Quantum Resonance viewed as Weak Measurement

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Quantum resonance, *i.e.*, amplification in transition probability available under certain conditions, offers a powerful means for determining fundamental quantities in physics, including the time duration of the second adopted in the SI units and neutron's electric dipole moment which is directly linked to CP violation. We revisit two of the typical examples, the Rabi resonance and the Ramsey resonance, and show that both of these represent the weak value amplification and that near the resonance points they share exactly the same behavior of transition probabilities except for the measurement strength whose difference leads to the known advantage of the Ramsey resonance in the sensitivity. Conversely, this suggests that the weak value may be measured, for instance, through the Ramsey resonance. In fact, we argue that previous measurements of neutron electric dipole moment have potentially determined the imaginary component of the weak value of the spin of neutrons with higher precision than the conventional weak value measurement via neutron beams by three orders of magnitude.

I. INTRODUCTION

One of the major goals of science is to determine fundamental physical quantities of nature, and to this end the phenomena of resonance are often invoked as a powerful means for measuring the quantities. Resonance occurs ubiquitously when a certain condition is met in oscillations, as we are all familiar with acoustic resonance of strings or electrical resonance in tuned circuits. As such, its characteristics have been widely exploited for various purposes, including the measurement of frequency of sound waves via acoustic resonance or realization of filters in electrical circuits via electrical resonance. In quantum physics, resonance is particularly useful to determine physical parameters accurately. For instance, in the SI units [1, 2] the time dulation of the second is defined from the ground-state hyperfine transition frequency of the cesium 133 atoms, and in particle physics the fundamental parameters such as the neutron electric dipole moment (EDM) in the Standard Model are constrained [3] significantly.

The quantum resonance emerges as a phenomenon of amplification in transition probability between initial and final states. One of the most well-known quantum resonances is the magnetic resonance. It was first studied by Rabi in 1937 in a general setting [4] and was subsequently used to measure nuclear magnetic moments [5–7]. Following this, in 1950 a new mechanism of resonance was devised theoretically by Ramsey [8] and also confirmed experimentally. This Ramsey resonance has an advantage over the Rabi resonance in that it possesses a smaller half-width of resonance (0.6 times that of the Rabi resonance) and accordingly higher sensitivity to deviations from its resonance point. Another advantage of the Ramsey resonance is that the resonance condition is more robust against disturbance of external electric/magnetic fields compared to the Rabi resonance.

As for the accurate (precision) measurement of fundamental parameters, we also have the approach of the weak measurement proposed in 1988 by Aharonov, Albert, and Vaidman [9] in which one measures a physical quantity weakly on the condition of reaching a prescribed state called the post-selected state at the end of transition. The measured value in the procedure is then called the weak value, and an optical setup was first devised to demonstrate that the weak measurement potentially serves as a technique for amplifying the weak value [10]. Subsequently, the weak value amplification was applied to precision measurements such as the observation of the spin Hall effect of light [11] and the detection of ultrasensitive beam deflection in a Sagnac interferometer [12] (for a review, see *e.g.*, Ref. [13]). A notable aspect of the weak value amplification is that it is achieved at the cost of suppressing the transition probability between initial and final states. In fact, the quantum resonance and the weak value amplification share common characteristics, *e.g.*, a transition probability between initial and final states is amplified or suppressed under the conditions of their occurrence. Although these two approaches of amplification have been extensively studied in the context of precision measurements, the relation between them has remained unnoticed to this day. This prompts us to revisit the quantum resonance in light of weak measurement and see the possible connection between the two.

In this paper, we revisit both the Rabi and the Ramsey resonances and point out that the two resonances indeed represent the weak value amplification and that around their resonance points the behaviors of transition probabilities are exactly the same except for the sensitivity which is proportional to the measurement strength. In more detail, we show that the Ramsey resonance amounts to the weak measurement $\pi/2 \simeq$

1/0.6 times stronger than the Rabi resonance, which is consistent with the relation of the half-width mentioned above. As a by-product, this allows us to measure the imaginary component of the weak value through the Ramsey resonance. It is argued that previous experiments of neutron EDM have potentially determined the imaginary component of the weak value of the spin of neutrons with higher precision than the standard weak value measurement via neutron beams by three orders of magnitude.

This paper is organized as follows: In Sec. II, we present formulae of the weak measurement of two-level systems involving a post-selection, and, in the following sections, we evaluate the resonances in light of the weak measurement. In Sec. III, we briefly review the Rabi and Ramsey resonances and then present a unified interpretation of these resonances in terms of the weak measurement. In Sec. IV, measurement of the weak value using the Ramsey resonance technique in neutron EDM experiment is discussed. We finish with our summary and outlooks in Sec. V.

II. PRELIMINARIES

To make things easier for later sections, let us summarize the formulas for the time evolution of two-level systems with a post-selection. Consider the system described by a time-independent Hamiltonian,

$$H := H_0 + V, \tag{1}$$

where H_0 denotes a part dominantly describing the time evolution of the system, and V is a part describing a perturbative effect on the system. In particular, we assume that V is an operator representing a direct measurement process in the system. In contrast to indirect measurements, direct measurements require no system of measurement device and extract information from the difference in the system itself between $V = 0$ and $V \neq 0$. As explained later with some examples, the measurement operator V is defined appropriately according to the situations we have. We parametrize the two terms of the Hamiltonian (1) as

$$H_0 = h\left(n_0^{(h)} + \sigma_h\right), \quad V = v\left(n_0^{(v)} + \sigma_v\right), \tag{2}$$

where h, v are real positive values, $n_0^{(h)}, n_0^{(v)}$ stand for the part proportional to the 2×2 identity matrix while $\sigma_h := \sum_{i=1}^3 n_i^{(h)}\sigma_i$ and $\sigma_v := \sum_{i=1}^3 n_i^{(v)}\sigma_i$ are defined with the Pauli matrices σ_i, $i = 1, 2, 3$. The vectors $\vec{n}^{(h)} := (n_1^{(h)}, n_2^{(h)}, n_3^{(h)})$ and $\vec{n}^{(v)} := (n_1^{(v)}, n_2^{(v)}, n_3^{(v)})$ are normalized as $\vec{n}^{(h)} \cdot \vec{n}^{(h)*} = 1$ and $\vec{n}^{(v)} \cdot \vec{n}^{(v)*} = 1$. These components $n_0^{(h)}, n_0^{(v)}$ and $n_i^{(h)}, n_i^{(v)}$ for $i = 1, 2, 3$ may take complex values when non-hermitian time evolution is considered.

Let $|\psi(t)\rangle$ be a quantum state of the system at time t. The time evolution of the state between t_0 and $t_0 + t$ is given by

$$|\psi(t_0 + t)\rangle = e^{-iHt}|\psi_i\rangle, \tag{3}$$

where $|\psi_i\rangle := |\psi(t_0)\rangle$ and we put $\hbar = 1$ throughout this paper for brevity. From the standard formula $e^{iX} = \cos\sqrt{(\vec{x})^2} + iX/\sqrt{(\vec{x})^2}\sin\sqrt{(\vec{x})^2}$ for $X := \sum_{i=1}^3 x_i\sigma_i$, it is straightforward to show that the time evolution can be approximated, up to the first order of v, as

$$|\psi(t_0 + t)\rangle \simeq e^{-i(hn_0^{(h)} + vn_0^{(v)})t}\left[e^{-ih\sigma_h t}\left(1 - ivt\frac{\vec{n}^{(h)} \cdot \vec{n}^{(v)}}{(\vec{n}^{(h)})^2}\sigma_h\right) - i\frac{v}{h}\frac{1}{\sqrt{(\vec{n}^{(h)})^2}}\left(\sigma_v - \frac{\vec{n}^{(h)} \cdot \vec{n}^{(v)}}{(\vec{n}^{(h)})^2}\sigma_h\right)\sin\left(h\sqrt{(\vec{n}^{(h)})^2}t\right)\right]|\psi_i\rangle$$

$$= e^{-ivn_0^{(v)}t}\left[e^{-iH_0 t}\left(1 - ivt\frac{\vec{n}^{(h)} \cdot \vec{n}^{(v)}}{(\vec{n}^{(h)})^2}\sigma_h\right) - i\frac{v}{2h}[\sigma_a, e^{-iH_0 t}]\right]|\psi_i\rangle. \tag{4}$$

Here, for our later convenience, we have introduced $\sigma_a := \sum_{i=1}^3 n_i^{(a)}\sigma_i$ with $\vec{n}^{(a)} := (n_1^{(a)}, n_2^{(a)}, n_3^{(a)})$ which fulfills the relation,

$$[\sigma_a, \sigma_h] = 2i\left(\sigma_v - \frac{\vec{n}^{(h)} \cdot \vec{n}^{(v)}}{(\vec{n}^{(h)})^2}\sigma_h\right), \tag{5}$$

and satisfies the normalization condition $\vec{n}^{(a)} \cdot \vec{n}^{(a)*} = 1$.

