Complex Probability Measure and Aharonov's Weak Value

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(Dated: October 5, 2012)

We present a complex probability measure relevant for double (pairs of) states in quantum mechanics, as an extension of the standard probability measure for single states that underlies Born's statistical rule. When the double states are treated as the initial and final states of a quantum process, we find that Aharonov's weak value, which has acquired a renewed interest as a novel observable quantity inherent in the process, arises as an expectation value associated with the probability measure. Despite being complex, our measure admits the physical interpretation as mixed processes, *i.e.*, an ensemble of processes superposed with classical probabilities.

PACS numbers: 03.65.-w, 02.50.Cw, 03.65.Aa, 03.65.Ta, 03.65.Ca

Introduction. The weak value, which has long been advocated by Aharonov *et al.* [1] as a novel and physically measurable quantity, is currently attracting much attention. The weak value arises as an intrinsic physical entity in the time-symmetric formulation of quantum mechanics [2], where one considers a quantum process specified by both initial and final states (obtained by pre- and post-selections), instead of only an initial state (preselection) in the usual formulation of quantum mechanics.

There are basically two reasons for the renewed interest in the weak value. One is that it offers an intriguing possibility of extending the scope of 'physical reality' in quantum mechanics, and thereby sheds a new light both on paradoxical phenomena such as Hardy's paradox [3, 4], and on fundamental issues such as the direct detection of wave function [5] or tracing the trajectories of Young's double slit experiment [6]. The other, which is more important for practical purposes, is that it serves to measure physical quantities by macroscopic apparatuses with significantly amplified precision, as has been demonstrated in the strength of interaction in the spin Hall effect [7] or in the beam diffraction [8].

Despite these impressive developments and the enhanced recognition of its significance, the weak value still remains obscure, not just because it is complex-valued and defies our direct perception as physical reality, but also because it lacks a solid basis for the uniqueness of the form, that is, the reason why it takes the form as it stands. The present work is an attempt to provide such a basis from probability theory. Specifically, we consider the possible form of probability measure that can be associated with pairs of states, called 'double states' in this paper, in a manner consistent with the given states. Employing an argument analogous to that used in Gleason's theorem [9] which determines the real probability measure and lays a foundation of Born's statistical rule for single states, we show that the measure for double states can also be determined properly. Although our measure turns out to be complex in general and contains a free parameter, it reduces in the single state limit uniquely to the conventional real measure crucial to Born's rule. When applied to quantum processes in which the double states are identified with the initial and final states, our complex measure is found to admit the interpretation of mixed processes, *i.e.*, ensemble of processes superposed with real probabilities, assuring that it is indeed physically testable. The weak value is then obtained as the expectation value of the observable measured in a particular ensemble corresponding to a pure process specified by the double states.

In what follows, we begin by invoking the probability measure for single states and proceed to discuss the probability measure for double states. We then apply our measure to quantum processes to see how the weak value arises there, and argue that the appropriately generalized measure admits an operational interpretation as mixed processes.

Probability Measure for Single States. For our purposes, we first recall Gleason's theorem [9] which deduces Born's statistical rule based on a probability measure fulfilling certain logical conditions. Given a Hilbert space $\mathcal{H}$ of a finite dimension $d \geq 3$, consider a real-valued measure μ which is a map from the space of projection operators $\mathcal{P}(\mathcal{H})$ of $\mathcal{H}$ to non-negatives, *i.e.*, $\mu\colon \mathcal{P}(\mathcal{H}) \to [0, \infty) \subset \mathbb{R}$. The theorem states that, if the map is

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bounded $|\mu(P)| < \infty$ and satisfies the partial additivity condition,

$$\mu\left(\sum_i P_i\right) = \sum_i \mu\left(P_i\right), \tag{1}$$

for a set of projection operators $\{P_i\}$ which are mutually orthogonal $P_iP_j = \emptyset$ (null operator) for $i \neq j$, then it has the form,

$$\mu(P) = \mathrm{tr}(WP), \tag{2}$$

with a positive self-adjoint trace class operator W.

