Exchange Symmetry and Multipartite Entanglement

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Entanglement of multipartite systems is studied based on exchange symmetry under the permutation group S_N. With the observation that symmetric property under the exchange of two constituent states and their separability are intimately linked, we show that anti-symmetric (fermionic) states are necessarily globally entangled, while symmetric (bosonic) states are either globally entangled or fully separable and possess essentially identical states in all the constituent systems. It is also shown that there cannot exist a fully separable state which is orthogonal to all symmetric states, and that full separability of states does not survive under total symmetrization unless the states are originally symmetric. Besides, anyonic states permitted under the braid group B_N must also be globally entangled. Our results indicate that exchange symmetry can be powerful in determining the entanglement of multipartite systems.

I. INTRODUCTION

One of the prominent properties of quantum states is entanglement [1], which is an essential element underlying the impossibility of emulating genuinely quantum phenomena based on classical local realism [2]. This impossibility has been experimentally confirmed in various physical systems [3, 4, 5, 6] over the last few decades, and now we are taking up the issue of non-local realism [7, 8]. Besides its significance in the foundation of quantum mechanics, entanglement provides an indispensable resource in many branches of quantum information science [9, 10]. In spite of the vital importance in these areas, entanglement defies our understanding both conceptually and theoretically, especially in multipartite systems. This is due to the inexplicably rich and intricate structure of multipartite quantum states, which results in the exponential increase in the number of ingredients of entanglement when the number N of the constituent systems becomes large. For example, it is known that for $N = 2$, qubit (level $n = 2$) systems have only a single class of entanglement ordered by Schmidt coefficients [9, 10], while for $N = 3$ the systems can have many classes of entanglement [11]. The obstacle for grasping entanglement is also seen in the construction of entanglement measures [12] for multipartite systems, where, for instance, the scalable measure proposed by Meyer and Wallach [13] (see also [14]) has been shown, despite its feasibility, to be flawed for quantifying the entanglement globally [15].

In physics, symmetry is a fundamental tool to analyze systems as well as to construct models without dealing with the insignificant detail of the structure [16]. The practical use of symmetry has been seen, *e.g.*, in obtaining the spectrum of the hydrogen atom by the hidden $O(4)$ symmetry [17, 18], or in classifying relativistic particle states based on the Lorentz group. One may then expect that a similar utility of symmetry will be found in the analysis of multipartite entangled systems as well. Among the various symmetries available to multipartite systems, the most fundamental is perhaps exchange symmetry, which in particle systems characterizes states of bosons, fermions, parafermions [19, 20, 21] and anyons [22, 23, 24]. In fact, exchange symmetry has been shown to be a key for realizing independence (or more precisely the i.i.d. property) of subsystems in a large system [25, 26], which is important for the validity of local experiments to infer the global property of the system.

The aim of the present paper is to demonstrate that exchange symmetry is indeed quite useful to pin down the entanglement property of multipartite states. After observing an intimate link between the symmetric property under the exchange of two constituent states and their separability, we shall see that fermionic (anti-symmetric) states are always globally entangled for general N and n, while bosonic (symmetric) states are either globally entangled or fully separable (Theorem 1). Anyonic states share the same entanglement property with the fermionic states. We also show that symmetric states are severely restricted with respect to the full separability of the subsystems. Some of the previous studies relevant to the subject, though not exploiting exchange symmetry as we do here, may be found in [27, 28, 29, 30, 31, 32, 33, 34].

This paper is organized as follows. In Sec. II, we provide necessary settings for our arguments of multipartite entanglement, and thereby present our first lemma on state decompositions. Permutation is introduced in Sec. III to obtain the second lemma, which gives a certain restriction to states obeying permutation symmetry. Combining the two lemmas, in Sec. IV we present Theorem 1. The symmetric states are further analyzed with regard to the full separability in Sec. V, where our results are given by two propositions. Sec. VI is dedicated to our conclusion and discussions.

