr-qc]
(贾丁)

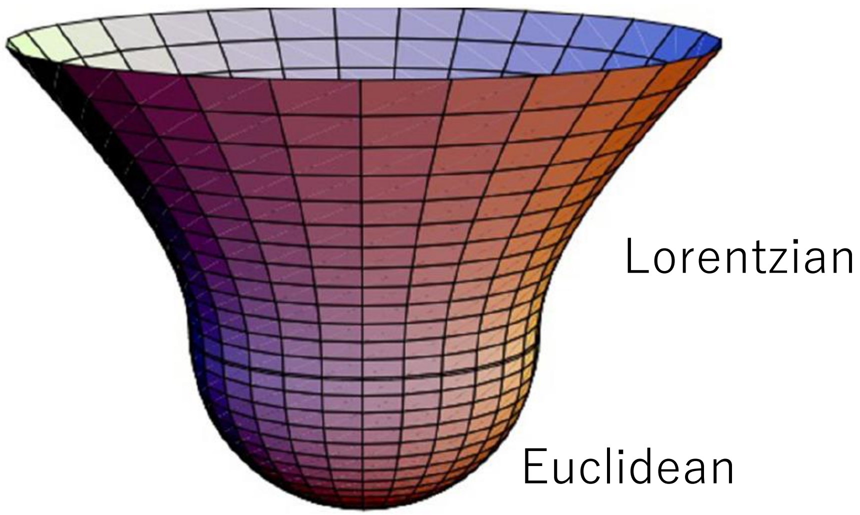
How did our Universe begin ?

A. Vilenkin
“Creation of Universes from Nothing”
Phys.Lett.B 117 (1982) 25



tunneling from nothing

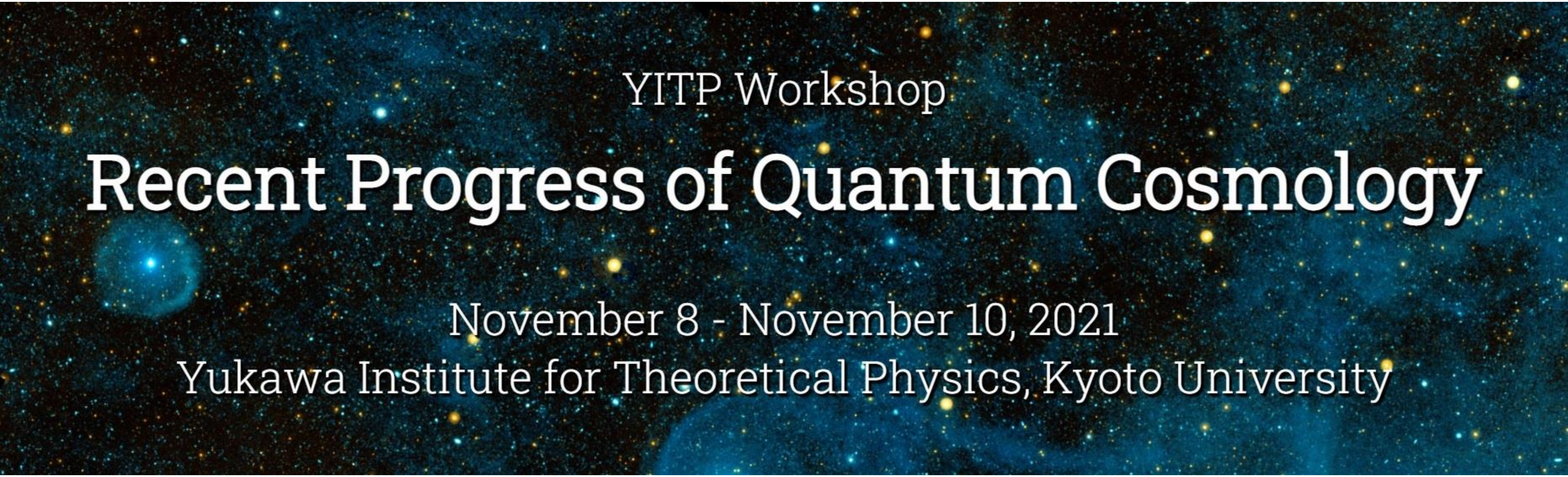
J. B. Hartle and S. W. Hawking
“Wave function of the Universe”
Phys. Rev. D 28,(1983) 2960



no boundary proposal

Quantum Cosmology

There was a YITP workshop recently dedicated to the whole subject.



Vilenkin, Hartle, Lehners, Ashtekar and many more.
Check out the videos!

non-renormalizability

QG is perturbatively **non-renormalizable.** $G_{\mathrm{N}} = [M^{-2}]$
This issue may be overcome in a nonperturbative approach.

Euclidean quantum gravity ('90s)

$$Z = \int \mathcal{D}g_{\mu\nu}\, e^{-S[g]} \qquad S = \int d^4x \sqrt{g}\left(\frac{1}{16\pi G_{\mathrm{N}}} R + \Lambda\right)$$

Dynamical Triangulation Ambjørn, Migdal, Yukawa, Hotta-Izubuchi-JN,⋯

various phases (crampled, branched polymer, crinkled) discovered

Regge Calculus Hamber, Holm-Janke, JN-Oshikawa,⋯

c.f.) pathologies of Euclidean QG

unbounded action $-\infty < \int d^4x \sqrt{g} R < \infty$

conformal mode instability $g_{\mu\nu}(x) = e^{\varphi(x)} \tilde{g}_{\mu\nu}(x) \qquad \mathcal{L} = -\partial_\mu \varphi(x) \partial_\mu \varphi(x)$

What about Lorentzian QG ?

Euclidean $Z = \int \mathcal{D}g_{\mu\nu}\, e^{-S[g]}$ $\qquad S = \int d^4x \sqrt{g} \left(\frac{1}{16\pi G_{\mathrm{N}}} R + \Lambda \right)$

Lorentzian $Z = \int \mathcal{D}g_{\mu\nu}\, e^{iS[g]}$ $\qquad S = \int d^4x \sqrt{-g} \left(\frac{1}{16\pi G_{\mathrm{N}}} R - \Lambda \right)$

Pathologies in Euclidean QG are gone!

~~unbounded action~~

~~conformal mode instability~~

However, the path integral is **NOT** absolutely convergent.

How can we make any sense out of this formal expression…

Recent excitement in Lorentzian Quantum Gravity

$$Z = \int \mathcal{D}g_{\mu\nu}\, e^{iS[g]} \qquad\qquad S = \int d^4x\sqrt{-g}\left(\frac{1}{G_{\mathrm{N}}}R - \Lambda\right)$$

Schematically, one deforms the integration contour as

$$g_{\mu\nu}(x,\tau) \in \mathbb{C} \qquad \tau\ :\ \text{deformation parameter}$$

$$\frac{dg_{\mu\nu}(x,\tau)}{d\tau} = i\,\overline{\frac{\partial S}{\partial g_{\mu\nu}(x,\tau)}} \qquad g_{\mu\nu}(x,0) = g_{\mu\nu}(x) \in \mathbb{R}$$

($\tau \to \infty$ gives Lefschetz thimbles.)

Lorentzian QG can thus be made **well-defined**. (Cauchy’s theorem). 😊

mini-superspace models (Lehners, Hartle, Turok, $\cdots$)
simplicial Lorentzian QG based on Regge calculus (Ding Jia 2110.05953)

Towards a well-defined quantum cosmology

- Discretize the Lorentzian geometry by using piecewise flat manifold (building blocks = simplices).

- Treat the link lengths as the dynamical variables using fixed triangulation (Regge calculus).

- Deform the integration contour ("thimbles") by complexifying the link lengths to define the path integral.

Monte Carlo simulation is possible just like lattice QCD !

This is the novelty of Ding Jia, arXiv:2110.05953[gr-qc].

