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Equation of State for the One-Dimensional Attractive δ-Potential Bose Gas in the Weak-Coupling Regime

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Approximated formulas for real quasimomentum and the associated energy spectrum are presented for one-dimensional Bose gas with weak attractive contact interactions. On the basis of the energy spectrum, we obtain the equation of state in the high-temperature region, which is found to be the van der Waals equation without volume correction.

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Introduction. One-dimensional quantum gas has been attracting considerable attention since its seminal experimental demonstrations by using cold atoms [1, 2]. To date, one of the bosonic gas models extensively studied is the Lieb-Liniger model [3], which is solvable and characterized by a tunable coupling constant.

In the case of repulsive interaction, much literature is found both for the zero-temperature region [3–6] and the finite-temperature region [7–10]. The Yang-Yang equations [7], whose solutions describe the thermodynamical behavior of the system, have been demonstrated experimentally [10]. Generalization to anyonic systems is also found in [11–13]. In parallel, the case of attractive interaction has been investigated [14–21] in the last decade, which uncovered various exotic features including bound states with complex quasimomentum [3, 14–17, 21]. Phase structure with respect to the coupling strength is found in the paradigm of Gross-Pitaevskii mean-field theory [19, 20].

In this Brief Report, we present two simple but notable results about the Lieb-Liniger model valid for *weak* attractive interactions. One of them is the explicit form of the quasimomentum and energy spectrum, and the other is the equation of state in the high-temperature region which resembles the van der Waals equation of state [22].

Energy Spectrum in the Weak Coupling Limit. The Lieb-Liniger model [3] describes an N-partite bosonic system in one-dimensional space with point interactions. It is governed by the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} - c\sum_{i<j}\delta(x_i - x_j), \tag{1}$$

where m is the mass of the particle, $c \geq 0$ is the coupling constant of the attractive δ interaction, and the variable $x_i \in [0, L]$ represents the coordinate of the ith particle. The eigenfunctions ψ are symmetric $\psi(\cdots, x_i, \cdots, x_j, \cdots) = \psi(\cdots, x_j, \cdots, x_i, \cdots)$ and obey the periodicity conditions $\psi(\cdots, x_i, \cdots) = \psi(\cdots, x_i + L, \cdots)$ for all i. For convenience we hereafter work with the unit $\hbar = m = 1$.

Following the standard treatment of the model, we adopt the Bethe ansatz,

$$\psi(x) = \sum_{\sigma} a_\sigma \exp\left(i\sum_{i=1}^{N} k_{\sigma(i)} x_i\right) \tag{2}$$

for the simplex region $0 < x_1 < x_2 < \cdots < x_N < L$. Here k_i is a quasi-momentum (rapidity) and the summation is over all permutations σ of the particles $i \mapsto \sigma(i)$ on which the coefficients a_σ depend. Plugging (2) into the Schrödinger equation $H\psi = E\psi$, we obtain the Bethe equations [3]

$$Lk_i = (2n_i + N - 1)\pi + 2\sum_{j\neq i}\arctan\delta_{i,j}. \tag{3}$$

Here, n_i is an integer, $\delta_{i,j}$ is the rescaled relative rapidity defined by

$$\delta_{i,j} = (k_i - k_j)/c \tag{4}$$

for $c \neq 0$, and the arctangent takes the principal value, $|\arctan\delta_{i,j}| \leq \pi/2$. The energy E then reads $E = \sum_i k_i^2/2$.

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In what follows, we consider the integer set $\{n_i\}$ so that $k_i \in \mathbb{R}$ for all i for the (dimensionless) weak coupling,

$$\epsilon := cL/\pi \ll 1. \tag{5}$$

We can confine ourselves to the case of the ordering,

$$k_1 < k_2 < \cdots < k_N, \tag{6}$$

by relabeling the indices i appropriately. Note that (6) implies $\delta_{i,j} > 0$ for $i > j$. We also assume that in the weak regime k_i are all regular with respect to c.