II. DECOMPOSITION OF MULTIPARTITE STATES

Consider the Hilbert space $\mathscr{H}$ given by the tensor product of the constituent Hilbert spaces $\mathscr{H}_i = \mathbb{C}^n$, $i = 1, \ldots, N$, for some positive integer n. Each space $\mathscr{H}_i$ is equipped with a set of orthonormal basis states,

$$|0\rangle_i, \ |1\rangle_i, \ \ldots, \ |n-1\rangle_i, \qquad i = 1, 2, \ldots, N, \tag{2.1}$$

with which a state $|\psi\rangle_i \in \mathscr{H}_i$ can be expanded as

$$|\psi\rangle_i = \sum_{a=0}^{n-1} c_a |a\rangle_i, \qquad c_a \in \mathbb{C}. \tag{2.2}$$

The total Hilbert space $\mathscr{H}$ is spanned by the set of N-fold direct product states,

$$\bigotimes_{i=1}^{N} |a_i\rangle_i, \qquad a_i \in \{0, 1, \ldots, n-1\}. \tag{2.3}$$

To discuss subsystems of the Hilbert space $\mathscr{H}$, we consider the set of labels of the constituents,

$$\mathcal{N} := \{1, 2, \cdots, N\}, \tag{2.4}$$

and introduce a partition $\mathscr{A}$ consisting of exclusive subsets $\mathscr{A} = \{\mathscr{A}_k\}_{k=1}^{s}$ covering $\mathcal{N}$, i.e.,

$$\bigcup_{k=1}^{s} \mathscr{A}_k = \mathcal{N} \quad \text{and} \quad \mathscr{A}_j \cap \mathscr{A}_k = \emptyset \quad \text{for} \quad j \neq k. \tag{2.5}$$

Each subset $\mathscr{A}_k$ determines the tensor product Hilbert space $\mathscr{H}(\mathscr{A}_k) := \bigotimes_{i \in \mathscr{A}_k} \mathscr{H}_i$ as a subsystem of $\mathscr{H}$. In order to treat the labels of the basis states in $\mathscr{H}(\mathscr{A}_k)$ collectively, we also introduce

$$\alpha_k := \{a_i \,|\, i \in \mathscr{A}_k\}, \tag{2.6}$$

and thereby denote the basis states in $\mathscr{H}(\mathscr{A}_k)$ as

$$|\alpha_k\rangle := \bigotimes_{i \in \mathscr{A}_k} |a_i\rangle_i. \tag{2.7}$$

A state $|A_k\rangle \in \mathscr{H}(\mathscr{A}_k)$ is then written as

$$|A_k\rangle = \sum_{\alpha_k} A_{\alpha_k} |\alpha_k\rangle, \qquad A_{\alpha_k} \in \mathbb{C}, \tag{2.8}$$

where the summation of α_k is over all $a_i = 0, 1, \cdots, n-1$ for $i \in \mathscr{A}_k$ appearing in α_k. For convenience, we assume that in each of the subsystems the state is normalized:

$$\langle A_k | A_k \rangle = 1. \tag{2.9}$$

By choosing a partition $\mathscr{A} = \{\mathscr{A}_k\}_{k=1}^{s}$ properly, and taking the direct product of the state $|A_k\rangle$ in (2.8) for all k, we can express any state $|\psi\rangle \in \mathscr{H}$ in the product form of subsystems,

$$|\psi\rangle = \bigotimes_{k=1}^{s} |A_k\rangle. \tag{2.10}$$

Note that the expression of the product (2.10) is not unique for a given state $|\psi\rangle \in \mathscr{H}$, since it depends on the choice of the bases in the subsystems used in the expansion. Moreover, even under the same choice, the state may still admit different product expressions having distinct partition number s due to the ambiguity in assigning the partition for the state. Among all possible s in those admitted partitions for a given $|\psi\rangle$, there exists the maximal partition number which gives the number of subsystems into which the state $|\psi\rangle$ can be decomposed at most. Adopting the notation $M(\psi)$ for the maximal partition number of the state $|\psi\rangle$, we say that the state with $M(\psi) = d$ is a *d-fold direct product state*. If $M(\psi) = N$, the state $|\psi\rangle$ is called *fully separable*, and if $M(\psi) = 1$, the state is *globally entangled*.