Since we have $\mu(\emptyset) = 0$ from (2), if the range of the map μ is restricted to $[0, 1]$, and if the condition $\mu(\mathbb{1}) = 1$ is further imposed, then obviously the map can be interpreted as a probability measure with the attached meaning that $\emptyset$ and $\mathbb{1}$ represent propositions which are identically false and true, respectively. Note that $\mu(\mathbb{1}) = 1$ implies $\mathrm{tr}(W) = 1$.

In quantum mechanics, the probability measure is indeed realized by such a measure μ, where the operator W corresponds to the density operator ρ that characterizes the state of the system. For instance, if the system is described by a pure state $\rho = P_\psi := |\psi\rangle\langle\psi|$ for some normalized $|\psi\rangle \in \mathcal{H}$, our probability measure is required to yield

$$\mu(P_\psi) = 1, \qquad \mu(P_{\psi^\perp}) = 0. \tag{3}$$

Here, the first condition states that the probability of being in the state $|\psi\rangle$ is unity, whereas the second states that there is no probability assigned for an arbitrary state $|\psi^\perp\rangle$ orthogonal to $|\psi\rangle$ for which the projection is given by $P_{\psi^\perp} := |\psi^\perp\rangle\langle\psi^\perp|$. Namely, the measure μ has no support for the subspace $\mathcal{P}(\mathcal{H}_\psi^\perp) \subset \mathcal{P}(\mathcal{H})$, where $\mathcal{H}_\psi^\perp$ is the orthogonal complement to the one-dimensional subspace $\mathcal{H}_\psi = \mathrm{span}\{|\psi\rangle\}$. We note that, because of (1) the second condition in (3) actually follows from the first for a non-negative map μ, but this will no longer be the case when the non-negativity is lifted.

From (3) one finds that W is uniquely determined as $W = \rho = P_\psi$, and this shows that the probability of the state $|\psi\rangle$ being in the subspace $\mathcal{H}_i \subset \mathcal{H}$ specified by the projection P_i reads $\mu(P_i) = \mathrm{tr}(\rho P_i)$, which is just Born's statistical rule. It follows that, if an observable A is measured in the pure state ρ, the expectation value is given by

$$\mathcal{E}(A) := \sum_i a_i\, \mu(P_i) = \langle\psi|A|\psi\rangle, \tag{4}$$

where a_i is an eigenvalue of A, and P_i is the corresponding projection appearing in the spectral decomposition $A = \sum_i a_i\, P_i$. One notable consequence of this is that the expectation value satisfies the sum rule, $\mathcal{E}(A+B) = \mathcal{E}(A) + \mathcal{E}(B)$ for any observables A, B which may not commute with each other. This implies that, although the sum of the individual measurement outcomes of A and B may not be an eigenvalue of $A+B$, on average they coincide.

Complex Measure for Double States. Now we extend the forgoing argument to a measure characterized by double states. Let $\{|\psi\rangle, |\phi\rangle\}$ be two states arbitrarily chosen from $\mathcal{H}$ except that they are neither identical (up to a phase) nor orthogonal to each other (*i.e.*, $\langle\phi|\,\psi\rangle \neq 0$).

Analogously to the single state case (3), given the two states $\{|\psi\rangle, |\phi\rangle\}$ we wish to require

$$\begin{array}{ll} \mu(P_\psi) = 1, & \mu(P_{\psi^\perp}) = 0, \\ \mu(P_\phi) = 1, & \mu(P_{\phi^\perp}) = 0, \end{array} \tag{5}$$

where $P_\phi = |\phi\rangle\langle\phi|$ and $P_{\phi^\perp} = |\phi^\perp\rangle\langle\phi^\perp|$ with $|\phi^\perp\rangle \in \mathcal{H}_\phi^\perp$. Obviously, in view of the uniqueness of W, this is impossible unless the two states are identical. However, the condition (5) can be met if one promotes the measure to a complex one.