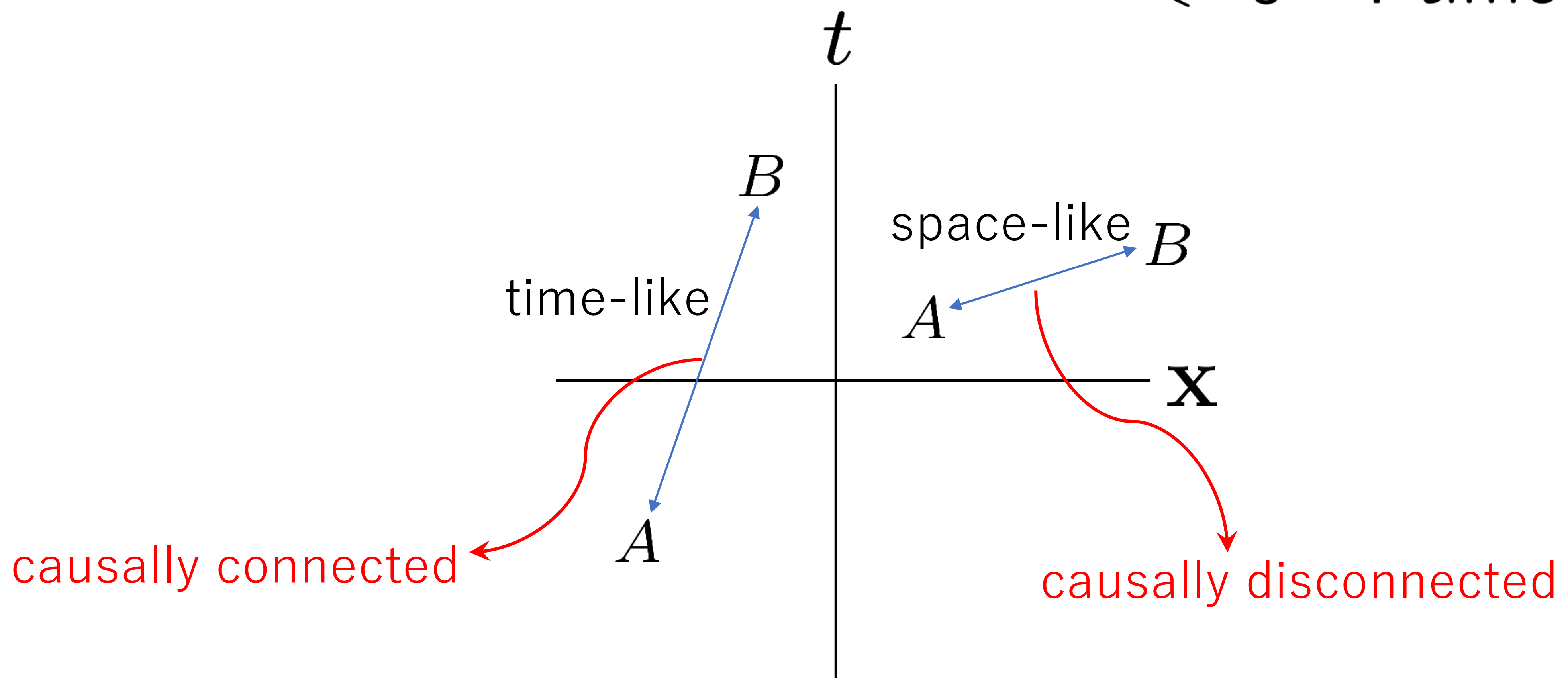
Plan of the talk

0. Introduction
1. What's needed in making Lorentzian QG well-defined
2. Constructing the action for discretized geometry
3. Monte Carlo calculations on Lefschetz thimbles
4. Summary and discussions

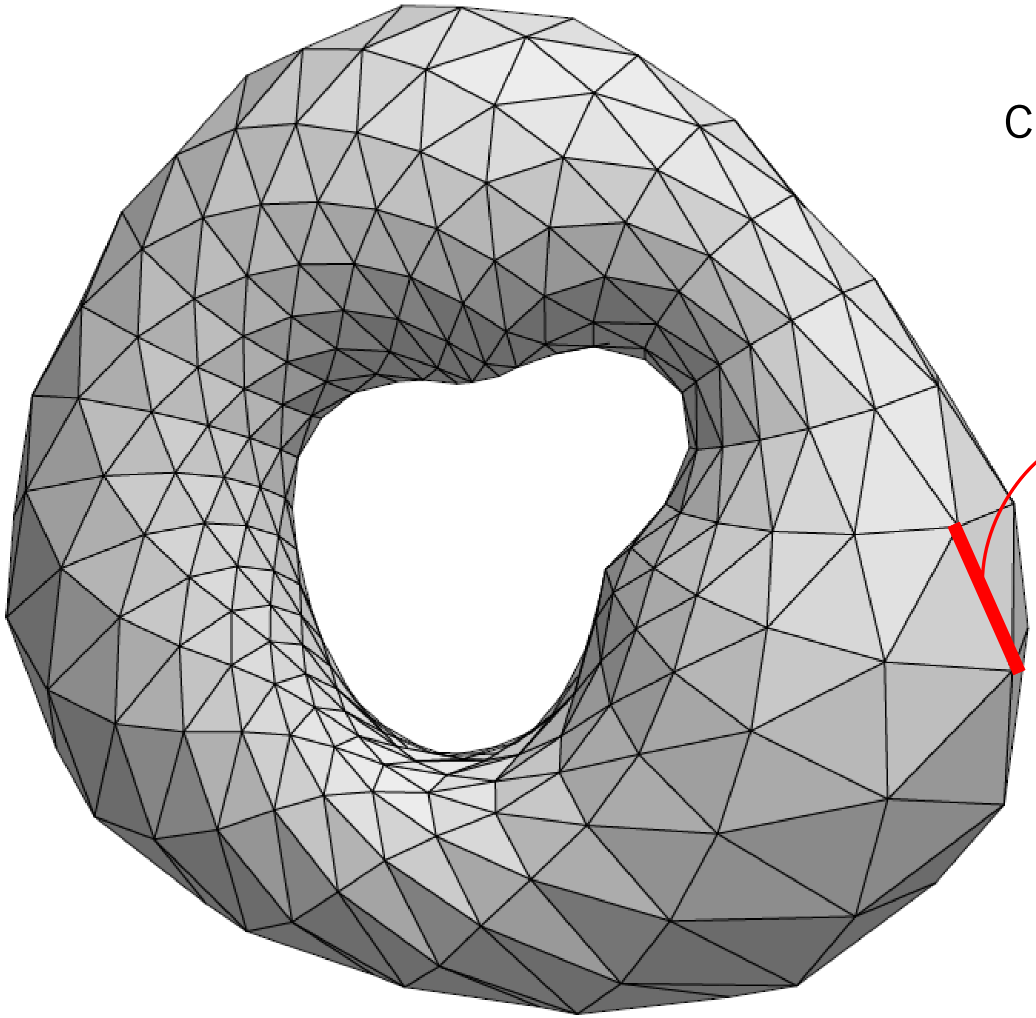
1. What’s needed in making Lorentzian QG well-defined

“distance” in Minkowski spacetime

$$d^2(A,B) = -(t_A - t_B)^2 + (\mathbf{x}_A - \mathbf{x}_B)^2 \quad \begin{array}{ll} > \ 0 & : \text{space-like} \\ < \ 0 & : \text{time-like} \end{array}$$



Lorentzian simplicial manifold



$$Z_{\mathrm{QG}} = \int \mathcal{D}g_{\mu\nu}\, e^{iS[g]}$$

continuum limit

$$Z = \int_{-\infty}^{\infty} \prod_{i:\mathrm{link}} d\sigma_i\, \mu(\sigma_i)\, e^{iS[\sigma]}$$

link $\sigma_i \in \mathbb{R}$ link length squared

$\sigma_i > 0$: space-like link

$\sigma_i < 0$: time-like link

Deform the integration contour by

$$\frac{d\sigma_i}{d\tau} = i\overline{\frac{\partial S_{\mathrm{eff}}[\sigma(\tau)]}{\partial \sigma_i}}$$

(Cauchy's theorem)

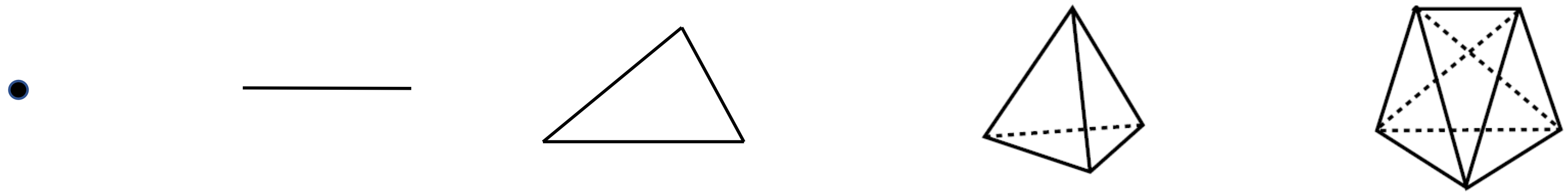
need to be holomorphic

Building block of simplicial manifold
“d-simplices”