In what follows we focuses on the case where v takes a small value so that our approximation up to the first order of v is valid. Note that the second terms in the first

and the second lines of Eq. (4) vanish for $\vec{n}^{(h)} \propto \vec{n}^{(v)}$ for which we have $\sigma_v = (\vec{n}^{(h)} \cdot \vec{n}^{(v)})/(\vec{n}^{(h)})^2\sigma_h$ and $[\sigma_a, e^{-iH_0 t}] = 0$. After the time evolution, we perform the post-selection, that is, we restrict ourselves to the case where the state $|\psi_f\rangle$ obtained in the measurement at the end is found to be the post-selected state we have specified in advance. This measurement through V involving the post-selection (with the approximation up to the first order of v) is referred to as *the weak measurement*. The transition amplitude from $|\psi_i\rangle$ to $|\psi_f\rangle$ then reads

$$\langle\psi_f|\psi(t_0+t)\rangle = \langle\psi_f|e^{-iHt}|\psi_i\rangle$$

$$\simeq e^{-ivn_0^{(v)}t}\langle\psi_f|e^{-iH_0t}|\psi_i\rangle\left(1 - ivt\frac{\vec{n}^{(h)}\cdot\vec{n}^{(v)}}{(\vec{n}^{(h)})^2}\sigma_h^W - i\frac{v}{2h}\left(\sigma_{a,L}^W - \sigma_{a,R}^W\right)\right), \tag{6}$$

where

$$\sigma_h^W := \frac{\langle\psi_f|\sigma_h e^{-iH_0t}|\psi_i\rangle}{\langle\psi_f|e^{-iH_0t}|\psi_i\rangle} = \frac{\langle\psi_f|e^{-iH_0t}\sigma_h|\psi_i\rangle}{\langle\psi_f|e^{-iH_0t}|\psi_i\rangle}, \quad \sigma_{a,L}^W := \frac{\langle\psi_f|\sigma_a e^{-iH_0t}|\psi_i\rangle}{\langle\psi_f|e^{-iH_0t}|\psi_i\rangle}, \quad \sigma_{a,R}^W := \frac{\langle\psi_f|e^{-iH_0t}\sigma_a|\psi_i\rangle}{\langle\psi_f|e^{-iH_0t}|\psi_i\rangle}. \tag{7}$$

are regarded as *the weak values* corresponding to σ_h and σ_a, respectively. For the latter the weak value may take different values at the initial and final times of the transition and hence requires the distinction with the labels 'R' and 'L', while for the former the two coincide. With these, the transition probability from $|\psi_i\rangle$ to $|\psi_f\rangle$ is writtien as

$$\mathrm{Pr}_{i\to f}(v) := |\langle\psi_f|\psi(t_0+t)\rangle|^2$$

$$\simeq \mathrm{Pr}_{i\to f}(0)\, e^{2vt\,\mathrm{Im}\,n_0^{(v)}}\left[1 + 2vt\left(\mathrm{Im}\left(\frac{\vec{n}^{(h)}\cdot\vec{n}^{(v)}}{(\vec{n}^{(h)})^2}\right)\mathrm{Re}\,\sigma_h^W + \mathrm{Re}\left(\frac{\vec{n}^{(h)}\cdot\vec{n}^{(v)}}{(\vec{n}^{(h)})^2}\right)\mathrm{Im}\,\sigma_h^W\right) + \frac{v}{h}\left(\mathrm{Im}\,\sigma_{a,L}^W - \mathrm{Im}\,\sigma_{a,R}^W\right)\right], \tag{8}$$

with $\mathrm{Pr}_{i\to f}(0) = \left|\langle\psi_f|e^{-iH_0t}|\psi_i\rangle\right|^2$. Below, we shall analyze the time evolution in more detail for the typical two cases: $\vec{n}^{(h)}\times\vec{n}^{(v)}=0$ and $\vec{n}^{(h)}\cdot\vec{n}^{(v)}=0$ both with $n_0^{(v)}=0$.

A. Commutative case: $\vec{n}^{(h)}\times\vec{n}^{(v)}=0$

In the case of $\vec{n}^{(h)}\times\vec{n}^{(v)}=0$, i.e., $\vec{n}^{(h)}\propto\vec{n}^{(v)}$, the two operators H_0 and V are commutative to each other, and $[H_0,V]=0$ holds. As will be explained later, the Ramsey resonance corresponds to this case. For $n_0^{(v)}=0$, up to the first order of v, the time evolution of Eq. (4) becomes

$$|\psi(t_0+t)\rangle \simeq e^{-iH_0t}\left(1 - ivt\sigma_v\right)|\psi_i\rangle, \tag{9}$$

and hence

$$\langle\psi_f|\psi(t_0+t)\rangle \simeq \langle\psi_f|e^{-iH_0t}|\psi_i\rangle\left(1 - ivt\sigma_v^W\right). \tag{10}$$

Substituting Eq. (10) into Eq. (8), we end up with

$$\mathrm{Pr}_{i\to f}(v) \simeq \mathrm{Pr}_{i\to f}(0)\left(1 + 2vt\,\mathrm{Im}\,\sigma_v^W\right). \tag{11}$$

This represents the effects of the measurement involving the post-selection up to the first order of v, i.e., the weak measurement on the transition probability. From Eq. (11), the imaginary part of the weak value of σ_v is given by

$$\mathrm{Im}\,\sigma_v^W = \frac{1}{2\mathrm{Pr}_{i\to f}(0)}\frac{d\mathrm{Pr}_{i\to f}(v)}{d(vt)}\bigg|_{v=0}. \tag{12}$$

This formula shows that the weak value $\mathrm{Im}\,\sigma_v^W$ can be measured by combining observable quantities $\mathrm{Pr}_{i\to f}(0)$ and $\mathrm{Pr}_{i\to f}(v)$. In other words, the weak value $\mathrm{Im}\,\sigma_v^W$ signifies the susceptibility with respect to the interaction V. The parameter vt appearing in Eq. (12) is recognized as the *measurement strength*, and our calculations carried out to the first order of vt are deemed to be applicable for the weak measurements satisfying $vt\ll 1$.

B. Non-commutative case: $\vec{n}^{(h)}\cdot\vec{n}^{(v)}=0$

In the case of $\vec{n}^{(h)}\cdot\vec{n}^{(v)}=0$, the two operators H_0 and V are not commutative, $[H_0,V]\neq 0$. As we shall see later, the Rabi resonance falls in this case. For $n_0^{(v)}=0$, up to the first order of v, the time evolution of Eq. (4) is expressed as

$$|\psi(t_0+t)\rangle \simeq \left(e^{-iH_0t} - i\frac{v}{2h}[\sigma_a, e^{-iH_0t}]\right)|\psi_i\rangle, \tag{13}$$

which implies

$$\langle\psi_f|\psi(t_0+t)\rangle \simeq \langle\psi_f|e^{-iH_0t}|\psi_i\rangle\left(1 - i\frac{v}{2h}\left(\sigma_{a,L}^W - \sigma_{a,R}^W\right)\right). \tag{14}$$

Substituting Eq. (14) into Eq. (8), we find

$$\mathrm{Pr}_{i\to f}(v) \simeq \mathrm{Pr}_{i\to f}(0)\left(1 + \frac{v}{h}\left(\mathrm{Im}\,\sigma_{a,L}^W - \mathrm{Im}\,\sigma_{a,R}^W\right)\right). \tag{15}$$

As before, Eq. (15) is rewritten as

$$\mathrm{Im}\,\sigma_{a,L}^W - \mathrm{Im}\,\sigma_{a,R}^W = \frac{1}{\mathrm{Pr}_{i\to f}(0)}\frac{d\mathrm{Pr}_{i\to f}(v)}{d(v/h)}\bigg|_{v=0}. \tag{16}$$

From Eq. (16), the parameter v/h is now recognized as the measurement strength of the weak measurement which provides the difference $\mathrm{Im}\,\sigma_{a,L}^W - \mathrm{Im}\,\sigma_{a,R}^W$. The

above results are applicable for the weak measurement satisfying the condition $v/h \ll 1$.

Comparing Eq. (15) with Eq. (11), it is clear that the transition probabilities between the initial and the final states are determined by distinct weak values depending on the commutative and non-commutative cases. Despite the difference, we shall observe later in Section. III that there is a close parallel in the transition probabilities between the Ramsey resonance and the Rabi resonance, which correspond to the commutative and the non-commutative case of the weak measurement, respectively.

III. RESONANCE

Quantum resonance is a phenomenon in which the transition amplitude is amplified when a specific condition holds. Here, using the formulae provided in Sec. II, we revisit the Rabi and Ramsey resonances to see that they admit a unified interpretation in light of the weak measurements associated with the two resonances.

A. Rabi resonance

Consider a system described by the following time-dependent Hamiltonian:

$$H_{\text{Rabi}}(t) := -\frac{\omega_0}{2}\sigma_3 + \omega_1 \left(\cos \omega t\, \sigma_1 - \sin \omega t\, \sigma_2\right), \quad (17)$$

where ω_0, ω_1 and ω are real constants. This describes a particle spin system under an external magnetic field with ω_0 being the product of a statistic magnetic field along the z axis and the particle's magnetic moment. In addition, the spin is under the influence of a rotating magnetic field with an angular frequency ω on the $x - y$ plane, and ω_1 represents the of the magnitude of the product of the rotating field and the particle's magnetic moment. The Rabi resonance occurs in this setting and is commonly referred to as magnetic resonance.