Otherwise, the state is *partially entangled*. To the former two extreme cases, we introduce the space $\mathscr{T}$ of all fully separable states and the space $\mathscr{G}$ of all globally entangled states, *i.e.*,

$$\mathscr{T} := \big\{ |\psi\rangle \,\big|\, M(\psi) = N \big\}, \qquad \mathscr{G} := \big\{ |\psi\rangle \,\big|\, M(\psi) = 1 \big\}. \tag{2.11}$$

Given two partitions, $\mathscr{A} = \{\mathscr{A}_k\}_{k=1}^s$ and $\mathscr{B} = \{\mathscr{B}_l\}_{l=1}^t$, we can construct another partition $\mathscr{C} = \{\mathscr{C}_m\}_{m=1}^u$ by collecting non-empty intersections of $\mathscr{A}_k$ and $\mathscr{B}_l$ for all k and l, *i.e.*, if $m = m(k,l) \in \{1, 2, \ldots, u\}$ labels such nonempty subsets formed by $\mathscr{A}_k \cap \mathscr{B}_l$, we have

$$\mathscr{C} = \{\mathscr{C}_m\}_{m=1}^u, \qquad \mathscr{C}_m = \mathscr{A}_k \cap \mathscr{B}_l \neq \emptyset, \tag{2.12}$$

where obviously $1 \le u \le st$. We denote the partition defined this way symbolically by $\mathscr{C} = \mathscr{A} \cap \mathscr{B}$.

We can also define the complement $\bar{\mathscr{A}}_k$ of $\mathscr{A}_k$ as a set satisfying

$$\mathscr{A}_k \cup \bar{\mathscr{A}}_k = \mathcal{N}, \qquad \mathscr{A}_k \cap \bar{\mathscr{A}}_k = \emptyset. \tag{2.13}$$

Clearly, $\{\mathscr{A}_k, \bar{\mathscr{A}}_k\}$ qualifies as a partition consisting of the two subsets, and accordingly we may also employ

$$\bar{\alpha}_k := \{a_i \,|\, i \in \bar{\mathscr{A}}_k\}, \tag{2.14}$$

for the basis states $|\bar{\alpha}_k\rangle$ in the subsystem $\mathscr{H}(\bar{\mathscr{A}}_k)$,

$$|\bar{\alpha}_k\rangle := \bigotimes_{i \in \bar{\mathscr{A}}_k} |a_i\rangle_i. \tag{2.15}$$

The partition just introduced is useful to express the density matrix associated with the subsystem $\mathscr{H}(\mathscr{A}_k)$. Indeed, from the density matrix $\rho = |\psi\rangle\langle\psi|$ of the state $|\psi\rangle \in \mathscr{H}$, we obtain the density matrix ρ_{α_k} for the subsystem $\mathscr{H}(\mathscr{A}_k)$ by the partial trace over the complemental subsystem $\mathscr{H}(\bar{\mathscr{A}}_k)$,

$$\rho_{\alpha_k} = \mathrm{Tr}_{\bar{\alpha}_k} \rho := \sum_{\bar{\alpha}_k} \langle \bar{\alpha}_k | \rho | \bar{\alpha}_k \rangle. \tag{2.16}$$

In particular, for the state $|\psi\rangle$ in the decomposed form (2.10), we have

$$\rho = \bigotimes_{k=1}^s \rho_{\alpha_k}, \qquad \rho_{\alpha_k} = |A_k\rangle\langle A_k|. \tag{2.17}$$