d-simplex(単体)の複数形

Euclidean　　Consider (d+1) points in the Euclidean space and connect them

$d=0$　$d=1$　$d=2$　$d=3$　$d=4$



Lorentzian　　Consider (d+1) points in the Minkowski spacetime and connect them

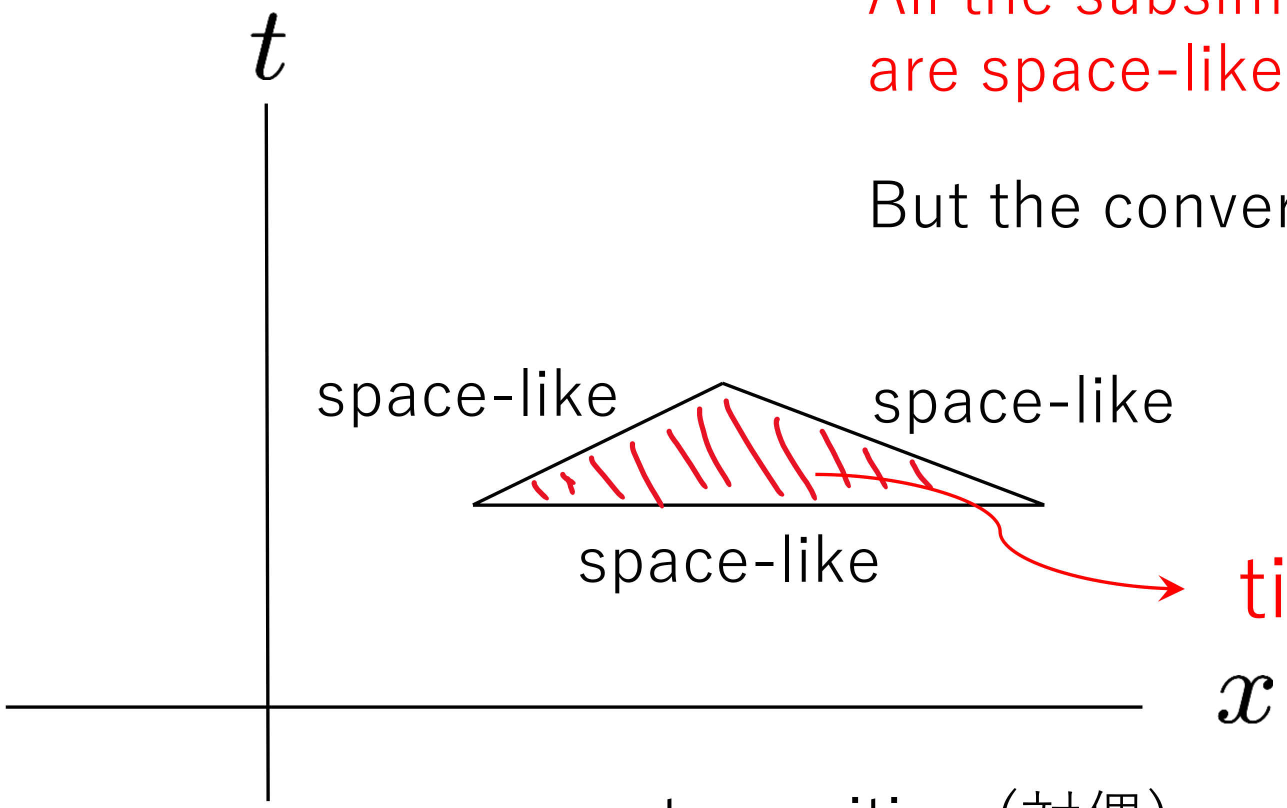
“space-like d-simplex” : <u>all the pairs of points</u> inside the simplex have space-like distances

subsimplices

p-simplices on the boundary of a d-simplex
$(p < d)$

All the subsimplices of a space-like d-simplex are space-like.

But the converse (逆) is not true.



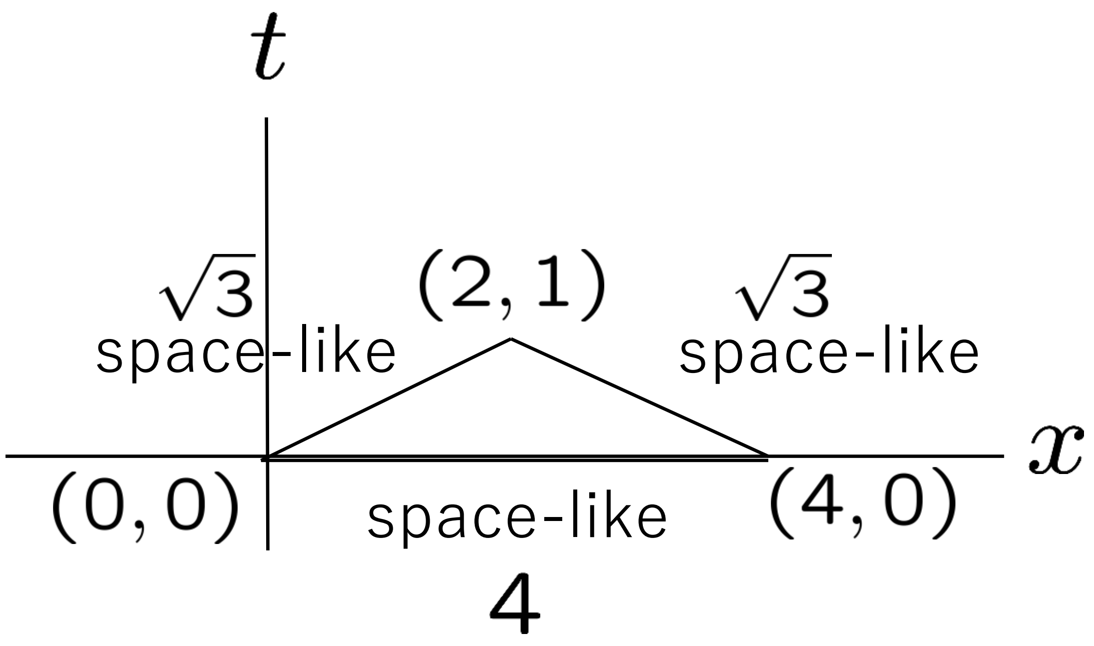
time-like 2-simplex!

contraposition (対偶) :
Any d-simplex containing a time-like subsimplex is time-like.

“triangle inequality”

Ordinary triangle inequality does not hold in Minkowski spacetime.

$$d^2(A,B) = -(t_A - t_B)^2 + (\mathbf{x}_A - \mathbf{x}_B)^2$$



$$\sqrt{3} + \sqrt{3} < 4$$

This is possible because
the triangle is actually time-like.

What is the Lorentzian version of the triangle inequality ?

What is the condition that should be satisfied by the link lengths
in order to form a *d*-simplex in Minkowski spacetime ?

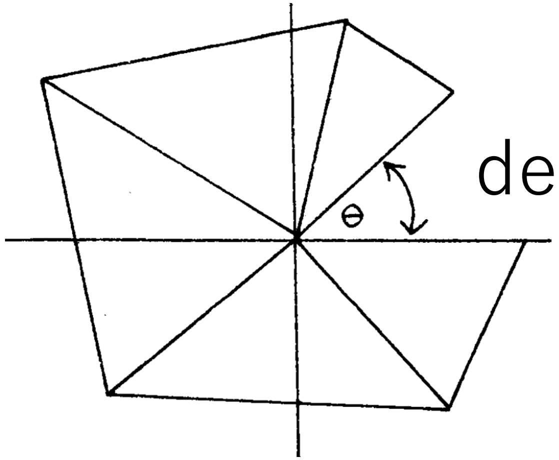
We also need to construct the action.

Lorentzian $Z = \int \mathcal{D}g_{\mu\nu}\, e^{iS[g]}$ $\quad S = \int d^4x \sqrt{-g}\left(\frac{1}{16\pi G_{\mathrm{N}}} R - \Lambda\right)$

cosmological const. term $\Lambda \int d^4x \sqrt{-g}$ volume of d-simplices

Einstein-Hilbert term $\frac{1}{16\pi G_{\mathrm{N}}} \int d^4x \sqrt{-g} R$

In simplicial manifolds,
the curvature is concentrated
on $(d-2)$-simplices



These are all well-known in Euclidean QG.
We need to extend them to Lorentzian QG respecting holomorphicity !

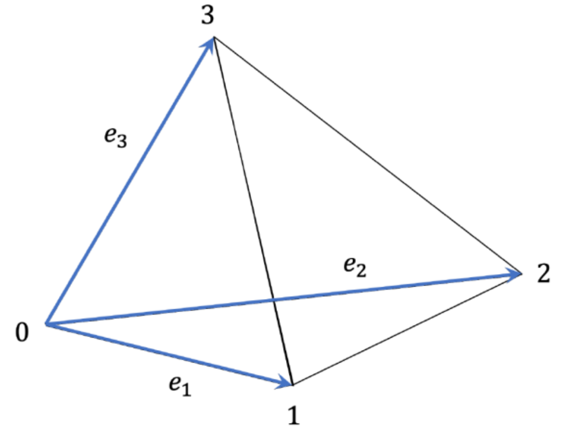
2. Constructing the action for discretized geometry

Introducing basis vectors in each simplex

Any point within the simplex :

$$x = \sum_{j=1}^{d} x_j e_j \quad (0 \le x_j \le 1,\ \sum_{j=1}^{d} x_j \le 1)$$

e.g.) 3-simplex



distance :

$$dx \cdot dx = \boxed{(e_i \cdot e_j)}\, dx_i\, dx_j$$

$\parallel$

metric : g_{ij}

link length squared : σ_{ij} $\quad i, j = 0, 1, 2, \cdots, d$

$$\sigma_{0i} = e_i \cdot e_i \qquad i = 1, 2, \cdots, d$$

$$\begin{aligned}\sigma_{ij} &= (e_i - e_j) \cdot (e_i - e_j) \qquad i, j = 1, 2, \cdots, d \\ &= \sigma_{0i} - 2 e_i \cdot e_j + \sigma_{0j}\end{aligned}$$

$$\begin{aligned} g_{ij} &= e_i \cdot e_j \\ &= \frac{1}{2}(\sigma_{0i} + \sigma_{0j} - \sigma_{ij}) \end{aligned}$$

No difference from the Euclidean case so far.