Two remarks are in order. First, there actually exist $\{n_i\}$ for few-partite systems on which the condition (5) is satisfied. For example, it is shown in [3] that $k_i \in \mathbb{R}$ for bipartite systems for $\epsilon < 4/\pi$. For tripartite systems, the same holds in the weak-coupling regime if the system has nonzero $\{n_i\}$ at the noninteracting limit $c \to 0$ [15]. Second, under the regularity assumption the ordering (6) implies no level crossing in the weak regime.

We now show the following proposition.

Proposition 1 *For $\epsilon \ll 1$, the rescaled relative rapidity and the coupling have the tradeoff relation,*

$$\epsilon\delta_{i,j} = 2(\bar{n}_i - \bar{n}_j) + \mathcal{O}(\epsilon) \tag{7}$$

with

$$\bar{n}_i := n_i + i - 1. \tag{8}$$

Proof. To show this, we first note that for $c \to 0$,

$$\arctan \delta_{i,j} \to \begin{cases} \pi/2 & \text{for } i > j, \\ -\pi/2 & \text{for } i < j. \end{cases} \tag{9}$$

Substituting this to the Bethe equations (3), we find

$$k_i \to \frac{2\pi}{L}\bar{n}_i, \tag{10}$$

with $\bar{n}_i$ given by (8). This implies that the number $\bar{n}_i$ is nothing but the quantum number of the free case $c = 0$. Since (6) holds for $c = 0$, for the weak coupling regime the regularity condition assures that

$$\bar{n}_1 < \bar{n}_2 < \cdots < \bar{n}_N, \tag{11}$$

and also that

$$k_i = \frac{2\pi}{L}\bar{n}_i + \frac{1}{L}\mathcal{O}(\epsilon), \tag{12}$$

on account of $\mathcal{O}(c) = (1/L)\mathcal{O}(\epsilon)$ from (5). We then obtain

$$\epsilon\delta_{i+1.i} = \frac{L}{\pi}(k_{i+1} - k_i) = 2(\bar{n}_i - \bar{n}_j) + \mathcal{O}(\epsilon), \tag{13}$$

which completes the proof. ■

The tradeoff relation (7) implies $|\delta_{i,j}| = 2(\bar{n}_i - \bar{n}_j)/\epsilon + \mathcal{O}(1) \gg 1$ or $1/|\delta_{i,j}| = \mathcal{O}(\epsilon)$, which is useful to approximate k_i. Indeed, with

$$\arctan x = \begin{cases} -1/x + \pi/2 + \mathcal{O}(1/x^3), & x > 1, \\ -1/x - \pi/2 + \mathcal{O}(1/x^3), & x < -1, \end{cases} \tag{14}$$

substituted for Eq. (3), we find

$$k_i = \frac{2\pi}{L}\bar{n}_i - \frac{2}{L}\sum_{j\neq i}^{N}\frac{1}{\delta_{i,j}} + \frac{1}{L}\mathcal{O}(\epsilon^3). \tag{15}$$

Utilizing (7) again, we can eliminate $\delta_{i,j}$ from (15) to arrive at the following theorem.

Theorem 1 *For $\epsilon \ll 1$, the rapidity is approximated by*

$$k_i = \frac{2\pi}{L}\bar{n}_i - \frac{\epsilon}{L}\sum_{j\neq i}^{N}\frac{1}{\bar{n}_i - \bar{n}_j} + \frac{1}{L}\mathcal{O}(\epsilon^2). \tag{16}$$

This is one of the announced results of this paper.