To see this, let us invoke the generalized Gleason's theorem [10] which extends the range of the map from $[0, \infty)$ to the entire reals $\mathbb{R}$. Demanding the condition (1), one finds that such a measure μ_R admits the same form,

$$\mu_R(P) = \mathrm{tr}(W_R P), \tag{6}$$

but now W_R is a self-adjoint trace class operator, not necessarily positive. In order to extend the range of the map to complex numbers $\mathbb{C}$, we choose two such real maps μ_R, $\mu'_R\colon \mathcal{P}(\mathcal{H}) \to \mathbb{R}$ and consider

$$\mu_C(P) = \mu_R(P) + i\mu'_R(P), \tag{7}$$

with the imaginary unit i. Clearly, the map μ_C still fulfills (1) by linearity and is written as

$$\mu_C(P) = \mathrm{tr}(W_C P), \qquad W_C = W_R + iW'_R, \tag{8}$$

where W_R and W'_R are the self-adjoint trace class operators associated with μ_R and μ'_R, respectively. We then have:

Theorem *If a map $\mu_C\colon \mathcal{P}(\mathcal{H}) \to \mathbb{C}$ satisfies the partial additivity condition (1) and the consistency condition (5) for two non-identical states $|\psi\rangle, |\phi\rangle$ with $\langle\phi|\,\psi\rangle \neq 0$, then it has the form,*

$$\begin{aligned} \mu_C(P) &= \mathrm{tr}(W_C P), \\ W_C &= \alpha\frac{|\psi\rangle\langle\phi|}{\langle\phi|\psi\rangle} + (1-\alpha)\frac{|\phi\rangle\langle\psi|}{\langle\psi|\phi\rangle}, \end{aligned} \tag{9}$$

for some $\alpha \in \mathbb{C}$.

Proof. The complex measure fulfilling (1) is given by (8) with a trace class operator W_C. Let $\{|e_i\rangle; i = 1, \ldots, d\}$ be a complete orthonormal basis in $\mathcal{H}$ with $|e_1\rangle = |\psi\rangle$. In terms of this we expand W_C and $|\psi^\perp\rangle \in \mathcal{H}_\psi^\perp$ as

$$W_C = \sum_{i,j=1}^{d} \beta_{ij}|e_i\rangle\langle e_j|, \qquad |\psi^\perp\rangle = \sum_{i=2}^{d} \gamma_i|e_i\rangle, \tag{10}$$

with $\beta_{ij}, \gamma_i \in \mathbb{C}$. From (5) we have

$$\begin{aligned} 0 = \mu_C(P_{\psi^\perp}) &= \sum_{i,j}\sum_{k,l\geq 2} \beta_{ij}\gamma_k\gamma_l^* \mathrm{tr}(|e_i\rangle\langle e_j|e_k\rangle\langle e_l|) \\ &= \sum_{i,j\geq 2} \beta_{ij}\gamma_j\gamma_i, \end{aligned} \tag{11}$$

which implies $\beta_{ij} = 0$ for $i, j \geq 2$ since γ_i can be chosen arbitrarily. The operator W_C is thus written, with some (unnormalized) states $|\xi_1\rangle, |\xi_2\rangle \in \mathcal{H}_\psi^\perp$, as

$$W_C = |\psi\rangle\langle\psi| + |\psi\rangle\langle\xi_1| + |\xi_2\rangle\langle\psi|. \qquad (12)$$

Defining

$$|\eta\rangle = |\phi\rangle - \langle\psi|\phi\rangle\,|\psi\rangle \in \mathcal{H}_\psi^\perp, \qquad (13)$$

we further decompose

$$|\xi_i\rangle = z_i\,|\eta\rangle + |\xi_i'\rangle, \qquad i = 1, 2, \qquad (14)$$

with $z_i \in \mathbb{C}$ so that $|\xi_i'\rangle \in \mathcal{H}_\eta^\perp$ in addition to $|\xi_i'\rangle \in \mathcal{H}_\psi^\perp$. Similarly, if we define

$$|\zeta\rangle = |\psi\rangle - \langle\phi|\psi\rangle|\phi\rangle \in \mathcal{H}_\phi^\perp, \qquad (15)$$

we find that both $|\phi\rangle$ and $|\zeta\rangle$ belong to the linear space spanned by $|\psi\rangle$ and $|\eta\rangle$. It follows that $|\xi_i'\rangle \in \mathcal{H}_\phi^\perp$ and $|\xi_i'\rangle \in \mathcal{H}_\zeta^\perp$ for $i = 1, 2$ as well. This observation motivates us to rewrite (12) in favor of $|\phi\rangle$, $|\zeta\rangle$ and $|\xi_i'\rangle$ to find