The time evolution of a state $|\psi(t)\rangle$ is described by the Schrödinger equation,

$$i\frac{d}{dt}|\psi(t)\rangle = H_{\text{Rabi}}(t)|\psi(t)\rangle. \quad (18)$$

In a rotating coordinate frame $|\psi'(t)\rangle := e^{-i\omega t\sigma_3/2}|\psi(t)\rangle$, Eq. (18) is recast into the Schrödinger equation, $id|\psi'(t)\rangle/dt = H'_{\text{Rabi}}|\psi'(t)\rangle$, with the time-independent Hamiltonian,

$$H'_{\text{Rabi}} := \omega_1 \left(\sigma_1 + \frac{\omega - \omega_0}{2\omega_1}\sigma_3\right). \quad (19)$$

The time evolution of the state between t_0 and $t_0 + t$ is given by

$$|\psi(t_0 + t)\rangle = e^{i\omega(t_0+t)\sigma_3/2}e^{-iH'_{\text{Rabi}}t}e^{-i\omega t_0\sigma_3/2}|\psi(t_0)\rangle. \quad (20)$$

We may thus regard H'_{Rabi} as the Hamiltonian H of Eq. (1).

Now we choose our initial state as $|\psi(t_0)\rangle = |\pm\rangle$, which is an eigenstate of σ_3 satisfying $\sigma_3|\pm\rangle = \pm|\pm\rangle$. Then, from Eq. (20), the transition amplitude from $|\pm\rangle$ to $|\pm\rangle$ is obtained as

$$\langle\pm|\psi(t_0 + t)\rangle = e^{\pm i\omega t/2}\langle\pm|e^{-iH'_{\text{Rabi}}t}|\pm\rangle. \quad (21)$$

From Eq. (21), the corresponding transition probability is given by

$$\text{Pr}_{\pm\to\pm}^{\text{Rabi}} = \left|\langle\pm|e^{-iH'_{\text{Rabi}}t}|\pm\rangle\right|^2. \quad (22)$$

A resonance arises when $\omega = \omega_0$ yielding $H'_{\text{Rabi}} = \omega_1\sigma_1$, for which we obtain

$$\text{Pr}_{\pm\to\pm}^{\text{Rabi}} = \left|\langle\pm|e^{-i\omega_1 t\sigma_1}|\pm\rangle\right|^2 \text{ for } \omega = \omega_0. \quad (23)$$

Then, for $\omega_1 t = \pi/2$ the amplitude Eq. (21) is suppressed and the probability of Eq. (23) reduces to zero. On the other hand, under the same condition the transition probability from $|\pm\rangle$ to $|\mp\rangle$ is amplified and becomes one on account of $\text{Pr}_{\pm\to\mp}^{\text{Rabi}} = 1 - \text{Pr}_{\pm\to\pm}^{\text{Rabi}}$. This shows that suppression and amplification occurs simultaneously when the resonance condition $\omega = \omega_0$ is fulfilled, and in this sense they are the two sides of the same coin. Note here that away from the resonance point, the unitary operator $e^{-iH'_{\text{Rabi}}t}$ implements a rotation of state around an axis slightly shifted from the z-axis, and the probability of Eq. (22) cannot be zero even if we tune ω_1.

A key aspect of this resonance is that the probability $\text{Pr}_{\pm\to\mp}^{\text{Rabi}}$ is amplified significantly at $\omega = \omega_0$ when $\omega_1 t$ is fixed at $\pi/2$. This enables us to measure the value of ω_0 through the resonance accurately. To see how this is achieved, let us imagine an experiment where we have three adjustable parameters ω, ω_1 and t, and one unknown parameter ω_0. It is clear that one can measure the unknown parameter ω_0 by tuning ω while keeping ω_1 and t and observing where the peak in the probability occurs, which leads to the value $\omega = \omega_0$ with accuracy determined from the width of the peak. This type of measurement process can be regarded as a weak measurement because it extracts information about the slight disturbance of the system around the resonance condition.

We now recosider the above resonance phenomena in the context of the weak measurement. For our convenience, let us decompose the parameter ω_0 as

$$\omega_0 := \overline{\omega}_0 + \epsilon, \quad (24)$$

where the parameter ϵ represents a small disturbance to be measured. We then consider a weak measurement of ϵ via the Rabi resonance in the following setup:

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Weak measurement via Rabi resonance

- Known parameters: $\overline{\omega}_0$, ω, ω_1, and t

- Unknown parameter: ϵ

- Condition of weak measurement: $\epsilon/\omega_1 \ll 1$

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Since the parameter ϵ characterizes a disturbance of the system with reference to the null case $\epsilon = 0$, we here consider $\epsilon/\omega_1 \ll 1$ to be the condition of the weak measurement. With this in mind, we rewrite the Hamiltonian of Eq. (19) as

$$H'_{\text{Rabi}} = \omega_1 \sigma_1 + \omega_1 (\phi_{\text{Rabi}} + \delta) \sigma_3, \quad (25)$$

with

$$\phi_{\text{Rabi}} := \frac{\omega - \overline{\omega}_0}{2\omega_1} \quad (26)$$

and $\delta := -\epsilon/2\omega_1$. Regarding $H'_{\text{Rabi}} = H$ in Eq. (1), we

choose the two operators H_0 and V as

$$H_0 = \omega_1 (\sigma_1 + \phi_{\text{Rabi}} \sigma_3), \quad V = \omega_1 \delta \sigma_3. \quad (27)$$

In the following, we restrict ourselves to a parameter region near the resonance point where $\phi_{\text{Rabi}} \ll 1$ and $\delta \ll 1$ hold. Up to the first order of ϕ_{Rabi} and δ, the coefficients of the Pauli matrices of Eq. (2) are found to be

$$h = \omega_1, \quad n_0^{(h)} = 0, \quad \vec{n}^{(h)} = (1, 0, \phi_{\text{Rabi}}),$$
$$v = \omega_1 \delta, \quad n_0^{(v)} = 0, \quad \vec{n}^{(v)} = (0, 0, 1). \quad (28)$$

Since $\sigma_h = \sigma_1 + \phi_{\text{Rabi}} \sigma_3$ and $\sigma_v = \sigma_3$ in our approximation, taking $\sigma_a = -\sigma_2$ we find

$$[\sigma_a, \sigma_h] = 2i(\sigma_3 + \phi_{\text{Rabi}} \sigma_1), \quad (29)$$

which is precisely the condition (5) required for σ_a within the same approximation. From Eq. (4), the time evolution of the system is found to be

$$|\psi(t_0 + t)\rangle \simeq \left(e^{-iH_0 t} + i\frac{\delta}{2} \left[\sigma_2, e^{-iH_0 t} \right] \right) |\psi_i\rangle$$
$$\simeq \left(e^{i\phi_{\text{Rabi}} \sigma_2/2} e^{-i\omega_1 t\sigma_1} e^{-i\phi_{\text{Rabi}} \sigma_2/2} + i\frac{\delta}{2} \left[\sigma_2, e^{i\phi_{\text{Rabi}} \sigma_2/2} e^{-i\omega_1 t\sigma_1} e^{-i\phi_{\text{Rabi}} \sigma_2/2} \right] \right) |\psi_i\rangle, \quad (30)$$

where we have used $e^{i\phi_{\text{Rabi}} \sigma_2/2} \sigma_1 e^{-i\phi_{\text{Rabi}} \sigma_2/2} = \sigma_1 \cos \phi_{\text{Rabi}} + \sigma_3 \sin \phi_{\text{Rabi}}$. Substituting Eq. (30) into Eq. (6), we obtain

$$\langle \psi_f | \psi(t_0 + t) \rangle \simeq \langle \psi_f(\phi_{\text{Rabi}}) | e^{-i\omega_1 t\sigma_1} | \psi_i(\phi_{\text{Rabi}}) \rangle \left(1 + i\frac{\delta}{2} \left(\sigma_{2,L}^W (\phi_{\text{Rabi}}) - \sigma_{2,R}^W (\phi_{\text{Rabi}}) \right) \right), \quad (31)$$

where $|\psi_i(\phi_{\text{Rabi}})\rangle := e^{-i\phi_{\text{Rabi}}\sigma_2/2}|\psi_i\rangle$, $|\psi_f(\phi_{\text{Rabi}})\rangle := e^{-i\phi_{\text{Rabi}}\sigma_2/2}|\psi_f\rangle$, and the weak values are given by

$$\sigma_{2,L}^W (\phi_{\text{Rabi}}) := \frac{\langle \psi_f(\phi_{\text{Rabi}})|\sigma_2 e^{-i\omega_1 t\sigma_1}|\psi_i(\phi_{\text{Rabi}})\rangle}{\langle \psi_f(\phi_{\text{Rabi}})|e^{-i\omega_1 t\sigma_1}|\psi_i(\phi_{\text{Rabi}})\rangle}, \quad \sigma_{2,R}^W (\phi_{\text{Rabi}}) := \frac{\langle \psi_f(\phi_{\text{Rabi}})|e^{-i\omega_1 t\sigma_1}\sigma_2|\psi_i(\phi_{\text{Rabi}})\rangle}{\langle \psi_f(\phi_{\text{Rabi}})|e^{-i\omega_1 t\sigma_1}|\psi_i(\phi_{\text{Rabi}})\rangle}. \quad (32)$$

Plugging $|\psi_i\rangle = |\pm\rangle$, $|\psi_f\rangle = |\pm\rangle$ and Eq. (31) into Eq. (8), we obtain the transition probability,

$$\text{Pr}_{\pm \to \pm}^{\text{Rabi}}(\epsilon) \simeq \text{Pr}_{\pm \to \pm}^{\text{Rabi}}(0) \left(1 + \frac{\epsilon}{2\omega_1} \left(\text{Im}\, \sigma_{2,L}^W (\phi_{\text{Rabi}}) - \text{Im}\, \sigma_{2,R}^W (\phi_{\text{Rabi}}) \right) \right). \quad (33)$$