Volume of a d-simplex

$$x = \sum_{j=1}^{d} x_j e_j \quad (0 \le x_j \le 1, \ \sum_{j=1}^{d} x_j \le 1)$$

Euclidean : $\displaystyle\int dx_1 \cdots dx_d \sqrt{\det g} = \frac{1}{d!}\sqrt{\det g}$

the squared volume of a d-simplex

$$\mathbb{V} = \frac{1}{(d!)^2} \det g \qquad \begin{cases} \mathbb{V} > 0 & \text{for space-like simplex} \\ \mathbb{V} < 0 & \text{for time-like simplex} \end{cases}$$

For $d = 1$, $\mathbb{V} = \sigma_{01}$ link length squared

For $d = 2$, $\mathbb{V} = \frac{1}{16}(-\sigma_{01}^2 - \sigma_{02}^2 - \sigma_{12}^2 + 2\sigma_{01}\sigma_{02} + 2\sigma_{02}\sigma_{12} + 2\sigma_{12}\sigma_{01})$

Heron's formula

Generalized triangle inequaltities

a simplex s ⟸ link length squared σ_{ij}

σ_{ij} must obey certain inequalities
to describe a simplex

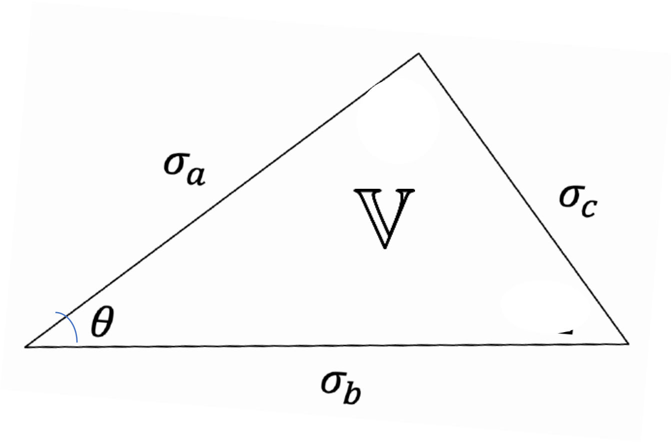
Euclidean :
$$\begin{cases} \mathbb{V}_s > 0 \\ \mathbb{V}_r > 0 \quad \text{for all subsimplies } r \text{ of } s \end{cases}$$

Lorentzian :
$$\begin{cases} \mathbb{V}_s < 0 \\ \mathbb{V}_r < 0 \Longrightarrow \mathbb{V}_t < 0 \quad \text{for all subsimplies } t \supset r \end{cases}$$

This is the Lorentzian version of the triangle inequalities !

angles as holomorphic functions of σ_i

<u>Euclidean case :</u>



cosine theorem
$$\cos\theta = \frac{\sigma_a + \sigma_b - \sigma_c}{2\sqrt{\sigma_a \sigma_b}}$$

area of the triangle

$$\begin{cases} \sqrt{\mathbb{V}} = \frac{1}{2}\sqrt{\sigma_a\sigma_b}\sin\theta \\ \mathbb{V} = \frac{1}{16}(-\sigma_a^2 - \sigma_b^2 - \sigma_c^2 + 2\sigma_a\sigma_b + 2\sigma_b\sigma_c + 2\sigma_c\sigma_a) \end{cases}$$

(Heron's formula)

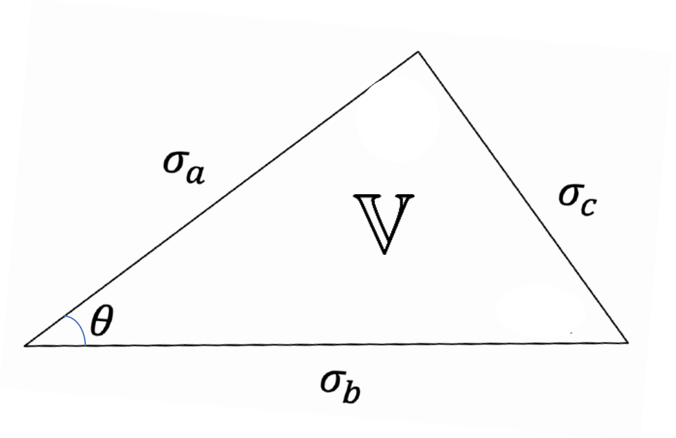
$$\Rightarrow \sin\theta = \frac{\sqrt{-\sigma_a^2 - \sigma_b^2 - \sigma_c^2 + 2\sigma_a\sigma_b + 2\sigma_b\sigma_c + 2\sigma_c\sigma_a}}{2\sqrt{\sigma_a\sigma_b}}$$

$$e^{i\theta} = \cos\theta + i\sin\theta = \frac{\sigma_a + \sigma_b - \sigma_c + i\sqrt{-\sigma_a^2 - \sigma_b^2 - \sigma_c^2 + 2\sigma_a\sigma_b + 2\sigma_b\sigma_c + 2\sigma_c\sigma_a}}{2\sqrt{\sigma_a\sigma_b}}$$

$$\boxed{\theta = -i\log\left\{\frac{\sigma_a + \sigma_b - \sigma_c + i\sqrt{-\sigma_a^2 - \sigma_b^2 - \sigma_c^2 + 2\sigma_a\sigma_b + 2\sigma_b\sigma_c + 2\sigma_c\sigma_a}}{2\sqrt{\sigma_a\sigma_b}}\right\}}$$

How to define “Lorentzian angle”

$$\theta = -i \log \left\{ \frac{\sigma_a + \sigma_b - \sigma_c + 4i\sqrt{\mathbb{V}}}{2\sqrt{\sigma_a \sigma_b}} \right\}$$



Note : $\mathbb{V} < 0$

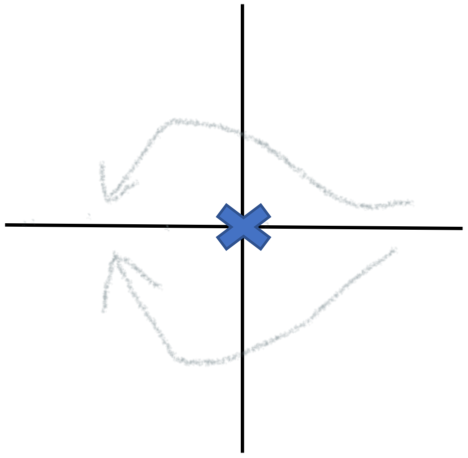
$\sigma_a, \ \sigma_b, \ \sigma_c$ can be negative

How should we choose the branch of $\sqrt{\ }$ and log ?

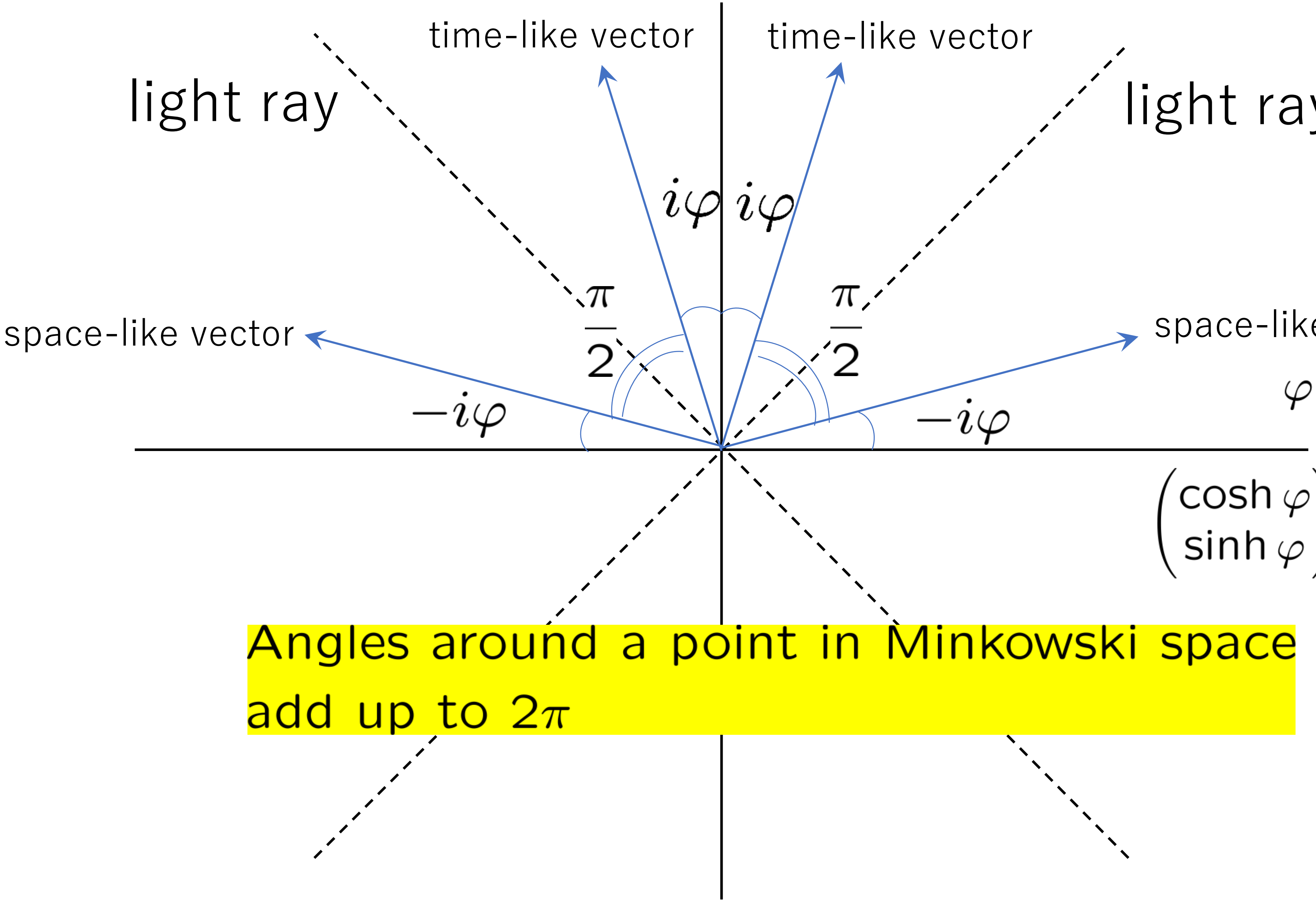
The author’s proposal :

$$\theta = -i \log \left\{ \frac{\sigma_a + \sigma_b - \sigma_c + 4i\sqrt{\mathbb{V} - 0i}}{2\sqrt{\sigma_a - 0i}\sqrt{\sigma_b - 0i}} \right\}$$

$$z = r\, e^{i\theta} \quad \left\{ \begin{array}{rll} \log z = \log r + i\theta & & -\pi < \theta \leq \pi \\ \sqrt{z} = \sqrt{r} e^{i\theta/2} & & -\pi < \theta \leq \pi \\ \sqrt{z - 0i} = \sqrt{r} e^{i\theta/2} & & -\pi \leq \theta < \pi \end{array} \right.$$