$$\begin{aligned} W_C &= \omega_{\zeta\zeta}|\zeta\rangle\langle\zeta| + \omega_{\phi\phi}|\phi\rangle\langle\phi| + \omega_{\phi\zeta}|\phi\rangle\langle\zeta| + \omega_{\zeta\phi}|\zeta\rangle\langle\phi| \\ &\quad + \omega_{\phi\xi_1}|\phi\rangle\langle\xi_1'| + \omega_{\xi_2\phi}|\xi_2'\rangle\langle\phi| + |\zeta\rangle\langle\xi_1'| + |\xi_2'\rangle\langle\zeta|, \end{aligned} \qquad (16)$$

where

$$\begin{aligned} &\omega_{\zeta\zeta} = 1 - z_1^*\langle\phi|\psi\rangle - z_2\langle\psi|\phi\rangle, \\ &\omega_{\phi\phi} = 1 - \omega_{\zeta\zeta} + \omega_{\zeta\zeta}|\langle\psi|\phi\rangle|^2, \\ &\omega_{\phi\zeta} = z_2 + \omega_{\zeta\zeta}\langle\phi|\psi\rangle, \\ &\omega_{\zeta\phi} = z_1^* + \omega_{\zeta\zeta}\langle\psi|\phi\rangle, \\ &\omega_{\phi\xi_1} = \langle\phi|\psi\rangle, \qquad \omega_{\xi_2\phi} = \langle\psi|\phi\rangle. \end{aligned} \qquad (17)$$

On the other hand, an analogous argument for the state $|\phi\rangle$ demanded by (5) shows that W_C must also be of the form,

$$W_C = |\phi\rangle\langle\phi| + |\phi\rangle\langle\chi_1| + |\chi_2\rangle\langle\phi|, \qquad (18)$$

with some (unnormalized) states $|\chi_1\rangle, |\chi_2\rangle \in \mathcal{H}_\phi^\perp$. Since $|\psi\rangle$ and $|\phi\rangle$ are not identical, we have $|\zeta\rangle \neq 0$. Comparing (16) and (18), we obtain

$$\omega_{\zeta\zeta} = 0, \qquad |\xi_1'\rangle = |\xi_2'\rangle = 0, \qquad (19)$$

which implies

$$\begin{aligned} W_C &= |\psi\rangle\langle\psi| + z_1^*|\psi\rangle\langle\eta| + z_2|\eta\rangle\langle\psi| \\ &= z_1^*|\psi\rangle\langle\phi| + z_2|\phi\rangle\langle\psi|. \end{aligned} \qquad (20)$$

Since z_1, z_2 are free parameters but subject to $\omega_{\zeta\zeta} = 0$, we arrive at (9) after putting $\alpha = z_1^*\langle\phi|\psi\rangle$. ■

Having found the complex measure μ_C for double states specified by (5), we may consider the expectation value of an observable A. Putting aside the question of the meaning of complex probability for the moment, we just follow the standard construction of the expectation value as we had in (4) to find

$$\begin{aligned} \lambda(A) &:= \sum_i a_i\,\mu_C(P_i) \\ &= \alpha\frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle} + (1-\alpha)\frac{\langle\psi|A|\phi\rangle}{\langle\psi|\phi\rangle}. \end{aligned} \qquad (21)$$

We then notice that Aharonov's weak value $A_w = \langle\phi|A|\psi\rangle/\langle\phi|\psi\rangle$ arises at the choice $\alpha = 1$ of the expectation value $\lambda(A)$. Although $\lambda(A)$ is complex in general, it becomes real at $\alpha = 1/2$ where W_C becomes self-adjoint. This shows that one can find the measure μ which meets the condition (5) by extending the range of the map only to the entire $\mathbb{R}$, but we shall soon realize that the particular measure obtained by $\alpha = 1/2$ does not account for all possible cases when applied to quantum processes. Notice also that, as for $\mathcal{E}(A)$ the sum rule holds, $\lambda(A + B) = \lambda(A) + \lambda(B)$, for any α.

Interestingly, in the single state limit, that is, in the limit $|\phi\rangle \to |\psi\rangle$ the ambiguity of α disappears and our complex measure μ_C reduces to the real measure μ in (2) with the condition (3) enforced. Accordingly, the expectation value $\lambda(A)$ also reduces to the conventional one $\mathcal{E}(A)$ in (4).