For $\omega_1 t = \pi/2$, the weak values become

$$\sigma_{2,L}^W (\phi_{\text{Rabi}}) = -\sigma_{2,R}^W (\phi_{\text{Rabi}}) = -i \cot \phi_{\text{Rabi}}. \quad (34)$$

Also, the probability for $\epsilon = 0$ is obtained as

$$\text{Pr}_{\pm \to \pm}^{\text{Rabi}}(0) = \left| \langle \psi_f(\phi_{\text{Rabi}})|e^{-i\omega_1 t\sigma_1}|\psi_i(\phi_{\text{Rabi}})\rangle \right|^2$$
$$= \sin^2 \phi_{\text{Rabi}}. \quad (35)$$

The transition probability (33) then turns out to be

$$\text{Pr}_{\pm \to \pm}^{\text{Rabi}}(\epsilon) \simeq \text{Pr}_{\pm \to \pm}^{\text{Rabi}}(0) \left(1 + \delta_{\text{Rabi}} \text{Im}\, \sigma_{2,L}^W (\phi_{\text{Rabi}}) \right),$$
$$(36)$$

where $\delta_{\text{Rabi}} := 2\epsilon t/\pi$ is the measurement strength of the Rabi resonance for $\omega_1 t = \pi/2$. In the limit $\phi_{\text{Rabi}} = 0$, the probability (36) vanishes, $\text{Pr}_{\pm \to \pm}^{\text{Rabi}}(\epsilon) = 0$, signaling the appearance of resonance[#1]. In this limit, the weak values (34) are simultaneously amplified. This shows that the Rabi resonance can be considered as the weak value

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[#1] The resonance condition remains unchanged up to the first order of ϵ, although it is modified from the second order of ϵ.

amplification with the measurement strength δ_{Rabi}, in which the Rabi resonance condition corresponds to the divergence condition of the weak value.

From Eq. (36), the imaginary component of the weak value of σ_2 is obtained as

$$\text{Im}\,\sigma_{2,L}^W(\phi_{\text{Rabi}}) = \frac{1}{\text{Pr}_{\pm\to\pm}^{\text{Rabi}}(0)}\frac{d\text{Pr}_{\pm\to\pm}^{\text{Rabi}}(\epsilon)}{d\delta_{\text{Rabi}}}\bigg|_{\delta_{\text{Rabi}}=0}, \quad (37)$$

which can be evaluated from the measured values of δ_{Rabi}. Eq. (37) implies that the measurement of ϵ based on the Rabi resonance potentially determines the imaginary component of the weak value of σ_2. Since the weak value of a physical quantity is defined in the limit of zero measurement strength, it is generally understood as the physical quantity possessed by the system when no disturbance is inflicted. The detail of experiments determining the weak value will be discussed in Sec. IV in the context of the Ramsey resonance.

We recall that the weak value amplification can be achieved by adjusting the pre- and post-selected states,

$|\psi_i\rangle$ and $|\psi_f\rangle$, in such a way that the denominators of Eq. (32) reduce to zero. Since in the present case $|\psi_i(\phi_{\text{Rabi}})\rangle$ and $|\psi_f(\phi_{\text{Rabi}})\rangle$ contain only the parameter $\phi_{\text{Rabi}} = (\omega - \overline{\omega}_0)/2\omega_1$ defined in a single time region of $t_0 < t$, the resonance condition $\omega = \omega_0$ is specified by the parameters in the single time region. It is, however, possible to alter the resonance condition by including other parameters associated with the pre-selected and post-selected states, enlarging the scope of parameters which affect the resonance condition. Indeed, the Ramsey resonance condition discussed below utilizes such parameters other than those in the original time region.

B. Ramsey resonance

In order to discuss a typical case of the Ramsey resonance[#2], in the context of weak measurement, let us consider the Hamiltonian describing the dynamics of a particle spin separated in three time regions,

$$H_{\text{Ramsey}}(t) := \begin{cases} -\frac{\omega}{2}\sigma_3 + \omega_1\left(\cos\omega t\sigma_1 - \sin\omega t\sigma_2\right), & t_0 \le t < t_0 + \tau/2, \\ -\frac{\omega_0}{2}\sigma_3, & t_0 + \tau/2 \le t < t_0 + \tau/2 + T, \\ -\frac{\omega}{2}\sigma_3 + \omega_1\left(\cos\omega t\sigma_1 - \sin\omega t\sigma_2\right), & t_0 + \tau/2 + T \le t < t_0 + \tau + T. \end{cases} \quad (38)$$

Since the Hamiltonian of the first and third time regions is the same as H_{Rabi} in Eq. (17) with $\omega_0 = \omega$, from

Eq. (20) the time evolution of the initial state $|\psi(t_0)\rangle$ is given by

$$|\psi(t_0 + \tau/2)\rangle = e^{i\omega(t_0+\tau/2)\sigma_3/2}e^{-i\omega_1\tau\sigma_1/2}e^{-i\omega t_0\sigma_3/2}|\psi(t_0)\rangle, \quad (39)$$

$$|\psi(t_0 + \tau/2 + T)\rangle = e^{i\omega_0\sigma_3 T/2}|\psi(t_0 + \tau/2)\rangle, \quad (40)$$

$$|\psi(t_0 + \tau + T)\rangle = e^{i\omega(t_0+\tau+T)\sigma_3/2}e^{-i\omega_1\tau\sigma_1/2}e^{-i\omega(t_0+\tau/2+T)\sigma_3/2}|\psi(t_0 + \tau/2 + T)\rangle. \quad (41)$$

Combining Eqs. (39)-(41), we find the final state,

$$|\psi(t_0 + \tau + T)\rangle = e^{i\omega(t_0+\tau+T)\sigma_3/2}e^{-i\omega_1\tau\sigma_1/2}e^{-i(\omega-\omega_0)T\sigma_3/2}e^{-i\omega_1\tau\sigma_1/2}e^{-i\omega t_0\sigma_3/2}|\psi(t_0)\rangle. \quad (42)$$

For our later convenience, we combine all unitary operators appearing in the time evolution into one:

$$U(t_0, t_0 + \tau + T) := e^{i\omega(t_0+\tau+T)\sigma_3/2}e^{-i\omega_1\tau\sigma_1/2}e^{-i(\omega-\omega_0)T\sigma_3/2}e^{-i\omega_1\tau\sigma_1/2}e^{-i\omega t_0\sigma_3/2}. \quad (43)$$

Now, as in the case of the Rabi resonance, let us consider the initial state $|\psi(t_0)\rangle = |\pm\rangle$. From Eq. (42), the transition amplitude from $|\pm\rangle$ to $|\pm\rangle$ is obtained as

$$\langle\pm|\psi(t_0 + \tau + T)\rangle = e^{\pm i\omega(\tau+T)/2}\langle\pm|e^{-i\omega_1\tau\sigma_1/2}e^{-i(\omega-\omega_0)T\sigma_3/2}e^{-i\omega_1\tau\sigma_1/2}|\pm\rangle. \quad (44)$$

___

[#2] A more general Ramsey resonance arises when the Hamiltonian of the first and third time regions is

$-\omega_0/2\sigma_3 + \omega_1\left(\cos\omega t\sigma_1 - \sin\omega t\sigma_2\right)$. However, for $\tau \ll T$,

From Eq. (44), the transition probability then reads

$$\mathrm{Pr}_{\pm\to\pm}^{\mathrm{Ramsey}} = \left| \langle\pm|e^{-i\omega_1\tau\sigma_1/2}e^{-i(\omega-\omega_0)T\sigma_3/2}e^{-i\omega_1\tau\sigma_1/2}|\pm\rangle \right|^2. \tag{45}$$

As before, we further consider the case $\omega = \omega_0$ for which we obtain

$$\mathrm{Pr}_{\pm\to\pm}^{\mathrm{Ramsey}} = \left| \langle\pm|e^{-i\omega_1\tau\sigma_1}|\pm\rangle \right|^2 \ \text{ for } \omega = \omega_0. \tag{46}$$

Eq. (46) shares the same form with Eq. (23), and for $\omega_1\tau = \pi/2$ the probability becomes zero. This indicates a resonance to arise there, which is referred to as the Ramsey resonance. It should be emphasized, however, that the Ramsey resonance is inherently different from the Rabi resonance. In fact, while the Rabi resonance arises between two oscillations ω and ω_0 in the same time region, the Ramsey resonance arises between two oscillations ω and ω_0 observed in different time regions.