Examples of Lorentzian angles

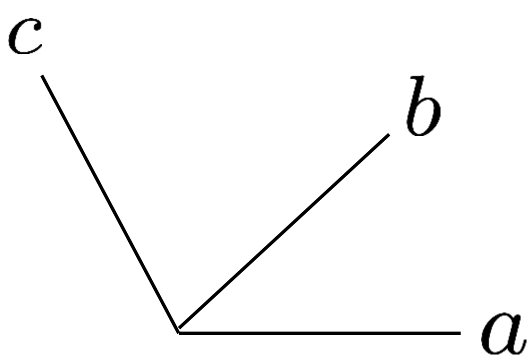


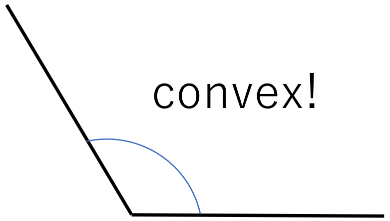
φ : boost angle

$$\begin{pmatrix} \cosh\varphi \\ \sinh\varphi \end{pmatrix} = \begin{pmatrix} \cos(-i\varphi) \\ \sin(-i\varphi) \end{pmatrix}$$

Angles around a point in Minkowski space add up to 2π

Important properties of convex Lorentzian angles





1) additivity

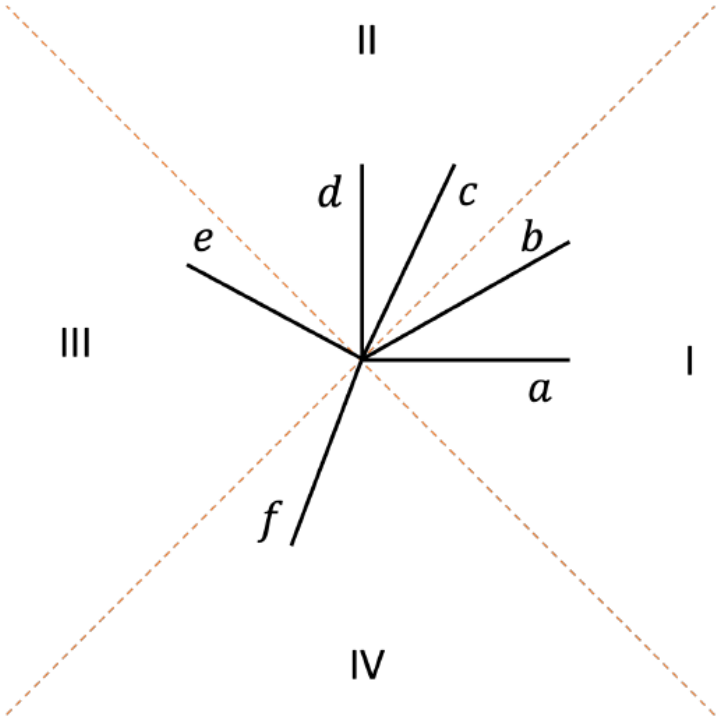
$$\theta(a,b)+\theta(b,c)=\theta(a,c)$$

2) relation to boost

$$\theta=\begin{cases}-i\varphi_{\text{boost}} & \text{angle between space-like vectors}\\ i\varphi_{\text{boost}} & \text{angle between time-like vectors}\end{cases}$$

3) angle across the light ray

$$\mathrm{Re}\,\theta=\frac{N\pi}{2} \qquad N:\ \#\text{ of light rays}$$



4) Angles around a point in Minkowski space (or angles of a Lorentzian triangle) add up to 2π

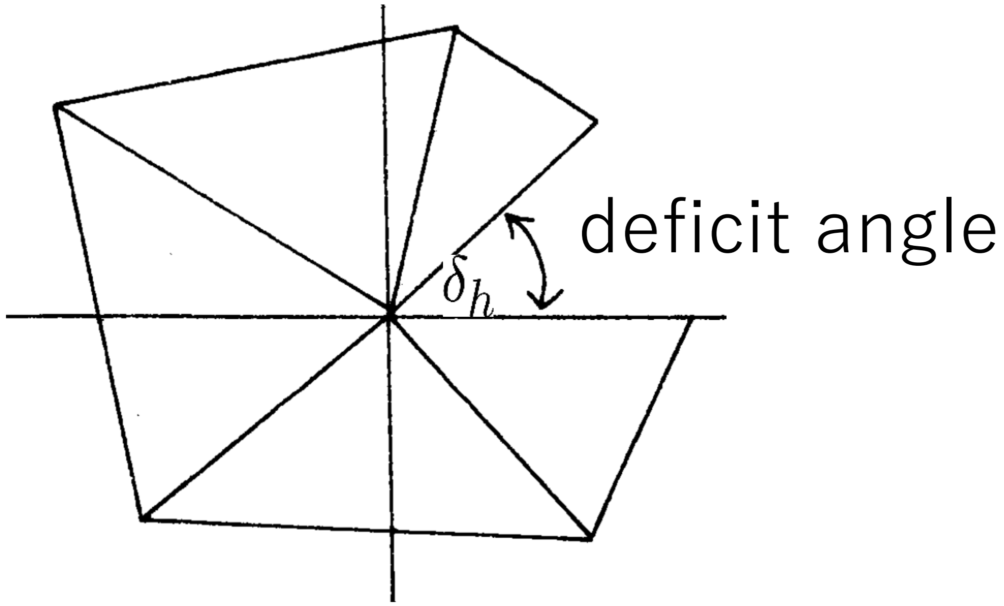
Constructing the action

cosmological constant term

$$\begin{aligned} S_{\text{c.c.}} &= -\Lambda \int d^4x \sqrt{-\det g} \\ &= i\Lambda \sum_s \sqrt{\mathbb{V}_s} \end{aligned}$$

Einstein-Hilbert term

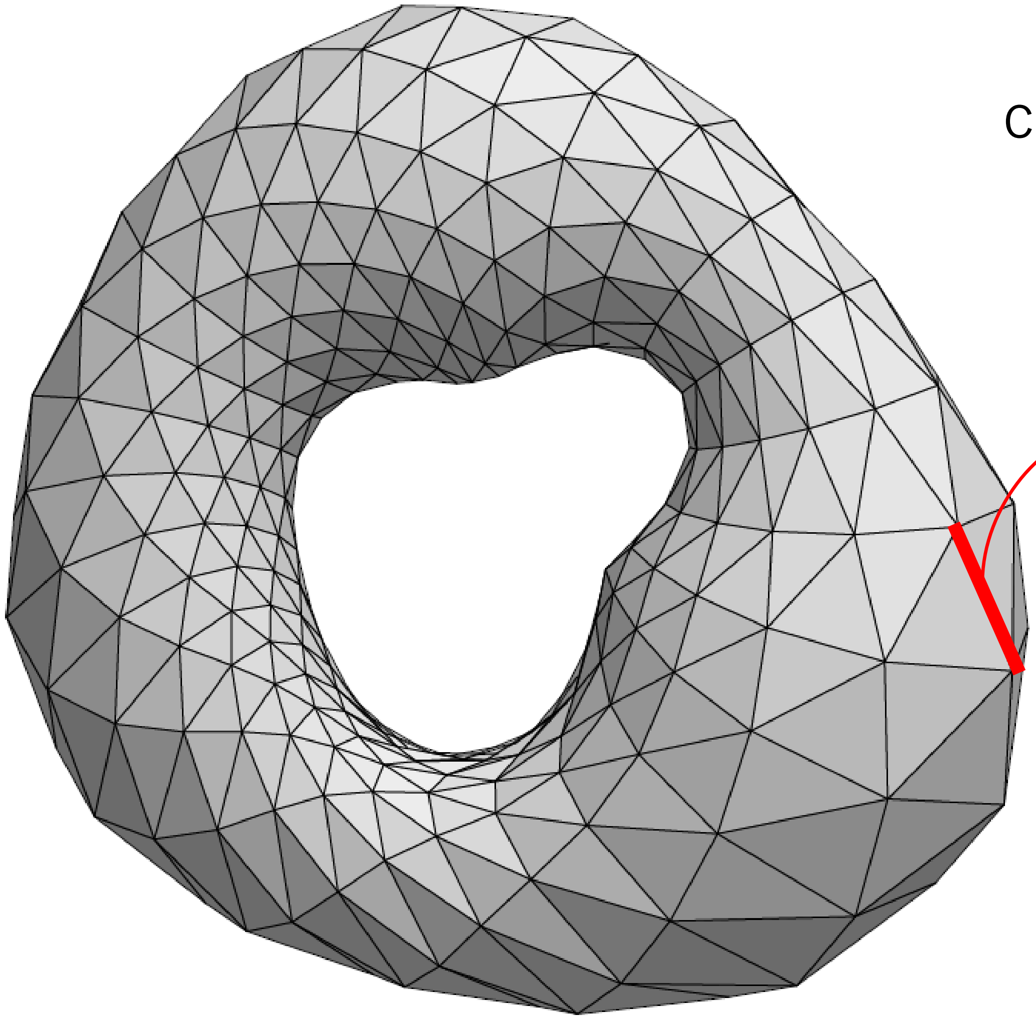
$$\begin{aligned} S_{\text{EH}} &= \frac{1}{16\pi G} \int d^4x \sqrt{-\det g R} \\ &= i \frac{1}{16\pi G} \sum_h \delta_h \sqrt{\mathbb{V}_h - 0i} \end{aligned}$$