Another observation worth mentioning is that, since under the single state $|\psi\rangle$ the probability $\mu(P_\phi) = |\langle\psi|\phi\rangle|^2$ represents the compatibility of the double states $|\phi\rangle$ and $|\psi\rangle$, one obtains the overall expectation value of A by the weighted product, $|\langle\psi|\phi\rangle|^2\lambda(A)$. The average value obtained after allowing the state $|\phi\rangle$ to vary freely may then be evaluated by

$$\sum_{|\phi\rangle\in\mathcal{B}} |\langle\psi|\phi\rangle|^2\lambda(A) = \langle\psi|A|\psi\rangle, \qquad (22)$$

where the summation is over the states of a complete basis $\mathcal{B}$ of $\mathcal{H}$. This outcome (22) implies that the average is actually independent of the choice of the basis set $\mathcal{B}$ as well as the parameter α. Similarly, if one averages over $|\psi\rangle$ instead, one obtains $\langle\phi|A|\phi\rangle$. In other words, the information of one of the double states will disappear once the average of the state is taken, and the outcome becomes the standard expectation value with respect to the other state left over.

So far, we have considered complex measures with W_C of the type (9) which fulfils (5). As one can extend the scope of single states from pure states to mixed states by allowing W to be any positive self-adjoint operators with unit trace, one may similarly extend the scope of double states by allowing W_C to be any operators with unit trace. If we let $\mathcal{T}(\mathcal{H})$ be the space of operators with unit trace, we have

$$\beta\,W_C + (1-\beta)\,W_C' \in \mathcal{T}(\mathcal{H}), \qquad (23)$$

for $W_C, W_C' \in \mathcal{T}(\mathcal{H})$ and $\beta \in \mathbb{C}$. This shows that, if we regard $\mathcal{T}(\mathcal{H})$ as the space of general double states, the space is 'convex' in the complex sense. The measure $\mu_C(P)$ also provides the map $\Pr(P; W_C) := \mu_C(P) = \mathrm{tr}(W_C P)$, which enjoys the affine property,

$$\begin{aligned} &\Pr(P; \beta\,W_C + (1-\beta)\,W_C') \\ &\quad = \beta\Pr(P; W_C) + (1-\beta)\Pr(P; W_C'), \end{aligned} \qquad (24)$$

analogous to the conventional probability map. Along with the property $\Pr(\emptyset; W_C) = 0$ and $\Pr(\mathbb{1}; W_C) = 1$, this may be considered as a formal support for $\mu_C(P)$ qualifying as a probability measure, albeit it is complex. To explore the possible use of the complex measure, and thereby examine the physical significance of the complex parameter α in (9) or (21), we now turn to the probability measure for a quantum process.

Probability Measure for a Process. The complex measure for double states can be used to furnish the probability measure for a quantum process $|\psi\rangle \to |\phi\rangle$ by taking the time dependence of the states properly into consideration. Let t_i and t_f be the initial time and the final time at which the states $|\psi\rangle$ and $|\phi\rangle$ are realized, respectively, and let t be the time of 'measuring' the obserbable A in the period, $t_i \le t \le t_f$. To evaluate the outcome of the measurement results, we would like to have the complex measure relevant at time t. Assuming that our system is closed during the process, we have a unitary operator U to describe the time development in the period. The forward time-developed state at t from the initial state is then given by $U(t-t_i)\,|\psi\rangle$, and the backward time-developed state at t from the final state is given by $U(t-t_f)\,|\phi\rangle$. This suggests that, instead of the two states $|\psi\rangle$, $|\phi\rangle$, we should use these forward and backward time-developed states to characterize the measure (9). We are thus led to

$$W_C(t) = \alpha \frac{U(t-t_i)\,|\psi\rangle\,\langle\phi|\,U^\dagger(t-t_f)}{\langle\phi|\,U(t_f-t_i)\,|\psi\rangle} + (1-\alpha)\frac{U(t-t_f)\,|\phi\rangle\,\langle\psi|\,U^\dagger(t-t_i)}{\langle\psi|\,U^\dagger(t_f-t_i)\,|\phi\rangle}, \tag{25}$$

from which we can obtain the time-dependent measure, $\mu_C(P;t) := \mathrm{tr}(W_C(t)P)$. The expectation value (21) then acquires the corresponding time-dependence by the use of $\mu_C(P;t)$, which is now characterized by the consistency condition at the initial and final times,

$$\begin{aligned} \mu_C(P_\psi;t_i) &= 1, \qquad \mu_C(P_{\psi^\perp};t_i) = 0, \\ \mu_C(P_\phi;t_f) &= 1, \qquad \mu_C(P_{\phi^\perp};t_f) = 0. \end{aligned} \tag{26}$$