Next, we discuss the Ramsey resonance from the viewpoint of weak measurement. To this end, similarly to Eq. (24) we write the parameter ω_0 as $\omega_0 := \overline{\omega}_0 + \epsilon$ and consider the weak measurement of ϵ in the following setup:

```
┌─ Weak measurement via Ramsey resonance ───

  • Known parameters: ω̄₀, ω, ω₁, τ, and T

  • Unknown parameter: ε

  • Condition of weak measurement: εT ≪ 1

└──────────────────────────────────────────
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Then, for $t_0 + \tau/2 \leq t < t_0 + \tau/2 + T$, the corresponding Hamiltonian in (38) is rewritten as

$$H_{\mathrm{Ramsey}}(t) = -\frac{\overline{\omega}_0}{2}\sigma_3 - \frac{\epsilon}{2}\sigma_3. \tag{47}$$

According to the above setup, we choose the two operators H_0 and V in (1) as

$$H_0 = -\frac{\overline{\omega}_0}{2}\sigma_3, \quad V = -\frac{\epsilon}{2}\sigma_3. \tag{48}$$

and also the coefficients appearing in Eq. (2) as

$$h := \frac{\overline{\omega}_0}{2}, \quad n_0^{(h)} := 0, \quad \vec{n}^{(h)} = (0,0,-1),$$
$$v := \frac{\epsilon}{2}, \quad n_0^{(v)} := 0, \quad \vec{n}^{(v)} := (0,0,-1). \tag{49}$$

Since $\vec{n}^{(h)} \times \vec{n}^{(v)} = 0$ holds, we are in the commutative case mentioned in Sec. II. If we choose $|\psi_i\rangle := e^{i\omega(t_0+\tau/2)\sigma_3/2}e^{-i\omega_1\tau\sigma_1/2}e^{-i\omega t_0\sigma_3/2}|\pm\rangle$, and

————

the following explanations are applicable even for such a general case.

$|\psi_f\rangle = e^{i\omega(t_0+\tau/2+T)\sigma_3/2}e^{i\omega_1\tau\sigma_1/2}e^{-i\omega(t_0+\tau+T)\sigma_3/2}|\pm\rangle$, then from Eqs. (11) and (47) the transition probability from $|\psi_i\rangle$ to $|\psi_f\rangle$ is given by

$$\mathrm{Pr}_{\pm\to\pm}^{\mathrm{Ramsey}}(\epsilon) \simeq \mathrm{Pr}_{\pm\to\pm}^{\mathrm{Ramsey}}(0)\left(1 - \epsilon T\,\mathrm{Im}\,\sigma_3^W\right), \tag{50}$$

where

$$\mathrm{Pr}_{\pm\to\pm}^{\mathrm{Ramsey}}(\epsilon) := \left| \langle\psi_f|e^{-i\int_{t_0+\tau/2}^{t_0+\tau/2+T} dt H_{\mathrm{Ramsey}}(t)}|\psi_i\rangle \right|^2, \tag{51}$$

and

$$\sigma_3^W = \frac{\langle\psi_f|e^{-iH_0T}\sigma_3|\psi_i\rangle}{\langle\psi_f|e^{-iH_0T}|\psi_i\rangle} = \frac{\langle\psi_f|\sigma_3 e^{-iH_0T}|\psi_i\rangle}{\langle\psi_f|e^{-iH_0T}|\psi_i\rangle}. \tag{52}$$

In particular, for $\omega_1\tau = \pi/2$, the following relations hold:

$$\langle\psi_f|e^{-iH_0T}\sigma_3|\psi_i\rangle = e^{\pm i\omega(\tau+T)/2}\langle\psi_f(\phi_{\mathrm{Ramsey}})|e^{-i\omega_1\tau\sigma_1}\sigma_2|\psi_i(\phi_{\mathrm{Ramsey}})\rangle, \tag{53}$$
$$\langle\psi_f|e^{-iH_0T}|\psi_i\rangle = e^{\pm i\omega(\tau+T)/2}\langle\psi_f(\phi_{\mathrm{Ramsey}})|e^{-i\omega_1\tau\sigma_1}|\psi_i(\phi_{\mathrm{Ramsey}})\rangle, \tag{54}$$

where

$$|\psi_i(\phi_{\mathrm{Ramsey}})\rangle = |\psi_f(\phi_{\mathrm{Ramsey}})\rangle := e^{-i\phi_{\mathrm{Ramsey}}\,\sigma_2/2}|\pm\rangle, \tag{55}$$

with

$$\phi_{\mathrm{Ramsey}} := (\omega - \overline{\omega}_0)\frac{T}{2} \tag{56}$$

————

From Eqs. (53) and (54), we find he weak value,

$$\sigma_3^W = -\sigma_{2,L}^W(\phi_{\mathrm{Ramsey}}) = \sigma_{2,R}^W(\phi_{\mathrm{Ramsey}}) = i\cot\phi_{\mathrm{Ramsey}}. \tag{57}$$

and also the transition probability for $\epsilon = 0$,

$$\mathrm{Pr}_{\pm\to\pm}^{\mathrm{Ramsey}}(0) = \left| \langle\psi_f(\phi_{\mathrm{Ramsey}})|e^{-i\omega_1\tau\sigma_1}|\psi_i(\phi_{\mathrm{Ramsey}})\rangle \right|^2$$
$$= \sin^2\phi_{\mathrm{Ramsey}}. \tag{58}$$

With these, Eq. (51) is evaluated as

$$\mathrm{Pr}_{\pm\to\pm}^{\mathrm{Ramsey}}(\epsilon) \simeq \mathrm{Pr}_{\pm\to\pm}^{\mathrm{Ramsey}}(0) \left(1 + \delta_{\mathrm{Ramsey}} \,\mathrm{Im}\,\sigma_{2,L}^{W}\left(\phi_{\mathrm{Ramsey}}\right)\right),$$
(59)

where $\delta_{\mathrm{Ramsey}} := \epsilon T$ is the measurement strength for the Ramsey resonance, and $\mathrm{Im}\,\sigma_{2,L}^{W}$ is given in Eq. (34). Similarly to the Rabi resonance, in the limit $\phi_{\mathrm{Ramsey}} = 0$ the probability (59) becomes zero and the weak value (57) diverges. This indicates that the Ramsey resonance can be regarded as weak value amplification with the measurement strength δ_{Ramsey}.

It is notable that, except for the magnitude of measurement strength, the two probability formulae, Eqs. (36) and (59), become identical at $T = t$. In particular, we obtain $\phi_{\mathrm{Ramsey}} = \phi_{\mathrm{Rabi}}$ as well as $\delta_{\mathrm{Ramsey}} = \delta_{\mathrm{Rabi}}$, if we replace T with $2t/\pi$. This implies that the weak measurement offers a unified framework for resonances in inherently different types, one being commutative and the other non-commutative in the classification given in Sec. II. It also shows that the measurement strength presents a means to quantify how sensitive the resonance is against small changes in resonance conditions. Such parallel becomes available due to our different choices of the initial and final states, $|\psi_i\rangle$ and $|\psi_f\rangle$, for the Ramsey and Rabi resonances.

We also note that from Eq. (59) the imaginary component of the weak value is obtained as

$$\mathrm{Im}\,\sigma_{2,L}^{W}(\phi_{\mathrm{Ramsey}}) = \frac{1}{\mathrm{Pr}_{\pm\to\pm}^{\mathrm{Ramsey}}(0)} \left.\frac{d\mathrm{Pr}_{\pm\to\pm}^{\mathrm{Ramsey}}(\epsilon)}{d\delta_{\mathrm{Ramsey}}}\right|_{\delta_{\mathrm{Ramsey}}=0}.$$
(60)

As in the case of the Rabi resonance, the right-hand side

of Eq. (60) can be evaluated from the measurement of δ_{Ramsey}, that is, the Ramsey resonance determines the imaginary component of the weak value of σ_2. Based on these results, we discuss the experiments determining the weak value in more detail in Sec. IV by taking some realistic factors into consideration.

C. Comparison between Rabi and Ramsey resonances in light of weak measurement

Before doing so, we revisit the Rabi and Ramsey resonances and compare them in light of the weak measurement. Recall that we have considered the case where the resonance condition is slightly broken by the unknown parameter ϵ and regarded that the disturbance of the system by ϵ is required in the weak measurement process. For our purpose, we may consider the particular choice $t = T$, which is allowed because the Ramsey resonance often assumes $\tau \ll T$ to ensure insensitivity to inhomogeneities of external electric/magnetic fields. Our comparison is based on the three key properties of resonance we found so far:

(i) *Resonance as weak value amplification* — Both the Rabi and Ramsey resonances are regarded as weak value amplification, and share a common feature that the their weak values (34) and (57) diverge as

$$\lim_{\phi\to0} \mathrm{Im}\,\sigma_{2,L}^{W}(\phi) = -\lim_{\phi\to0} \mathrm{Im}\,\sigma_{2,R}^{W}(\phi) = -\infty, \quad (61)$$

for $\phi = \phi_{\mathrm{Rabi}}$ or ϕ_{Ramsey}.

(ii) *Probability near resonance* — For $t = T$ and $\omega \simeq \overline{\omega}_0$, the formula for the transition probability from $|\pm\rangle$ to $|\pm\rangle$ is formally identical for the Rabi and Ramsey resonances:

$$\mathrm{Pr}_{\pm\to\pm}^{\mathrm{Rabi}}(\epsilon) \simeq \mathrm{Pr}_{\pm\to\pm}^{\mathrm{Rabi}}(0) \left(1 + \delta_{\mathrm{Rabi}} \,\mathrm{Im}\,\sigma_{2,L}^{W}\left(\phi_{\mathrm{Rabi}}\right)\right) \quad \text{[Rabi resonance]}, \tag{62}$$