Now we consider the density matrix $\rho_{\beta_l} = \mathrm{Tr}_{\bar{\beta}_l} \rho$ of the subsystem $\mathscr{H}(\mathscr{B}_l)$ defined by another partition $\mathscr{B} = \{\mathscr{B}_l\}_{l=1}^t$. Analogous to the subsystems $\mathscr{H}(\mathscr{A}_k)$ where we use α_k to denote collectively the basis states, we shall use β_l to denote the basis states in the subsystems $\mathscr{H}(\mathscr{B}_l)$. Since $\mathscr{A} = \{\mathscr{A}_k\}_{k=1}^s$ is a partition we have $\bar{\mathscr{B}}_l = \cup_k(\mathscr{A}_k \cap \bar{\mathscr{B}}_l)$, and hence the trace of the density matrix ρ over $\mathscr{H}(\bar{\mathscr{B}}_l)$ may also be realized by successive traces over the subsystems $\mathscr{H}(\mathscr{A}_k \cap \bar{\mathscr{B}}_l)$ for all k as

$$\mathrm{Tr}_{\bar{\beta}_l} \rho = \mathrm{Tr}_{\alpha_1 \cap \bar{\beta}_l} \mathrm{Tr}_{\alpha_2 \cap \bar{\beta}_l} \cdots \mathrm{Tr}_{\alpha_s \cap \bar{\beta}_l} \rho. \tag{2.18}$$

Combining with (2.17), we find for $|\psi\rangle$ in (2.10) the identity,

$$\rho_{\beta_l} = \mathrm{Tr}_{\bar{\beta}_l} \rho = \mathrm{Tr}_{\alpha_1 \cap \bar{\beta}_l} \mathrm{Tr}_{\alpha_2 \cap \bar{\beta}_l} \cdots \mathrm{Tr}_{\alpha_s \cap \bar{\beta}_l} \bigotimes_{k=1}^s \rho_{\alpha_k} = \bigotimes_{k=1}^s \mathrm{Tr}_{\alpha_k \cap \bar{\beta}_l} \rho_{\alpha_k}. \tag{2.19}$$

With this preparation, we obtain

Lemma 1 *If a pure state $|\psi\rangle \in \mathscr{H}$ can be decomposed in two ways by partitions $\mathscr{A} = \{\mathscr{A}_k\}_{k=1}^s$ and $\mathscr{B} = \{\mathscr{B}_l\}_{l=1}^t$,*

$$|\psi\rangle = \bigotimes_{k=1}^s |A_k\rangle = \bigotimes_{l=1}^t |B_l\rangle, \tag{2.20}$$

then, the state can be further decomposed into subsystems defined by the partition $\mathscr{C} = \mathscr{A} \cap \mathscr{B}$, that is, it can be written as

$$|\psi\rangle = \bigotimes_{m=1}^u |C_m\rangle, \qquad |C_m\rangle \in \mathscr{H}(\mathscr{C}_m), \tag{2.21}$$

where $\mathscr{H}(\mathscr{C}_m)$ are the subsystems associated with the partition $\mathscr{C} = \{\mathscr{C}_m\}_{m=1}^u$ defined in (2.12).

Proof. Consider the density matrix $\rho = |\psi\rangle\langle\psi|$ and the reduced density matrix ρ_{β_l} defined in (2.16). Since the state admits the decomposition under the partition $\mathscr{A}$, we have (2.19). But since $\rho_{\beta_l} = |B_l\rangle\langle B_l|$ is a density matrix for a pure state, we have $\rho_{\beta_l}^2 = \rho_{\beta_l}$ and, hence, from (2.19) and (2.18) used for β_l instead of $\bar{\beta}_l$, we find

$$1 = \mathrm{Tr}_{\beta_l}\rho_{\beta_l} = \mathrm{Tr}_{\beta_l}\rho_{\beta_l}^2 = \prod_{k=1}^{s} \mathrm{Tr}_{\alpha_k \cap \beta_l} \left(\mathrm{Tr}_{\alpha_k \cap \bar{\beta}_l}\rho_{\alpha_k}\right)^2. \tag{2.22}$$