Let us now derive the approximated energy spectrum E to the order of $\mathcal{O}(\epsilon)$. Summing up all the squared rapidities (16), we obtain

$$\begin{aligned} E_{\bar{n}_1,\bar{n}_2,\ldots,\bar{n}_N} &= \sum_{i=1}^{N}\frac{k_i^2}{2} \\ &\approx \sum_{i=1}^{N}\frac{2\pi^2\bar{n}_i^2}{L^2} - 2\pi\frac{\epsilon}{L^2}\sum_{i=1}^{N}\sum_{j\neq i}\frac{\bar{n}_i}{\bar{n}_i - \bar{n}_j} \\ &= \sum_{i=1}^{N}\frac{2\pi^2\bar{n}_i^2}{L^2} - c\frac{N(N-1)}{L}, \end{aligned} \tag{17}$$

where we have used

$$\begin{aligned} 2\sum_{i=1}^{N}\sum_{j\neq i}\frac{\bar{n}_i}{\bar{n}_i - \bar{n}_j} &= \sum_{i=1}^{N}\sum_{j\neq i}\frac{\bar{n}_i}{\bar{n}_i - \bar{n}_j} + \sum_{j=1}^{N}\sum_{i\neq j}\frac{\bar{n}_j}{\bar{n}_j - \bar{n}_i} \\ &= \sum_{i=1}^{N}\sum_{j\neq i}\frac{\bar{n}_i - \bar{n}_j}{\bar{n}_i - \bar{n}_j} \\ &= N(N-1). \end{aligned} \tag{18}$$

The energy spectrum (17) is additive for the particles, and we may rewrite it as $E_{\bar{n}_1,\bar{n}_2,\cdots\bar{n}_N} \approx \sum_{i=1}^{N} E_{\bar{n}_i}$ with

$$E_{\bar{n}_i} = \frac{2\pi^2}{L^2}\bar{n}_i^2 - c\frac{N-1}{L}. \tag{19}$$

The system thus behaves as an assembly of non-interacting particles of the energy $E_{\bar{n}}$, consisting of the kinetic part $2\pi^2\bar{n}^2/L^2$ and the averaged potential part $-c(N-1)/L$ which is proportional to the number density $(N-1)/L$.

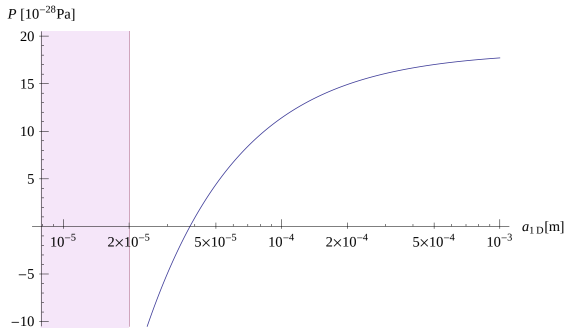

FIG. 1. (Color online) Pressure of one-dimensional Cs gas as a function of the one-dimensional scattering length $a_{1\mathrm{D}}$, evaluated for $N = 25$, $L = 30\,\mu\mathrm{m}$, $T = 0.16\,\mathrm{nK}$. For lower values of $a_{1\mathrm{D}}$, we find that P may become negative, which occurs before reaching the shaded range $a_{1\mathrm{D}} < 2 \times 10\,\mu\mathrm{m}$ where the weak coupling condition (5) is no longer valid.

Equation of State. Let us consider the equation of state of the system for the high-temperature region:

$$\beta := 1/k_{\mathrm{B}}T \ll L^2/2\pi^2, \tag{20}$$

where k_{B} is the Boltzmann constant and T is the temperature. Under the Maxwell-Boltzmann distribution, the partition function Z becomes

$$\begin{aligned} Z &\approx \int_{[-\infty,\infty]^N} \prod_{i=1}^{N} dp_i\, e^{-\beta E_{p_1,p_2,\ldots,p_N}} \\ &\approx L^N \left(2\pi\beta\right)^{-N/2} \exp\frac{\beta c N^2}{L}. \end{aligned} \tag{21}$$

To obtain this, we approximated the sum over $\{\bar{n}_i\}$ by the integral over $p_i := 2\pi\bar{n}_i/L$, ignoring the case of complex rapidities (*i.e.*, $p_i = p_j$ for some i, j) which is measure zero.