$$\mathrm{Pr}_{\pm\to\pm}^{\mathrm{Ramsey}}(\epsilon) \simeq \mathrm{Pr}_{\pm\to\pm}^{\mathrm{Ramsey}}(0) \left(1 + \delta_{\mathrm{Ramsey}} \,\mathrm{Im}\,\sigma_{2,L}^{W}\left(\phi_{\mathrm{Ramsey}}\right)\right) \quad \text{[Ramsey resonance]}, \tag{63}$$

with $\delta_{\mathrm{Rabi}} = 2\epsilon t/\pi$, $\delta_{\mathrm{Ramsey}} = \epsilon T$ and $\phi_{\mathrm{Rabi}} = (\omega - \overline{\omega}_0)t/\pi$, $\phi_{\mathrm{Ramsey}} = (\omega-\overline{\omega}_0)T/2$ as given in (26) and (56). We have then $\mathrm{Pr}_{\pm\to\pm}^{\mathrm{Rabi}}(0) = \sin^2\phi_{\mathrm{Rabi}}$ and $\mathrm{Pr}_{\pm\to\pm}^{\mathrm{Ramsey}}(0) = \sin^2\phi_{\mathrm{Ramsey}}$. For $t = T$ and $\phi_{\mathrm{Rabi}} = \phi_{\mathrm{Ramsey}} = \phi$, the difference between Eqs. (62) and (63) arises only from the difference between δ_{Rabi} and δ_{Ramsey}.

(iii) *Measurement strength in resonance* — For $t = T$, the measurement strengths of the Rabi and Ramsey resonances are $\delta_{\mathrm{Rabi}} = 2\epsilon t/\pi$ and $\delta_{\mathrm{Ramsey}} = \epsilon t$, respectively. Given this, the measurement strength of the Ramsey resonance is $\pi/2 \simeq 1/0.6$ times stronger than that of the Rabi resonance, implying that the Ramsey resonance has higher sensitivity to the shift of the resonance condition than the Rabi resonance.

From (i), (ii), and (iii), we see that the perspective of the weak measurement provides a unified understanding of the Rabi and Ramsey resonances where the sensitivity to the shift of the resonance condition is characterized by the measurement strength, in spite of the fact that the two resonances fall in different types of weak measurements, one being commutative and the other non-commutative.

IV. WEAK VALUE IN NEUTRON EDM EXPERIMENT

We have seen that resonance can generally be regarded as weak value amplification. Given this, we now propose

a weak value measurement of neutron EDM.

A. Imperfection effects of pre- and post-selections on Ramsey resonance

In preparation for the discussion of neutron EDM measurement, we will expand on the Ramsey resonance in Section. III B by taking realistic factors into consideration, that is, the possible imperfection of pre- and post-selection. For this, rather than the pure states considered in Section. III B, we consider a mixed state for the initial state of the system,

$$\rho_i := \frac{1 + P_i}{2}|+\rangle\langle+| + \frac{1 - P_i}{2}|-\rangle\langle-|, \tag{64}$$

where P_i denotes the statistical disparity of spin characterizing the imperfection of the pre-selected state. We also consider the imperfection of post-selection, which is dealt with the POVM operator:

$$E_+ := (1 - \epsilon_f)|+\rangle\langle+| + \epsilon_f|-\rangle\langle-|, \tag{65}$$
$$E_- := \epsilon_f|+\rangle\langle+| + (1 - \epsilon_f)|-\rangle\langle-|, \tag{66}$$

where E_+ and E_- correspond to the spin-up and spin-down states, respectively, and ϵ_f represents the imperfection of the spin observations.

Recall that the time evolution consists of the three time-regions, and the unitary time evolution is dictated by Eq. (43) for $\omega_0 = \overline{\omega}_0 + \epsilon$. The probability that the spin-up or -down state is observed at $t_0 + \tau + T$ is then found to be

$$\begin{aligned}
\mathrm{Pr}_{i\to f}^{\mathrm{Ramsey}}(\pm;\epsilon) &= \mathrm{Tr}\left[E_\pm U(t_0, t_0 + \tau + T)\rho_i U^\dagger(t_0, t_0 + \tau + T)\right] \\
&= \frac{1 + P_i}{2}\left((1 - \epsilon_f)\mathrm{Pr}_{+\to\pm}^{\mathrm{Ramsey}}(\epsilon) + \epsilon_f\mathrm{Pr}_{+\to\mp}^{\mathrm{Ramsey}}(\epsilon)\right) + \frac{1 - P_i}{2}\left((1 - \epsilon_f)\mathrm{Pr}_{-\to\pm}^{\mathrm{Ramsey}}(\epsilon) + \epsilon_f\mathrm{Pr}_{-\to\mp}^{\mathrm{Ramsey}}(\epsilon)\right),
\end{aligned} \tag{67}$$

where

$$\mathrm{Pr}_{+\to-}^{\mathrm{Ramsey}}(\epsilon) = |\langle-|U(t_0, t_0 + \tau + T)|+\rangle|^2 = \frac{1}{2}\left(1 + \cos\left(2\phi_{\mathrm{Ramsey}} - \epsilon T\right)\right) \text{ for } \omega_1\tau = \frac{\pi}{2}, \tag{68}$$

$$\mathrm{Pr}_{+\to+}^{\mathrm{Ramsey}}(\epsilon) = |\langle+|U(t_0, t_0 + \tau + T)|+\rangle|^2 = \frac{1}{2}\left(1 - \cos\left(2\phi_{\mathrm{Ramsey}} - \epsilon T\right)\right) \text{ for } \omega_1\tau = \frac{\pi}{2}, \tag{69}$$

$$\mathrm{Pr}_{-\to-}^{\mathrm{Ramsey}}(\epsilon) = |\langle-|U(t_0, t_0 + \tau + T)|-\rangle|^2 = \frac{1}{2}\left(1 - \cos\left(2\phi_{\mathrm{Ramsey}} - \epsilon T\right)\right) \text{ for } \omega_1\tau = \frac{\pi}{2}, \tag{70}$$

$$\mathrm{Pr}_{-\to+}^{\mathrm{Ramsey}}(\epsilon) = |\langle+|U(t_0, t_0 + \tau + T)|-\rangle|^2 = \frac{1}{2}\left(1 + \cos\left(2\phi_{\mathrm{Ramsey}} - \epsilon T\right)\right) \text{ for } \omega_1\tau = \frac{\pi}{2}, \tag{71}$$

with ϕ_{Ramsey} given in (56). Up to the first order of ϵ, the above probabilities for $\omega_1\tau = \pi/2$ can be expressed by the weak values as

$$\mathrm{Pr}_{+\to-}^{\mathrm{Ramsey}}(\epsilon) = \mathrm{Pr}_{-\to+}^{\mathrm{Ramsey}}(\epsilon) \simeq \left(1 - \mathrm{Pr}_{+\to+}^{\mathrm{Ramsey}}(0)\right)\left(1 + \epsilon T\frac{\mathrm{Pr}_{+\to+}^{\mathrm{Ramsey}}(0)}{1 - \mathrm{Pr}_{+\to+}^{\mathrm{Ramsey}}(0)}\,\mathrm{Im}\,\sigma_3^W\right), \tag{72}$$

$$\mathrm{Pr}_{+\to+}^{\mathrm{Ramsey}}(\epsilon) = \mathrm{Pr}_{-\to-}^{\mathrm{Ramsey}}(\epsilon) \simeq \mathrm{Pr}_{+\to+}^{\mathrm{Ramsey}}(0)\left(1 - \epsilon T\,\mathrm{Im}\,\sigma_3^W\right), \tag{73}$$

with the weak value σ_3^W given in Eq. (52). Substituting Eqs. (68)-(73) into Eq. (67), for $\omega_1\tau = \pi/2$, we obtain

$$\mathrm{Pr}_{i\to f}^{\mathrm{Ramsey}}(\pm;\epsilon) = \frac{1}{2}\left(1 \mp \alpha\cos\left(2\phi_{\mathrm{Ramsey}} - \epsilon T\right)\right) \tag{74}$$

$$\simeq \frac{1}{2}\left[1 \mp \alpha\left(1 - 2\mathrm{Pr}_{+\to+}^{\mathrm{Ramsey}}(0) + 2\epsilon T\,\mathrm{Pr}_{+\to+}^{\mathrm{Ramsey}}(0)\,\mathrm{Im}\,\sigma_3^W\right)\right], \tag{75}$$

where

$$\alpha := P_i(1 - 2\epsilon_f) \tag{76}$$

represents the overall degree of perfection in state preparation and measurement currently considered.

From Eq. (75), the imaginary part of the weak value can be retrieved as

$$\mathrm{Im}\,\sigma_3^W = \frac{2}{1 \mp \alpha - 2\mathrm{Pr}_{i\to f}^{\mathrm{Ramsey}}(\pm;0)}\frac{d\mathrm{Pr}_{i\to f}^{\mathrm{Ramsey}}(\pm;\epsilon)}{d(\epsilon T)}\bigg|_{\epsilon=0}. \tag{77}$$

Since the right-hand side of Eq. (77) consists of measurable quantities, we realize that the weak value $\operatorname{Im}\sigma_3^W$ can be determined through experiment realizing the Ramsey resonance. Below, we discuss a neutron EDM measurement in which $\operatorname{Im}\sigma_3^W$ can be obtained using the Ramsey resonance technique even when the presence of imperfection cannot be ignored.

B. Neutron EDM measurement experiment

In the neutron EDM measurement, the above Ramsey resonance technique has actually been widely applied, where the parameters in Eqs. (24) and (38) correspond to

$$\overline{\omega}_0 = 2\mu_{\rm n}B_0, \ \epsilon = 2d_{\rm n}E_0, \ \omega_1 = \mu_{\rm n}B_1/2. \qquad (78)$$

Here, $\mu_{\rm n}$ is the magnetic dipole moment of the neutron, $d_{\rm n}$ is the EDM of the neutron, B_0 and E_0 are static magnetic and electric fiedls, respectively, and B_1 denotes the magnitude of rotating magnetic field. Currently, the value of the neutron EDM has not been determined through experiments, and only its upper limit has been measured [3]. Namely, to date, ϵ is not inconsistent with zero in the experiments. We follow Refs. [14, 15] and briefly review some of the key points on technical descriptions of the experiment.