This shows that $\mathrm{Tr}_{\alpha_k \cap \beta_l} \left(\mathrm{Tr}_{\alpha_k \cap \bar{\beta}_l}\rho_{\alpha_k}\right)^2 = 1$ for all k and l, because the trace of a squared density matrix has an upper bound equal to unity, and this must be attained in (2.22). Since the upper bound is attained if and only if the state represented by the density matrix is pure, we learn that $\mathrm{Tr}_{\alpha_k \cap \bar{\beta}_l}\rho_{\alpha_k}$ corresponds to a pure state. It follows that the state has to be a direct product of the traced part and the rest if none of the subsets, $\mathscr{A}_k \cap \mathscr{B}_l$ and $\mathscr{A}_k \cap \bar{\mathscr{B}}_l$, is empty. Consequently, the state $|\psi\rangle \in \mathscr{H}$ can be decomposed into three subsystems as

$$|\psi\rangle = |C\rangle \otimes |C'\rangle \otimes |\bar{A}\rangle, \qquad |C\rangle \in \mathscr{H}(\mathscr{A}_k \cap \mathscr{B}_l), \quad |C'\rangle \in \mathscr{H}(\mathscr{A}_k \cap \bar{\mathscr{B}}_l), \quad |\bar{A}\rangle \in \mathscr{H}(\bar{\mathscr{A}}_k). \tag{2.23}$$

Since this is true for all k, l which generate non-empty intersections, and since the corresponding subsets $\mathscr{A}_k \cap \mathscr{B}_l$ are exclusive from each other and comprise the partition $\mathscr{C} = \mathscr{A} \cap \mathscr{B}$ in (2.12), we arrive at (2.21). ∎

III. DECOMPOSED STATES WITH PERMUTATION SYMMETRY

Next we discuss a consequence of permutation symmetry on the specification of states admitting a certain type of decompositions. To this end, given an elemant σ of the symmetric group S_N consisting of all permutations $i \to \sigma(i)$ of the elements of $\mathcal{N}$, we first define the corresponding linear operator π_σ by the action of σ on the label of the basis states,

$$\pi_\sigma \bigotimes_{i=1}^{N} |a_i\rangle_i = \bigotimes_{i=1}^{N} |a_{\sigma(i)}\rangle_i. \tag{3.1}$$

Let $\mathscr{I}_\sigma$ be a set of labels of the constituents which are invariant under the action of the given $\sigma \in S_N$,

$$\mathscr{I}_\sigma := \{i \,|\, \sigma(i) = i\}. \tag{3.2}$$

The complement of $\mathscr{I}_\sigma$ in $\mathcal{N}$ is

$$\bar{\mathscr{I}}_\sigma = \{i \,|\, \sigma(i) \neq i\}. \tag{3.3}$$

Observe that both $\mathscr{I}_\sigma$ and $\bar{\mathscr{I}}_\sigma$ are invariant under the multiple actions of σ,

$$\sigma^p \mathscr{I}_\sigma = \mathscr{I}_\sigma, \qquad \sigma^p \bar{\mathscr{I}}_\sigma = \bar{\mathscr{I}}_\sigma, \qquad \forall\, p \in \mathbb{Z}. \tag{3.4}$$

With respect to the actions under σ^p, one can introduce an equivalence relation between $i, j \in \bar{\mathscr{I}}_\sigma$ by

$$i \sim j \quad \Leftrightarrow \quad i = \sigma^p(j), \quad \exists\, p \in \mathbb{Z}, \tag{3.5}$$

which provides the equivalent classes $\bar{\mathscr{I}}_\sigma / \sim$. We note that $\{\mathscr{I}_\sigma, \bar{\mathscr{I}}_\sigma\}$ also give a partition with the corresponding subsystems $\mathscr{H}(\mathscr{I}_\sigma)$ and $\mathscr{H}(\bar{\mathscr{I}}_\sigma)$, respectively.