However, the identification of the operator W_C by (25) with the process $|\psi\rangle \to |\phi\rangle$ in the period $[t_i, t_f]$ is not quite correct, because our measure for double states is originally given at a single time t and does not involve the time direction in any intrinsic manner. In fact, one can also associate the same W_C in (25) with the 'dual' process $U(t_i-t_f)\,|\phi\rangle \to U(t_f-t_i)\,|\psi\rangle$ in the same period, since the two states that determine the double states at time t are equivalent in both cases (see Fig. 1). In fact, with $|\tilde\phi\rangle := U(t_i-t_f)\,|\phi\rangle$, $|\tilde\psi\rangle := U(t_f-t_i)\,|\psi\rangle$, one can equally characterize our measure by

$$\begin{aligned} \mu_C(P_{\tilde\phi};t_i) &= 1, \qquad \mu_C(P_{\tilde\phi^\perp};t_i) = 0, \\ \mu_C(P_{\tilde\psi};t_f) &= 1, \qquad \mu_C(P_{\tilde\psi^\perp};t_f) = 0, \end{aligned} \tag{27}$$

instead of (26). This indicates that the proper interpretation of W_C is that it is the measure corresponding to a linear superposition of the two processes, with

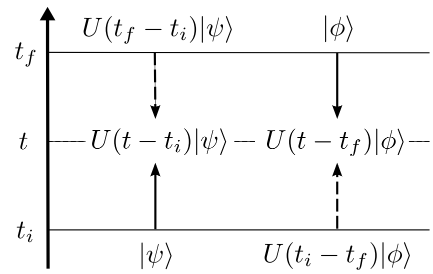

FIG. 1: The double state $W_C(t)$ in (25) can equally be associated with either of the two pure processes, $|\psi\rangle \to |\phi\rangle$ shown by the solid arrows or its dual $U(t_i-t_f)\,|\phi\rangle \to U(t_f-t_i)\,|\psi\rangle$ shown by the dashed arrows, representing the forward and backward time developments.

the parameter of the superposition α. In particular, the choice $\alpha = 1$ yields the pure process $|\psi\rangle \to |\phi\rangle$ whereas the choice $\alpha = 0$ yields another pure process $|\tilde\phi\rangle \to |\tilde\psi\rangle$. Through this identification we can establish a one-to-one correspondence between the measure and the superposed process, and thereby remove the ambiguity in the expectation value $\lambda(A)$. As a result, $\lambda(A)$ agrees precisely with Aharonov's weak value A_w for $W_C(t)$ when the process is $|\psi\rangle \to |\phi\rangle$.

On physical grounds, one may argue that this interpretation is marred by the complex nature of the superposition. This concern can be dismissed by invoking the fact that any operator $W_C \in \mathcal{T}(\mathcal{H})$ admits the expansion,

$$W_C = \sum_{i=1}^{d+1} p_i \frac{|\psi_i\rangle\,\langle\phi_i|}{\langle\phi_i|\psi_i\rangle}, \qquad p_i \in [0,1], \tag{28}$$

in terms of pairs of states $|\psi_i\rangle$, $|\phi_i\rangle \in \mathcal{H}$ for which $\langle\phi_i|\psi_i\rangle \ne 0$ for $i = 1, \dots, d+1$ (see the Appendix). Since $\mathrm{tr}(W_C) = \sum_i p_i = 1$, one can interpret the double states as a superposition of processes $|\psi_i\rangle \to |\phi_i\rangle$ with the real probability p_i (hereafter we omit the time dependence for simplicity, pretending $U = \mathbb{1}$). One therefore sees that the complex superposition of processes can actually be viewed as a classical probability mixture of different processes. Note that, since the expansion (28) is far from unique, so is the set of processes superposed. We may call those double states with $W_C^2 = W_C$ 'pure', and otherwise call them 'mixed', knowing that the former has only one term in the expansion (28) and reads $W_C = |\psi\rangle\,\langle\phi|\,/\,\langle\phi|\psi\rangle$ for some pair of states, $|\psi\rangle$ and $|\phi\rangle$.