sum over “hinge” [=(d-2)-simplex]

3. Monte Carlo calculations on Lefschetz thimbles

Lorentzian simplicial manifold



$$Z_{\text{QG}} = \int \mathcal{D}g_{\mu\nu}\, e^{iS[g]}$$

continuum limit

holomorphic

$$Z = \int_{-\infty}^{\infty} \prod_{i:\text{link}} d\sigma_i\, \mu(\sigma_i)\, e^{iS[\sigma]}$$

link $\sigma_i \in \mathbb{R}$ link length squared

$\sigma_i > 0$: space-like link

$\sigma_i < 0$: time-like link

Deform the integration contour by

$$\frac{d\sigma_i}{d\tau} = i\overline{\frac{\partial S_{\text{eff}}[\sigma(\tau)]}{\partial \sigma_i}}$$

(Cauchy's theorem)

How to deform the integration contour→ NEXT

We consider a general model defined by a multi-variable integral

$$Z = \int_{\mathbb{R}^N} dx\, e^{-S(x)}$$

$$x = (x_1, \cdots, x_N) \in \mathbb{R}^N$$

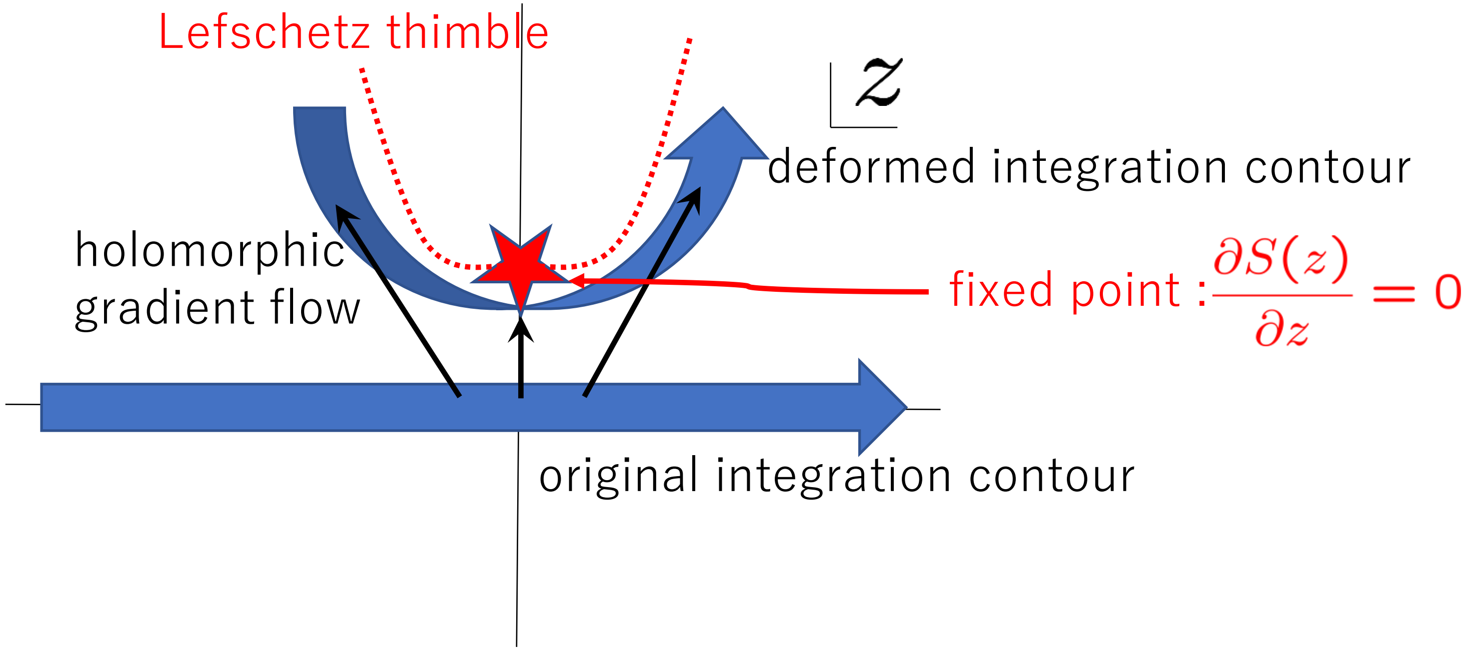
$$S(x) \in \mathbb{C}$$

$$\langle \mathcal{O}(x) \rangle = \frac{1}{Z} \int_{\mathbb{R}^N} dx\, \mathcal{O}(x)\, e^{-S(x)}$$

Difficult to evaluate due to the sign problem.

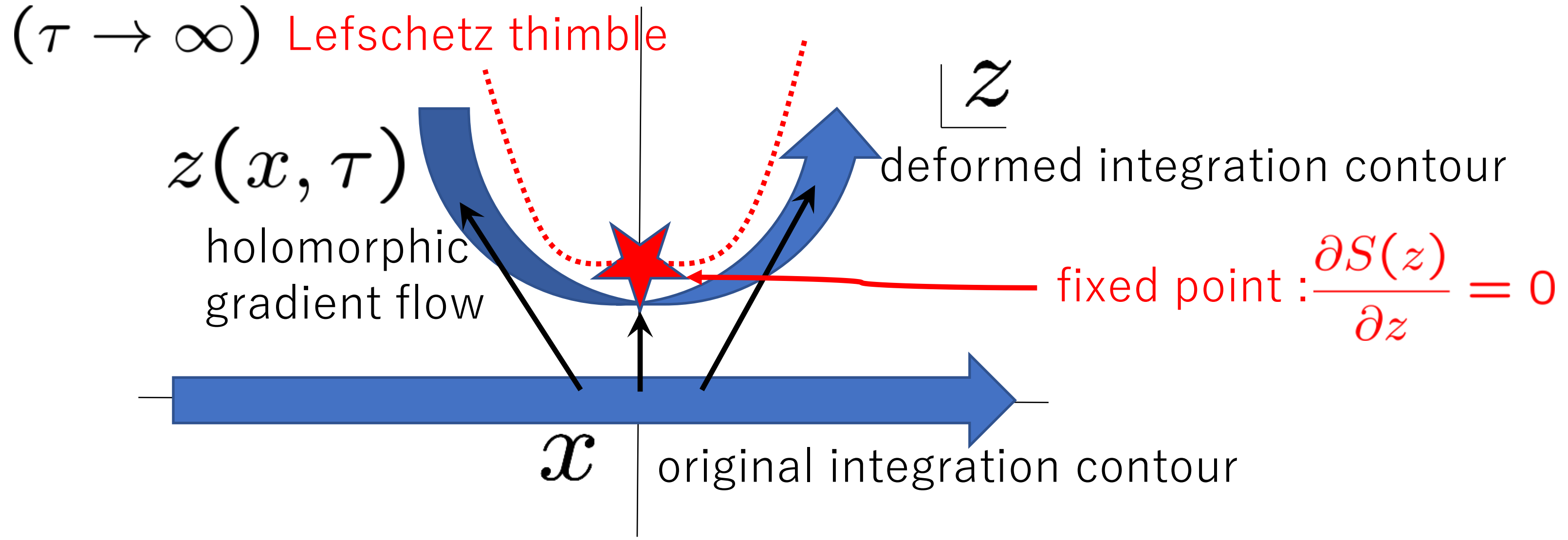
The generalized Lefschetz thimble method (GLTM)

A.Alexandru, G.Basar, P.F.Bedaque, G.W.Ridgway, N.C.Warrington, JHEP 1605 (2016) 053



As a result of the property of the holomorphic gradient flow, the sign problem becomes milder on the deformed contour !