Given two states in different constituent spaces, $|\psi\rangle_i \in \mathscr{H}_i$ and $|\varphi\rangle_j \in \mathscr{H}_j$, we say that these states are *equipollent* and write $|\psi\rangle_i \simeq |\varphi\rangle_j$ if all coefficients of $|\psi\rangle_i$ and $|\varphi\rangle_j$ coincide up to a phase in the expansion under some fixed orthonormal bases (2.2) in correspondence. Namely, if we expand the states as $|\psi\rangle_i = \sum_{k=0}^{n-1} c_k |k\rangle_i$ and $|\varphi\rangle_j = \sum_{k=0}^{n-1} d_k |k\rangle_j$ with some bases $\{|k\rangle_i\}$ and $\{|k\rangle_j\}$ in the two constituent spaces which are regarded to be correspondent to each other (*i.e.*, $|k\rangle_i \leftrightarrow |k\rangle_j$ for all k), then

$$|\psi\rangle_i \simeq |\varphi\rangle_j \quad \Leftrightarrow \quad c_k = \gamma\, d_k, \quad \forall\, k, \tag{3.6}$$

where γ is a phase factor $|\gamma| = 1$ common for all k. In particular, if (3.6) holds with $\gamma = 1$, we say that the two states are *strictly equipollent*. We now show

Lemma 2 *Suppose that a state $|\psi\rangle \in \mathscr{H}$ satisfies*
(i) $|\psi\rangle$ is an eigenstate for a given π_σ with eigenvalue λ,

$$\pi_\sigma |\psi\rangle = \lambda |\psi\rangle, \tag{3.7}$$

(ii) $|\psi\rangle$ is decomposed into states in $\mathscr{H}(\mathscr{I}_\sigma)$ and $\mathscr{H}(\bar{\mathscr{I}}_\sigma)$,

$$|\psi\rangle = |I\rangle \otimes |\bar{I}\rangle, \qquad |I\rangle \in \mathscr{H}(\mathscr{I}_\sigma), \qquad |\bar{I}\rangle \in \mathscr{H}(\bar{\mathscr{I}}_\sigma), \tag{3.8}$$

such that $|\bar{I}\rangle$ is a direct product of states $|\psi_i\rangle_i \in \mathscr{H}_i$ in the constituent spaces,

$$|\bar{I}\rangle = \bigotimes_{i \in \bar{\mathscr{I}}_\sigma} |\psi_i\rangle_i. \tag{3.9}$$

Then the eigenvalue is unity $\lambda = 1$, and the states of the constituents whose labels belong to the same equivalent class $\bar{\mathscr{I}}_\sigma/\sim$ are all equipollent,

$$|\psi_i\rangle_i \simeq |\psi_j\rangle_j \quad \text{for} \quad i \sim j, \quad i, j \in \bar{\mathscr{I}}_\sigma, \tag{3.10}$$

in the strict sense.

Proof. We first expand $|\psi_i\rangle_i$ for all $i \in \bar{\mathscr{I}}_\sigma$ as

$$|\psi_i\rangle_i = \sum_{a_i=0}^{n-1} c_{a_i}^i |a_i\rangle_i, \qquad c_{a_i}^i \in \mathbb{C}, \tag{3.11}$$

and observe that the left hand side of (3.7) becomes

$$\pi_\sigma |\psi\rangle = |I\rangle \otimes \bigotimes_{i \in \bar{\mathscr{I}}_\sigma} \left(\sum_{a_i=0}^{n-1} c_{a_i}^i |a_{\sigma(i)}\rangle_i \right) = |I\rangle \otimes \bigotimes_{i \in \bar{\mathscr{I}}_\sigma} \left(\sum_{a_{\sigma(i)}=0}^{n-1} c_{a_{\sigma(i)}}^{\sigma(i)} |a_{\sigma(i)}\rangle_i \right), \tag{3.12}$$

because under the direct product over all elements in $\bar{\mathscr{I}}_\sigma$ we can exchange the coefficients within $\bar{\mathscr{I}}_\sigma$ freely. Changing the labels $a_{\sigma(i)} \to a_i$ which are dummy indices of the summations, we find that (3.7) reads

$$\pi_\sigma |\psi\rangle = |I\rangle \otimes \bigotimes_{i \in \bar{\mathscr{I}}_\sigma} \left(\sum_{a_i=0}^{n-1} c_{a_i}^{\sigma(i)} |a_i\rangle_i \right) = \lambda |\psi\rangle. \tag{3.13}$$