The probabilistic interpretation admitted by the expansion (28) explains how the expectation value of the observable A can be obtained. Indeed, if we consider the measurement of A in an ensemble of processes $|\psi_i\rangle \to |\phi_i\rangle$ each of which prepared with probability p_i, the expectation value reads

$$\sum_i p_i \frac{\langle\phi_i|A|\psi_i\rangle}{\langle\phi_i|\psi_i\rangle} = \lambda(A), \tag{29}$$

in agreement with (21).

Our double states W_C in the context of quantum processes are closely related to the 'two-states' proposed in [11] which are defined by $W_T = \sum_{i,j} c_{ij} |e_i\rangle \langle f_j|$ with $c_{ij} \in \mathbb{C}$. The states appearing in the expansion are from the bases $\mathcal{B}_e = \{|e_i\rangle\}$, $\mathcal{B}_f = \{|f_j\rangle\}$ in $\mathcal{H}$ which are mutually non-orthogonal $\langle e_i|f_j\rangle \neq 0$ for any $|e_i\rangle \in \mathcal{B}_e$, $|f_j\rangle \in \mathcal{B}_f$. Due to the non-orthogonality condition, these W_T, if normalized, form a subspace of $\mathcal{T}(\mathcal{H})$ and are argued to represent processes of an open system which is realized as a subsystem of an extended closed system [11]. In contrast, our W_C is given a direct operational interpretation of a probability ensemble of processes with no need of extension, and it represents the most general situations determined by a pair of states in quantum mechanics.

Conclusion and Discussions. In this paper we have considered the complex probability measure for double states as an extension of the real probability measure for single states. Our intention is to provide an appropriate framework for discussing probabilities and physical quantities in a given circumstance specified by two conditions, which are basically determined by two states such as the initial and final states prescribing a quantum process. In order to find out the form of the measure, we have employed an analogous argument used for Gleason's theorem which establishes Born's rule of statistical interpretation for single states, implementing the complex extension and the additional condition required by the double states. The complex measure μ_C turns out to be rather simple (9) and contains a free parameter α. With this measure we have evaluated the expectation value $\lambda(A)$ of an observable A and found that it coincides with Aharonov's weak value at $\alpha = 1$.

In the application of the measure for describing quantum processes, we are alluded to the interpretation that the measure represents a superposition of two processes with the weight specified by α. The case $\alpha = 1$ is then found to be one of the two exceptional situations of realizing a pure process (the other being $\alpha = 0$), in accordance with the original interpretation of the weak value [1]. We also have extended our measure to accommodate the most general case of superpositions, in parallel with the case of single states. Our double states admit a direct operational interpretation in terms of classical probability ensembles of processes, which is important to assure that the complex measure is meaningful in describing actual physical situations.

We mention that the notion of complex probability in quantum mechanics appeared a long time ago in the expansion of density states in terms of two complete basis states [12], where the coefficients are interpreted as complex joint probability (see also [13]). In fact, this may be thought of just as our complex measure with positive W_C in such decomposition. The double states W_C, at $\alpha = 1/2$, also appear in the weak measurement tomography as the transient density matrix [14]. These instances suggest that our complex probability measure based on the double states may have more applications than the one mentioned here.

In our probabilistic treatment, the weak value A_w is obtained as the expectation value $\lambda(A)$. However, it has been argued that A_w should actually be regarded as an intrinsic value possessed by the system in the given process, which can be confirmed by measuring it without (almost) disturbing the system, *i.e.*, by weak measurement. The gap between the two realizations of A_w, as an expectation value and an intrinsic value, seems to indicate that a deeper understanding of the value is still in need. Along this line, the contextual value of the observable A deemed in the process has been studied in [15] requiring a set of conditions similar but stricter than ours and also asymmetric in time (in contrast to ours which are symmetric). Despite this difference, both of them have pointed to the unique status of the weak value A_w as a physical value consistent in the quantum process.