Since the volume of the system is given by L in our one dimensional model, the pressure P is evaluated as $P = \frac{1}{\beta}\frac{d}{dL}\ln Z$. We then obtain the following theorem.

Theorem 2 *The equation of state in the high-temperature region reads*

$$\left(P + c\frac{N^2}{L^2}\right) L \approx N k_{\mathrm{B}} T. \tag{22}$$

This is in fact the van der Waals equation without the volume correction. The absence of the volume correction is understood by the fact that the Lieb-Liniger model adopts only point particles.

The one-dimensional system of bosons with attractive interactions of our interest has actually been realized as the super-Tonks-Girardeau (sTG) phase of an ultracold gas of cesium (Cs) atoms using the confinement-induced resonance [23]. In this experiment, one observes a multiple of elongated tubes, each consisting of approximately 25 atoms along the size of $3 \times 10\,\mu\mathrm{m}$ on average. To analyze this in our context, we take these numbers for N and L, respectively, with the mass $m \approx 2.2 \times 10^{-25}\,\mathrm{kg}$. From (20), we then find that our equation of state (22) is valid for $T \gg 2\pi^2\hbar^2/mL^2k_{\mathrm{B}} \approx 8 \times 10\,\mathrm{pK}$. On the other hand, according to [24] our coupling c is related to the tunable one-dimensional scattering length $a_{1\mathrm{D}} > 0$ by $c = 2\hbar^2/ma_{1\mathrm{D}}$. Thus our weak coupling condition (5) implies $a_{1\mathrm{D}} \gg 2L/\pi \approx 2 \times 10\,\mu\mathrm{m}$. In contrast, we find $a_{1\mathrm{D}} \approx 3 \times 10^2\,\mathrm{nm}$ in the experiment [23], which is way out of the valid range.

If we are allowed to extend the range by tuning the parameters of experiment such as the strength of applied magnetic field appropriately without spoiling the sTG phase up to the valid range, then our formula suggests an intriguing possibility. This is seen in the behavior of pressure P (see Fig. 1) which becomes negative for lower values of $a_{1\mathrm{D}}$. This takes place when the pressure correction term surmounts the ideal gas term in (22), suggesting a phase transition there. This may also be realized, *e.g.*, by increasing the number density N/L even if $a_{1\mathrm{D}}$ is fixed.

Conclusion and Discussions. In this paper we obtained the approximated energy spectrum of the one-dimensional attractive Bose gas where particles interact weakly with each other via the contact δ potential. The coupling ϵ and the rescaled relative rapidity $\delta_{i,j}$ fulfill the tradeoff relation (Proposition 1). As a result, we obtained the closed expression of the approximated rapidity (Theorem 1). Based on this, we found that the equation of state is the van der Waals equation without the volume correction (Theorem 2).

The induced correction term is proportional to the attractive coupling c with the squared number density, which is precisely the same as the standard correction. This correction is conventionally ascribed to the average interparticle potentials in the classical thermodynamics derivation, whereas in our quantum-mechanical model the effect is taken care of by the nontrivial boundary conditions of the δ potential imposed at the positions of the particles. The valid range of our analysis does not seem to overlap with the parameter range of the present experiment with ultracold Cs gas, but if it can be extended we may observe a novel phase transition as a consequence of the quantum boundary effect.

In closing, we mention that the δ potential is not the only possible point interactions admitted in quantum mechanics. Indeed, it has been known that in one dimension we have a U(2) family of point interactions each characterized by distinct boundary conditions [25] including the δ potential as a special case. Despite that all of them are equally zero-range, these potentials can in principle give rise to different spectra through the non-trivial boundary conditions, bearing a variety of palpable physical effects as pointed out in [26–29]. We may therefore expect similar exotic outcomes to appear, for instance, in the equation of state when such potentials other than the δ are considered.

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