Taking the inner products with $\bigotimes_{i \in \bar{\mathscr{I}}_\sigma} {}_i\langle a|$ for $a \in \{0, 1, \ldots, n-1\}$ in (3.13), we obtain

$$\prod_{i \in \bar{\mathscr{I}}_\sigma} c_a^{\sigma(i)} = \lambda \prod_{i \in \bar{\mathscr{I}}_\sigma} c_a^i. \tag{3.14}$$

Since the product in (3.14) is over all elements of $\bar{\mathscr{I}}_\sigma$, we have the identity,

$$\prod_{i \in \bar{\mathscr{I}}_\sigma} c_a^{\sigma(i)} = \prod_{\sigma(i) \in \bar{\mathscr{I}}_\sigma} c_a^{\sigma(i)} = \prod_{i \in \bar{\mathscr{I}}_\sigma} c_a^i. \tag{3.15}$$

Combining (3.15) with (3.14), we find $\lambda = 1$.

Now choose an element $i \in \bar{\mathscr{I}}_\sigma$ and consider the state $|\tilde{\psi}\rangle = |a\rangle_i {}_i\langle a|\psi\rangle$ which is equivalent to $|\psi\rangle$ except that its constituent state $|\psi_i\rangle_i$ is replaced by $|a\rangle_i$. The inner product of $|\tilde{\psi}\rangle$ with $|\psi\rangle$ in (3.13) then yields

$$c_a^{\sigma(i)} = c_a^i. \tag{3.16}$$

Since this is true for all $i \in \bar{\mathscr{I}}_\sigma$ and $a \in \{0, 1, \ldots, n-1\}$, we find (3.10). ∎

IV. GLOBAL ENTANGLEMENT AND SYMMETRY OF STATES

In this section, we focus on multipartite states possessing particular symmetric properties under transpositions. We first introduce the set of all transpositions $\mathcal{T}_N \subset S_N$ by

$$\mathcal{T}_N := \{\sigma \,|\, \sigma = (i\,j),\, \forall\, i, j \in \mathcal{N}\}. \tag{4.1}$$

Combining the Lemmas in the previous sections and applying them to transpositions, we obtain

Proposition 1 *Suppose that there exists a state $|\psi\rangle$ decomposed as (2.10) under a partition $\mathscr{A}$ consisting of two subsets, $\mathscr{A} = \{\mathscr{A}_1, \mathscr{A}_2\}$. Suppose further that $|\psi\rangle$ is an eigenstate of π_σ,*

$$\pi_\sigma|\psi\rangle = \lambda|\psi\rangle, \tag{4.2}$$

for some transposition $\sigma = (i\,j) \in \mathcal{T}_N$ such that

$$i \in \mathscr{A}_1 \quad\text{and}\quad j \in \mathscr{A}_2. \tag{4.3}$$

Then the state $|\psi\rangle$ can be decomposed further by the partition $\mathscr{C} = \mathscr{A} \cap \mathscr{B}$, where $\mathscr{B} = \{\mathscr{B}_1, \mathscr{B}_2\}$ is the partition defined from $\mathscr{A}$ with the elements i and j interchanged. Moreover, the eigenvalue is unity $\lambda = 1$ and the states $|\psi_i\rangle_i$ and $|\psi_j\rangle_j$ of $|\psi\rangle$ in the constituent spaces $\mathscr{H}_i$ and $\mathscr{H}_j$ are equipollent $|\psi_i\rangle_i \simeq |\psi_j\rangle_j$ in the strict sense.