The holomorphic gradient flow



solve $\quad \frac{\partial}{\partial t} z_k(x,t) = \overline{\left(\frac{\partial S(z(x,t))}{\partial z_k} \right)}$

from $t = 0$ to $t = \tau$

with the initial condition $z(x,0) = x \in \mathbb{R}^N$

One obtains a one-to-one map from x to $z(x,\tau)$

An important property of the holomorphic gradient flow

$$\frac{d}{dt}S(z(x,t)) = \frac{\partial S(z(x,t))}{\partial z_k}\frac{\partial z_k(x,t)}{\partial t}$$

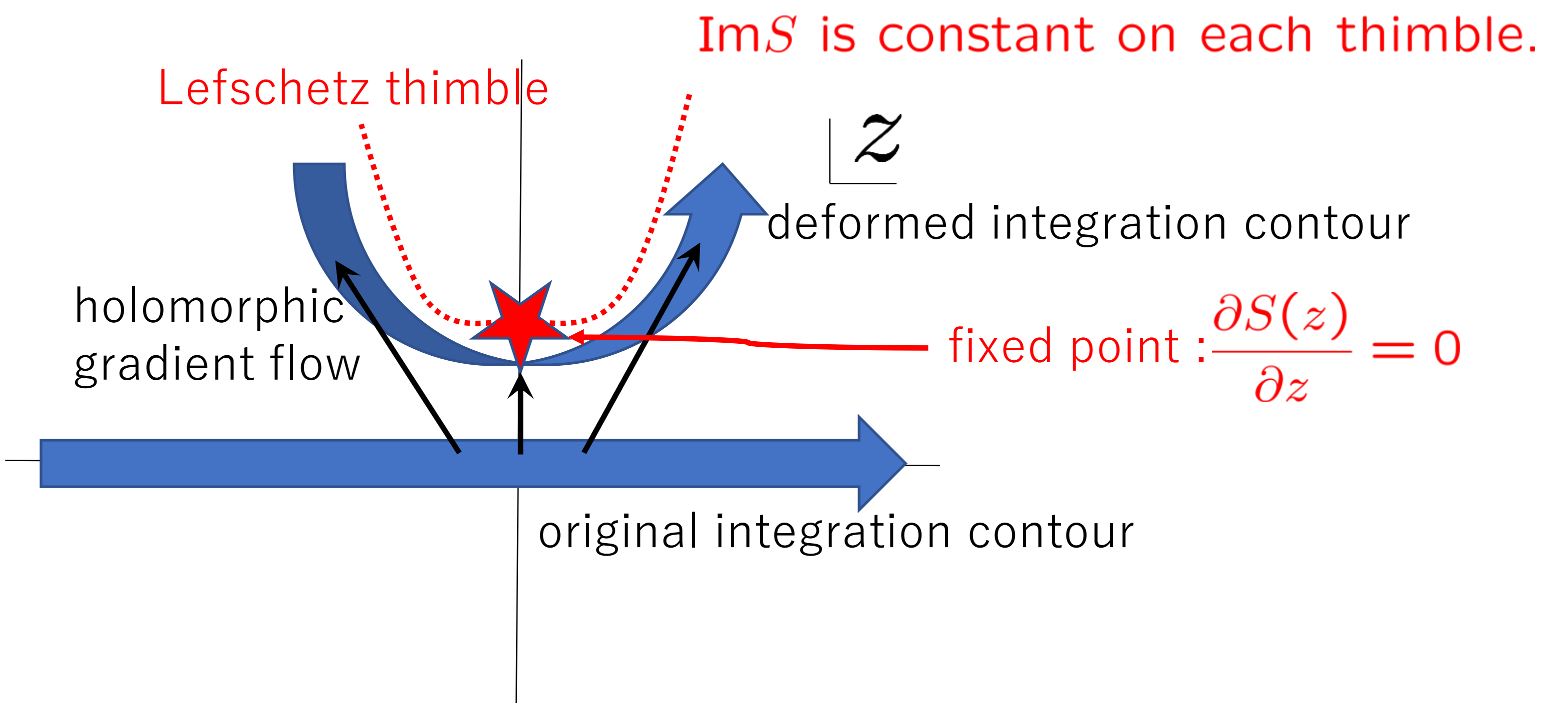
$$= \frac{\partial S(z(x,t))}{\partial z_k}\overline{\left(\frac{\partial S(z(x,t))}{\partial z_k}\right)}$$

$$= \left|\frac{\partial S(z(x,t))}{\partial z_k}\right|^2$$

real positive !

Real part of the action increases along the flow, while the imaginary part is kept constant.

The integration is dominated by a small region of x as the flow-time increases.



As a result, the sign problem becomes milder !

4. Summary and discussions

Summary

- Lorentzian QG can be made well-defined !

$$Z = \int \mathcal{D}g_{\mu\nu}\, e^{iS[g]} \qquad S = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi G_{\mathrm{N}}}R - \Lambda\right)$$

- Lorentzian manifold can be discretized by simplices respecting holomorphicity.

$$S_{\mathrm{c.c.}} = i\Lambda\sum_s\sqrt{\mathbb{V}_s} \qquad S_{\mathrm{EH}} = i\frac{1}{16\pi G}\sum_h \delta_h\sqrt{\mathbb{V}_h - 0i}$$

😊

- Deform the integration contour using “thimbles”.

$$\frac{d\sigma_\ell}{d\tau} = i\overline{\frac{\partial S_{\mathrm{eff}}[\sigma(\tau)]}{\partial\sigma_\ell}}$$

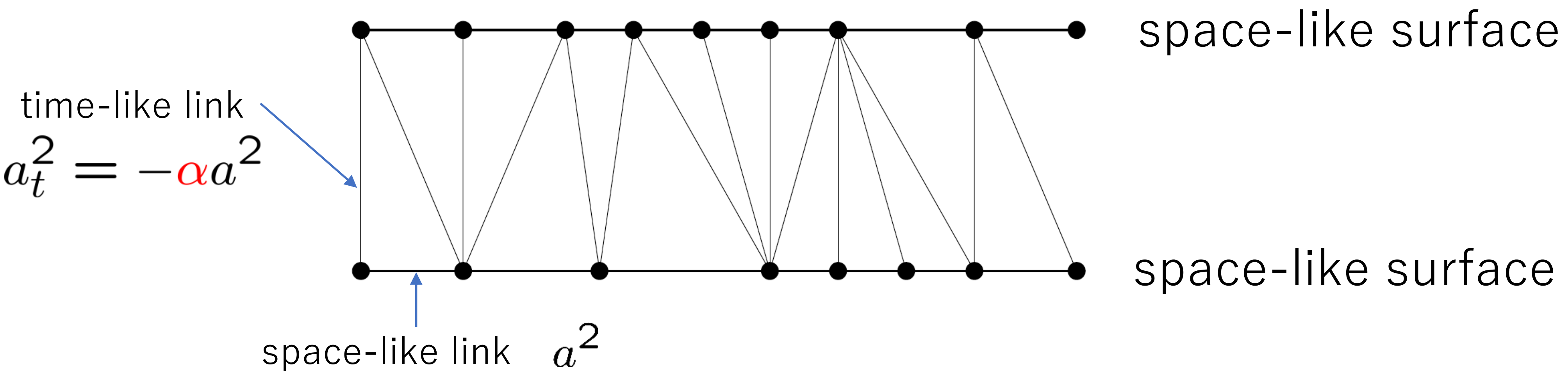
$$Z = \int \prod_{\ell:\mathrm{link}} d\sigma_\ell\, \mu(\sigma_\ell)\, e^{iS[\sigma]}$$

$\sigma_\ell \in \mathbb{R}$ link length square

Lorentzian triangle inequality

Relationship to the Causal Dynamical Triangulation

J.Ambjørn, A.Görlich, J.Jurkiewicz, R.Loll, Phys.Rept. 519 (2012) 127



Sum over all possible triangulations with a fixed.