Acknowledgments

We appreciate Professors Y. Aharonov, H. F. Hofmann, A. Hosoya, M. Iinuma, N. Imoto, Y. Shikano and A. Wipf for useful discussions.

Appendix

In this Appendix we show that any double state W_C can be expanded as (28) in terms of states $|\psi_i\rangle, |\phi_i\rangle \in \mathcal{H}$ for which $\langle \phi_i|\psi_i\rangle \neq 0$ under the probabilities $p_i \in [0,1]$ with $\Sigma_{i=1}^{d+1} p_i = 1$. We do this by furnishing the states explicitly as follows.

Let $\{|w_i\rangle ; i = 1, ..., d\}$ be a complete orthogonal basis $\mathcal{B}$ of $\mathcal{H}$. A double state W_C is then expanded as

$$W_C = \sum_{i,j=1}^{d} w_{ij} |w_i\rangle \langle w_j| , \tag{30}$$

with $w_{ij} \in \mathbb{C}$ and $\sum_i w_{ii} = 1$. Consider then the states $|\psi_i\rangle, |\phi_i\rangle$ for $i = 1, \dots, d$ given by

$$\begin{aligned} |\psi_i\rangle &= |w_i\rangle , \\ |\phi_i\rangle &= \sum_{j=1, j\neq i}^{d} \beta_{ij}^* |w_j\rangle + |w_i\rangle , \end{aligned} \tag{31}$$

with $\beta_{ij} \in \mathbb{C}$ and also for $i = d + 1$ by

$$\begin{aligned} |\psi_{d+1}\rangle &= \sum_{i=1}^{d-1} \alpha_i |w_i\rangle + |w_d\rangle , \\ |\phi_{d+1}\rangle &= \sum_{i=1}^{d-1} |w_i\rangle + \Big(1 - \sum_{j=1}^{d-1} \alpha_j^*\Big) |w_d\rangle , \end{aligned} \tag{32}$$

with $\alpha_i \in \mathbb{C}$. Note that these pairs of states fulfill $\langle \phi_i|\psi_i\rangle = 1$ for all $i = 1, \dots, d + 1$. Choose $p_i \in [0,1]$

such that $p_{d+1} \neq 0$, $\sum_{i=1}^{d+1} p_i = 1$, and $p_i = 0$ iff $w_{ik} = 0$ for all k. We then put

$$\alpha_i = \frac{1}{p_{d+1}} \left(w_{ii} - p_i \right), \qquad i = 1, \ldots, d-1, \tag{33}$$

and thereby set

$$\begin{aligned} \beta_{ij} &= \frac{1}{p_i}(w_{ij} - p_{d+1}\alpha_i), \qquad i \neq j, \\ \beta_{id} &= \frac{1}{p_i}\left(w_{id} - p_{d+1}\alpha_i \left(1 - \sum_{j=1}^{d-1} \alpha_i \right) \right), \\ \beta_{dj} &= \frac{1}{p_d}(w_{dj} - p_{d+1}), \end{aligned} \tag{34}$$

for $i, j = 1, \ldots, d-1$. These coefficients allow us to obtain W_C in (30) from (28) as required.

This shows that W_C can generically be decomposed into $\mathrm{rank}(W_C) + 1$ processes, with properly chosen $\{p_i\}$, and as such the processes are highly non-unique, as are the density operators which generically admit non-unique decompositions. Incidentally, we remark that, if we employ the singular value decomposition, we have the unique expansion $W_C = \sum_i^d r_i \left| u_i \right\rangle \left\langle v_i \right|$ with two bases $\mathcal{B}_u = \{\left| u_i \right\rangle\}$ and $\mathcal{B}_v = \{\left| v_i \right\rangle\}$ and $r_i \geq 0$ in terms of d pairs of states at most. However, the two bases $\mathcal{B}_u$ and $\mathcal{B}_v$ may have $\langle u_i | v_i \rangle = 0$ for some i, in which case the two states cannot appear as the initial and final states of a process (*i.e.*, the latter cannot be observed by postselection when the former is preselected), and accordingly they cannot specify a physical process. Such unphysical cases are excluded in our expansion (28).

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