Most recent experiments on neutron EDM use ultra-cold neutrons (UCNs) which move at a velocity of less than 7 m/s. At the start of the Ramsey resonance technique, the neutrons pass through the magnetized polarizing foil in order to prepare the initial state (64), and the tranmitted neutrons are stored in an evacuated chamber that has walls reflecting the neutrons. The rotating magnetic field B_1, perpendicular to B_0, is applied during the interval $\tau/2 = 2\,{\rm s}$. Subsequently, we let the neutrons precess during the interval $T = 130\,{\rm s}$ until the second rotating magnetic field B_1. After this, the neutrons are released from the evacuated chamber, and the number of neutrons with spin up (and those with spin down) is counted by using the magnetized polarizing foil, which implements the measurement of Eqs. (65) and (66).

In one of the measurements this cycle was repeated many times, and each cycle yielded about 14000 UCNs. Namely, these are obtained after cuts were applied to data gained in 545 runs containing 175,217 measurement cycles with 2.5×10^9 neutrons remained [15]. It is apparent from Eq. (74) that to measure the EDM effect ϵ, the magnetic field B_0 must be accurately controlled. A $^{199}{\rm Hg}$ cohabiting magnetometer is utilized to attain a precise magnetic field, that is, polarized $^{199}{\rm Hg}$ vapor is filled with the same chamber as the UCNs and the magnetic field B_0 is measured by monitoring the mercury precession frequency $\omega_{\rm Hg}$. By using the cohabiting magnetometer, the first-order estimate of the neutron precession frequency is determined as $\omega_{\rm n,Hg} :=$

$|\mu_{\rm n}/\mu_{\rm Hg}|\omega_{\rm Hg}$ where $\mu_{\rm Hg}$ is the magnetic dipole moment of the mercury.

Other experiments have precisely measured the ratio of magnetic dipole moments, which has a known value $\mu_{\rm n}/\mu_{\rm Hg} = -3.8424574(30)$ [16]. From Eq. (74), the number of counted spin-up or down neutrons $N_\pm$ reads[#3]

$$N_\pm = \overline{N}\left(1 \mp \alpha\cos\left(2\phi_{\rm Ramsey} - \epsilon T\right)\right), \qquad (79)$$

in which we have

$$
\begin{aligned}
2\phi_{\rm Ramsey} - \epsilon T &= (\omega - \omega_0)\,T \\
&= \left[(\omega - \omega_{\rm n,Hg}) - (\omega_0 - \omega_{\rm n,Hg})\right]T \\
&= (\delta\omega - \Phi)\,T, \qquad (80)
\end{aligned}
$$

where $\delta\omega := \omega - \omega_{\rm n,Hg}$ and $\Phi := \omega_0 - \omega_{\rm n,Hg}$. The true neutron frequency ω_0 may differ from the first-order estimate $\omega_{\rm n,Hg}$ measured from the mercury cohabiting magnetometer for several reasons, such as the EDM effect, the inherent uncertainty in the ratio $\mu_{\rm n}/\mu_{\rm Hg}$, and the difference in the spatial distribution in the chamber between UCNs and $^{199}{\rm Hg}$ via gravity. The above measurements were conducted by tuning $\delta\omega$ and varying the directions of the electric and magnetic fields depending on the cycles, and each run, typically consisting of several hundred measurement cycles, contains the cycles with different $\delta\omega$ and directions of the fields. If the electric and magnetic fields are parallel or anti-parallel, the precession frequency ω_0 is expressed as $\omega_0 = -2\mu_{\rm n}|B_0| \mp 2d_{\rm n}|E_0|$. In the approach of Ref. [15], for each run, the neutron counts $N_\pm$ were fitted to Eq. (79) for each of spin-up and -down states, and thereby determined $\overline{N}, \alpha$ and Φ for each of the two spin states. Considering this, for each run, Eq. (79) is now be written as

$$N_\pm = \overline{N}_\pm\left(1 \mp \alpha_\pm\cos\left(\delta\omega - \Phi_\pm\right)T\right). \qquad (81)$$

Note here that $\Phi_\pm$ is averaged over each run, during which the effects of EDM are expected to cancel out. In contrast, for each measurement cycle j and for each spin state, Eq. (79) should become

$$N_{\pm,j} = \overline{N}_\pm\left(1 \mp \alpha_\pm\cos\left(\delta\omega - \Phi_{\pm,j}\right)T\right). \qquad (82)$$

Comparing Eqs. (81) and (82), we find that the EDM contribution to the measurement cycle j may be given by

$$
\begin{aligned}
\epsilon_{\pm,j} &:= \Phi_{\pm,j} - \Phi_\pm \\
&= \delta\omega - \Phi_\pm - \frac{1}{T}\arccos\left[\frac{N_{\pm,j} - \overline{N}_\pm}{\overline{N}_\pm\alpha_\pm}\right]. \qquad (83)
\end{aligned}
$$

Meanwhile, we notice from Eq. (79) that

$$\frac{dN_\pm}{d\omega} = \pm\overline{N}\alpha T\sin\left(\omega - \omega_0\right)T, \qquad (84)$$

[#3] The dark counts may occur due, *e.g.*, to possible neutron decay. Such effects will be included in the prefactor $\overline{N}$.
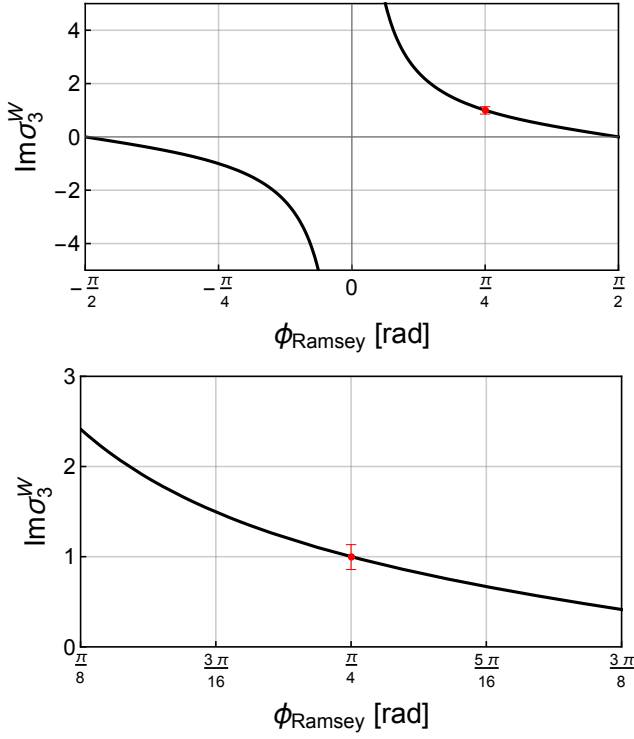
FIG. 1. The imaginary component of the weak value of σ_3 of Eq. (85) (black line) for various ϕ_{Ramsey}, together with a past experimental value [17] (red point). The upper plot is shown in a range $[-\pi/2, \pi/2]$, while the bottom depicts the enlarged plot focused in the range $[\pi/8, 3\pi/8]$ with an error bar given by the uncertainty.

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which implies that measurements at $\omega \simeq \omega_0 \pm \pi/2T$ possess the maximal sensitivity for $N_\pm$ against ϵ. In the neutron EDM experiment, the statistical uncertainty has been known to dominate over the systematic uncertainty. Since the fractional uncertainty in the number of neutrons counted is at best $1/\sqrt{N}$ with $N := N_+ + N_-$, we see from Eq. (84) that the uncertainty in the measurement of the frequency Φ is not better than $1/\sqrt{N}\alpha T$. Similarly, the statistical uncertainty in the EDM due to neutron counting is estimated as $1/2\alpha E_0 T\sqrt{N}$. According to Ref. [15], combining the applied electric field of $E_0 = 7$ kV/cm, the averaged polarization $\alpha = 0.58$, $T = 130$ s and $N = 2.5 \times 10^9$, and also removing from consideration the measurement cycles with $E_0 = 0$ kV/cm, the statistical uncertainty is estimated as $1.34 \times 10^{-26}\, e$cm, which is consistent with a more sophisticated estimation at a few percent level. Similarly, the statistical uncertainty of ΦT is also estimated as $\sigma_{\Phi T} \simeq 3.71 \times 10^{-5}$. The point is that the phase of the cosine function of Eq. (79) is kept under control within the estimated uncertainty of $\sigma_{\Phi T}$ in the experiment [15].

Now, let us return to the weak value of Eq. (77). Using Eqs. (81), (82) and (77), for each measurement cycle j,
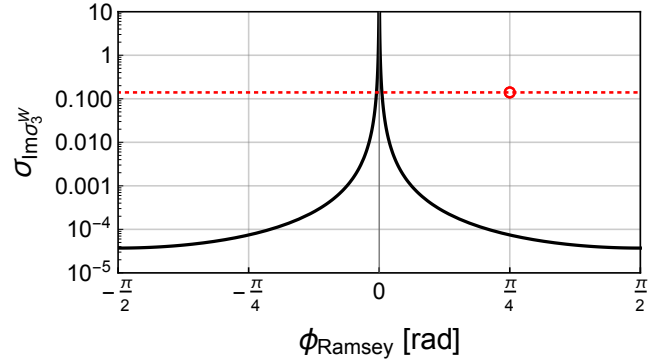

FIG. 2. The uncertainty of the imaginary component of the weak value of σ_3 of Eq. (86) (black line), together with a past experimental value [17] (dotted red line), which corresponds to the error bar of Fig. 1. We assumed $\sigma_{\Phi T} \simeq 3.71 \times 10^{-5}$ in the whole range.

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the imaginary part of the weak value σ_3^W is obtained as

$$
\begin{aligned}
(\mathrm{Im}\,\sigma_3^W)_j &= \frac{1}{\epsilon_{\pm,j} T} \frac{\cos\left(\delta\omega - \Phi_{\pm,j}\right)T - \cos\left(\delta\omega - \Phi_\pm\right)T}{1 - \cos\left(\delta\omega - \Phi_{\pm,j}\right)T} \\