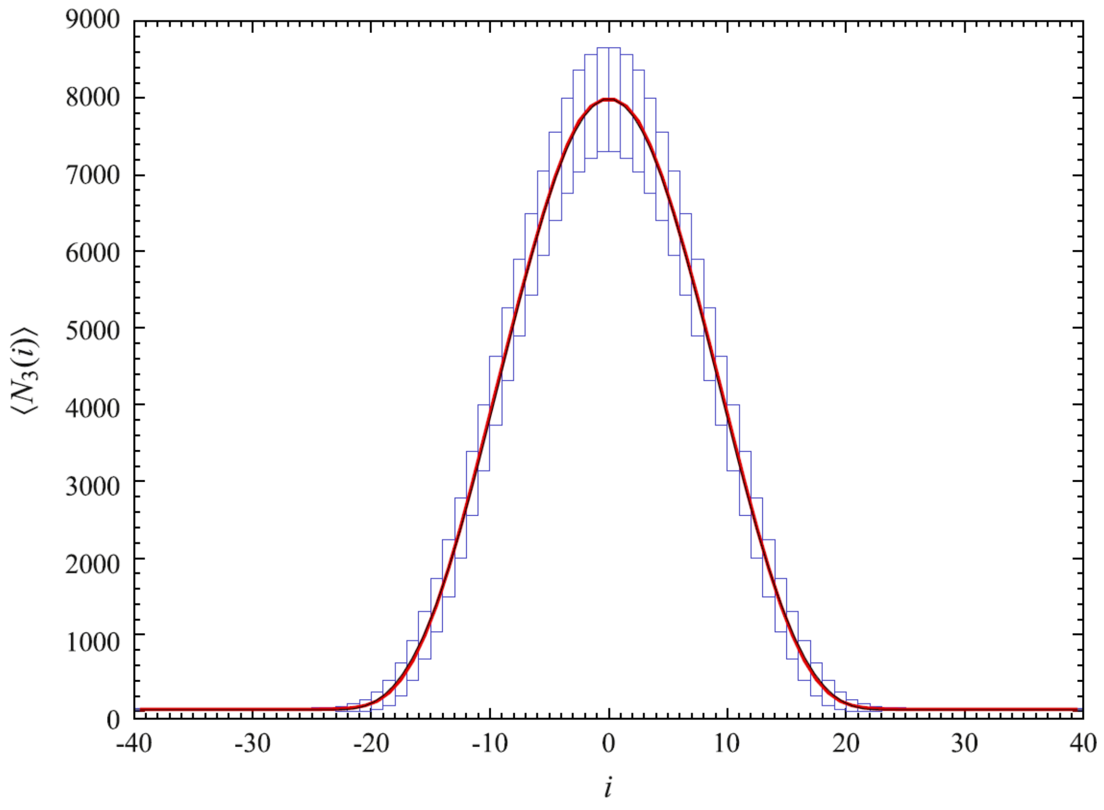
$$
\begin{aligned}
Z(\alpha) &= \sum_T e^{i\,S[T,\alpha]} \\
&= \sum_T e^{-S_{\mathrm{E}}[T,\tilde{\alpha}]}
\end{aligned}
\qquad
\begin{aligned}
&\alpha = -\tilde{\alpha} \\
&(\tilde{\alpha} > 0)
\end{aligned}
$$

Wick rotation

Monte Carlo simulation becomes possible!

Emergence of de Sitter space

J.Ambjørn, A.Görlich, J.Jurkiewicz, R.Loll, Phys.Rept. 519 (2012) 127



"imaginary time"-evolution of 3-volume consistent with macroscopic S^4 geometry.

"Euclidean de Sitter space"

In general, it is not straightforward to obtain quantities in Lorentzian QG due to the "Wick rotation" involved in CDT.

$$f(\tilde{\alpha}) = \langle \mathcal{O} \rangle_{\mathrm{CDT},\tilde{\alpha}}$$

$$\langle \mathcal{O} \rangle_{\mathrm{LQG},\alpha} = f(-\alpha)$$

analytic continuation

Regge calculus v.s. Dynamical Triangulation

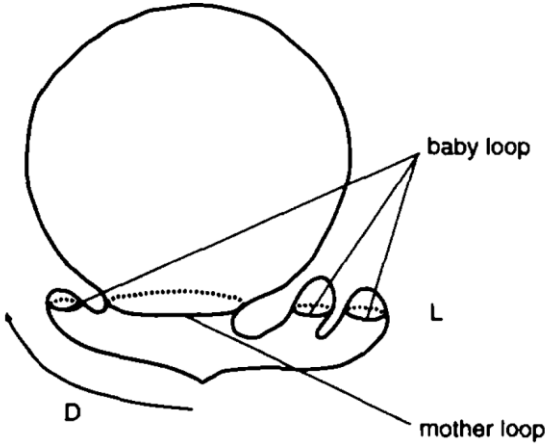
Test in 2d Euclidean QG

JN-Oshikawa, PLB338 ('94) 187

$$Z = \int \prod_i \frac{d\ell_i}{\ell_i} e^{-\lambda A} \Theta(\text{triangle inequalities})$$

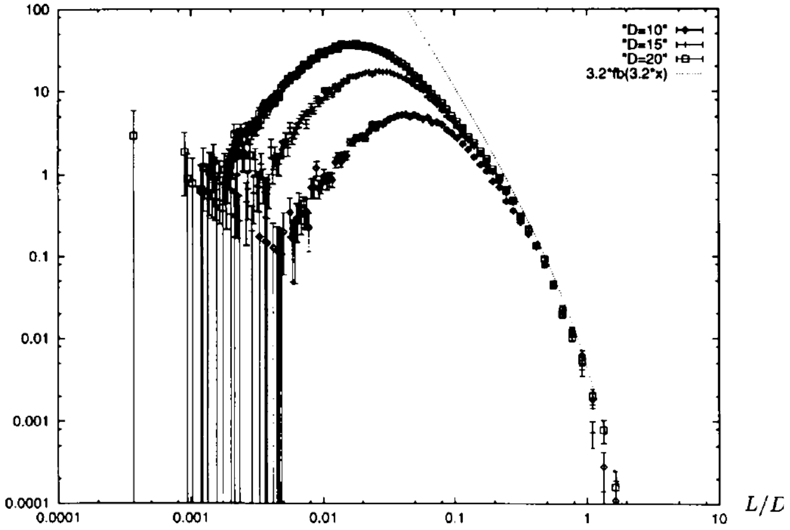
scale invariant measure

loop length distribution

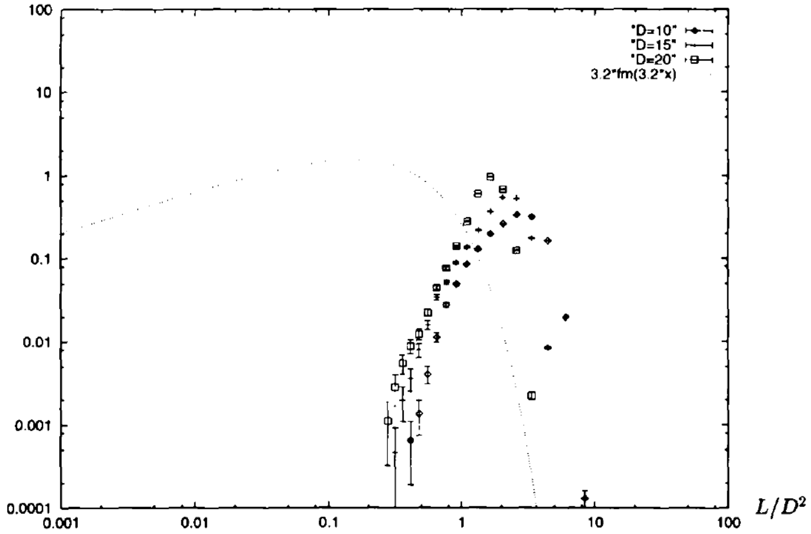


calculated exactly by
Kawai-Kawamoto-Mogami-Watabiki ('93)

distribution of the baby loops



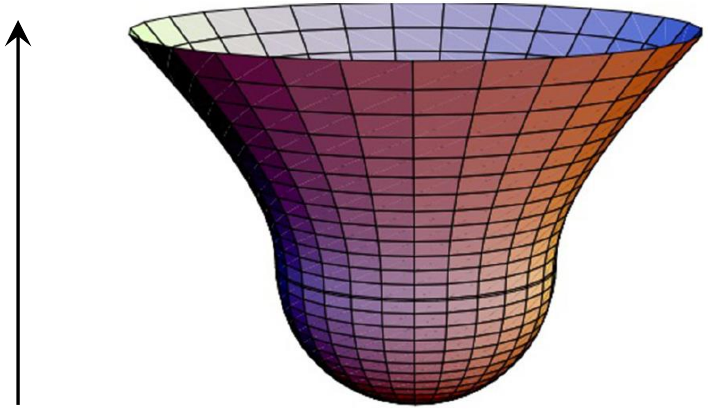
distribution of the mother loops



Regge calculus reproduces local fluctuations of the 2d surface
if the scale invariant measure is adopted.

Perspectives

- Monte Carlo simulation of Lorentzian QG on $\mathbb{S}^{d-1} \times [0, 1]$



- ➢ Is there a sensible continuum limit ?

  perturbative non-renormalizability

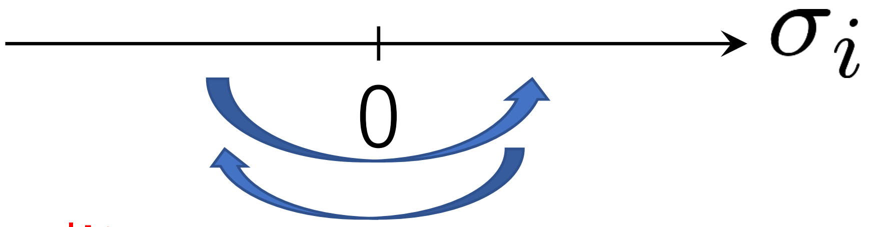
- ➢ Emergence of de Sitter space ?

  as suggested indirectly by CDT

- ➢ How did our Universe begin ?

  Tunneling? Bounce?

- possible technical problem



Link length square need to switch signs
without violating the Lorentzian triangle inequality.

How can we make this occur frequently during Monte Carlo simulation ?
(ergodicity problem)

5. Backup slides

What about the non-renormalizability ?

QG is perturbatively **non-renormalizable.** $G_{\mathrm{N}} = [M^{-2}]$

This issue may be overcome in a nonperturbative approach such as simplicial QG. (Renata Loll's talk)

However, it could be that the non-renormalizability calls for a fully UV complete theory of QG such as superstring theory.

The idea of contour deformation enables us to define superstring theory in (9+1)-dimensions nonperturbatively.