&\simeq \frac{\sin\left(\delta\omega - \Phi_\pm\right)T}{1 - \cos\left(\delta\omega - \Phi_\pm\right)T} \\
&= \cot\phi_{\mathrm{Ramsey}},
\end{aligned}
\tag{85}
$$

where in the second line we omitted the first order of the EDM effects. Fig. 1 shows the values of $\mathrm{Im}\,\sigma_3^W$ as a function of ϕ_{Ramsey}. The neutron EDM experiments determine the phases of trigonometric functions of the right-hand side of Eq. (85), which in turn determines the weak value with the uncertainty,

$$
\begin{aligned}
\sigma_{\mathrm{Im}\,\sigma_3^W} &:= \left| \csc\left(\frac{\left(\delta\omega - \Phi_\pm\right)T}{2} \right) \right|^2 \sigma_{\Phi T} \\
&= |\csc\phi_{\mathrm{Ramsey}}|^2 \, \sigma_{\Phi T},
\end{aligned}
\tag{86}
$$

as shown in Fig. 2 as a function of ϕ_{Ramsey}. For $\phi_{\mathrm{Ramsey}} = \pm\pi/4$ at which the red point of Fig. 2 resides, we have the theoretical uncertainty $\sigma_{\mathrm{Im}\,\sigma_3^W} \simeq 7.42 \times 10^{-5}$ under the reference value $\sigma_{\Phi T}$ mentioned before, which is far below the uncertainty in the preceding experiment of Ref. [17].

C. Comparison with previous works

Before closing, we wish to compare our method of measuring the weak value through the Ramsey resonance to the conventional weak value measurement briefly. The weak value of the Pauli spin operator σ_3 of neutrons has been previously measured at the high-flux reactor of the Institute Laue-Langevin (ILL) [17] in Grenoble, France, by using the neutron interferometer and neutron beams, where the real and imaginary components of the weak

value,

$$\langle \sigma_3 \rangle_{\mathrm{w}} := \frac{\langle \Psi_f | \sigma_3 | \Psi_i \rangle}{\langle \Psi_f | \Psi_i \rangle}, \qquad (87)$$

has been reported. There, the pre- and post-selected states are chosen as $|\Psi_i\rangle = e^{-i\pi\sigma_2/4}|+\rangle$ and $|\Psi_f\rangle = e^{-i\alpha\sigma_3/2}e^{-i\beta\sigma_2/2}|+\rangle$. In particular, for $\beta = -\pi/2$, we

obtain

$$\langle \Psi_f | \Psi_i \rangle = \langle + | e^{-i\pi\sigma_1/4} e^{i\alpha\sigma_3/2} e^{-i\pi\sigma_1/4} |+\rangle, \qquad (88)$$

$$\langle \Psi_f | \sigma_3 | \Psi_i \rangle = \langle + | e^{-i\pi\sigma_1/4} e^{i\alpha\sigma_3/2} \sigma_3 e^{-i\pi\sigma_1/4} |+\rangle. \qquad (89)$$

Recall that, for $|\psi_i(\phi_{\mathrm{Ramsey}})\rangle = |\psi_f(\phi_{\mathrm{Ramsey}})\rangle = e^{-i\phi_{\mathrm{Ramsey}}\sigma_2/2}|+\rangle$, our weak value (52) becomes

$$\sigma_3^W = \frac{\langle \psi_f(\phi_{\mathrm{Ramsey}}) | e^{-i\omega_1\tau\sigma_1} \sigma_2 | \psi_i(\phi_{\mathrm{Ramsey}}) \rangle}{\langle \psi_f(\phi_{\mathrm{Ramsey}}) | e^{-i\omega_1\tau\sigma_1} | \psi_i(\phi_{\mathrm{Ramsey}}) \rangle}.$$

For $\omega_1\tau = \pi/2$, the numerator and denominator are found to be

$$\langle \psi_f(\phi_{\mathrm{Ramsey}}) | e^{-i\omega_1\tau\sigma_1} | \psi_i(\phi_{\mathrm{Ramsey}}) \rangle = \langle + | e^{-i\pi\sigma_1/4} e^{-i\phi_{\mathrm{Ramsey}}\sigma_3} e^{-i\pi\sigma_1/4} |+\rangle, \qquad (90)$$

$$\langle \psi_f(\phi_{\mathrm{Ramsey}}) | e^{-i\omega_1\tau\sigma_1} \sigma_2 | \psi_i(\phi_{\mathrm{Ramsey}}) \rangle = \langle + | e^{-i\pi\sigma_1/4} e^{-i\phi_{\mathrm{Ramsey}}\sigma_3} \sigma_3 e^{-i\pi\sigma_1/4} |+\rangle, \qquad (91)$$

with ϕ_{Ramsey} given in (56). Observe then that, for $\alpha = -2\phi_{\mathrm{Ramsey}}$, Eqs. (90) and (91) are equal to Eqs. (88) and (89), respectively. Thus, for $\alpha = -2\phi_{\mathrm{Ramsey}}$, $\beta = -\pi/2$, and $\omega_1\tau = \pi/2$, $\langle \sigma_3 \rangle_{\mathrm{w}} = \sigma_3^W$ holds. In Ref. [17], Im $\langle \sigma_3 \rangle_{\mathrm{w}}$ was measured for $\alpha = -\pi/2$ and $\beta = -\pi/2$ as shown in Fig. 1 and 2. From Fig. 2, it is clear that the weak value measurement via the Ramsey resonance may improve the uncertainty of the imaginary component of the neutron's σ_3 by three orders of magnitude compared to Ref. [17].

However, it should be emphasized that, while neutron EDM experiments may be suited to precisely measure the imaginary part of σ_3, the conventional neutron beam experiments can measure both the real and imaginary parts of weak values. Besides, the conventional experiments using the neutron interferometer are indirect measurements and might be more suited for physical realization of quantum paradoxes such as the quantum Cheshire Cat [18].

V. SUMMARY

In the present paper we revisited the two typical resonance phenomena, the Rabi resonance and the Ramsey resonance, and showed that their response to deviation from the resonance point can be interpreted as the perturbative effect (represented by the term V in the Hamiltonian) in the weak measurement. Our main results given in Sec. III are: (i) Both of the Rabi and Ramsey resonances amount to the weak value amplification. (ii) Near the resonance point, the transition probability becomes identical in form for both the Rabi and Ramsey resonances differing only in their measurement strengths. (iii) The differences between the two resonances in the measurement strength and in the sensitivity to disturbance can be understood

based on the probability formula shared by them. In short, we found that the viewpoint of weak measurement provides a unified understanding of the Rabi and Ramsey resonances, where the the resonance phenomena are characterized by the behavior of the measurement strength under external conditions.

We also argued that by exploiting the data of previous neutron EDM experiments one can determine the imaginary component of the weak value of the spin of the neutrons. Moreover, we find that the precision of the obtained weak value turns out to be much higher (by three orders of magnitude) than that of the conventional weak value measurements using neutron beams. Although the conventional method has its merit in being capable of measuring the real and imaginary parts of weak values, our observation indicates that novel advantage may be uncovered by shedding a new light on resonances from the viewpoint of weak value amplification.

We end with a few remarks on the outlooks of this work. First, weak value measurements through ground-state hyperfine transition frequency of the cesium 133 atoms may be possible because they are also based on the Ramsey resonance. It would be interesting there to investigate the weak values associated with the minimal disturbance of the SI time definition. Second, the fundamental parameter measurements via the resonance might lead to a technology that allows one to conduct quantum paradoxes, including the quantum Cheshire Cat and the Three Box Paradox [19], which involve the intriguing aspect of peculiar quantum existence and have been discussed primarily in the context of weak measurement. Third, it could be possible to find the counterpart of resonance in indirect measurement through the connection between weak value amplification and resonances mentioned here. In fact, since all the

weak measurements discussed in this paper are direct measurements which are rather uncommon, we may expect more versatile outcomes for common indirect measurements, including a novel procedure of precision measurement based on the resonance phenomena.

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