Proof. With no loss of generality, we may consider the partition

$$\mathscr{A}_1 = \{1, \cdots, i, \cdots, M\} \quad\text{and}\quad \mathscr{A}_2 = \{M+1, \cdots, j, \cdots, N\}. \tag{4.4}$$

Observe that since $\pi_\sigma^2 = I$ (identity operator), we have $\lambda = \pm 1$. Then (4.2) implies $|\psi\rangle = \lambda^{-1}\pi_\sigma|\psi\rangle$, that is, the state $|\psi\rangle$ admits the decomposition (2.10) under another partition $\mathscr{B} = \{\mathscr{B}_1, \mathscr{B}_2\}$, where

$$\mathscr{B}_1 = \{1, \cdots, j, \cdots, M\} \quad\text{and}\quad \mathscr{B}_2 = \{M+1, \cdots, i, \cdots, N\}. \tag{4.5}$$

Lemma 1 then assures that the state $|\psi\rangle$ can be decomposed further by the partition $\mathscr{C} = \mathscr{A} \cap \mathscr{B}$. Noting that $\mathscr{A}_1 \cap \mathscr{B}_2 = \{i\}$ and $\mathscr{A}_2 \cap \mathscr{B}_1 = \{j\}$, we find that $|\psi\rangle$ has the constituent states $|\psi_i\rangle_i \in \mathscr{H}_i$ and $|\psi_j\rangle_j \in \mathscr{H}_j$ in the decomposition. Since these constituent states belong to the subsystem $\mathscr{H}(\mathscr{I}_\sigma)$, the rest of the statements of Proposition 1 follows from Lemma 2. ∎

This proposition implies

Corollary 1 *If a state $|\psi\rangle$ satisfies $\pi_\sigma|\psi\rangle = -|\psi\rangle$ under a transposition $\sigma = (i\,j)$, then $|\psi\rangle$ cannot be separable with respect to the constituent subspaces $\mathscr{H}_i$ and $\mathscr{H}_j$. On the other hand, if $|\psi\rangle$ satisfies $\pi_\sigma|\psi\rangle = |\psi\rangle$ under $\sigma = (i\,j)$, then two possibilities arise: either $|\psi\rangle$ is not separable with respect to $\mathscr{H}_i$ and $\mathscr{H}_j$, or otherwise it is separable with respect to $\mathscr{H}_i$ and $\mathscr{H}_j$ where the constituent states are strictly equipollent to each other.*

To proceed, we introduce the totally symmetrized subspace,

$$\mathscr{S} := \big\{\, |\psi\rangle \,\big|\, \pi_\sigma|\psi\rangle = |\psi\rangle,\, \forall\, \sigma \in S_N \big\}, \tag{4.6}$$

and the subset,

$$\mathscr{D} = \mathscr{T} \cap \mathscr{S} \tag{4.7}$$

which consists of N-fold direct product states belonging to $\mathscr{S}$. Consider now a state $|\psi\rangle \in \mathscr{H}$ fulfilling

$$\pi_\sigma|\psi\rangle = \lambda_\sigma|\psi\rangle, \quad \lambda_\sigma = \pm 1, \tag{4.8}$$

for all $\sigma \in \mathcal{T}_N$. Since $|\psi\rangle$ is a simultaneous eigenstate of the linear operators of the transpositions, multiple operations of the different transpositions on the state is naturally induced. It is well known that a non-trivial $|\psi\rangle$ with the condition (4.8) must be in one dimensional representations of S_N, and from this we have

$$\lambda_{\sigma\tau} = \lambda_\sigma \lambda_\tau, \qquad \forall\, \sigma, \tau \in S_N. \tag{4.9}$$

There are two different one dimensional representations for S_N: one is $\lambda_\sigma = -1$ for all $\sigma \in \mathcal{T}_N$, the other is $\lambda_\sigma = 1$ for all $\sigma \in \mathcal{T}_N$. A state realizing the former is called *anti-symmetric* whereas a state realizing the latter is called *symmetric* [16], and in particle systems these correspond to states of fermions and bosons, respectively. We then present

Theorem 1 *The following two statements hold:*
(i) Any anti-symmetric state $|\psi\rangle$ is globally entangled $|\psi\rangle \in \mathscr{G}$.
(ii) Any symmetric state $|\psi\rangle$ is either globally entangled $|\psi\rangle \in \mathscr{G}$, or otherwise it is fully separable $|\psi\rangle \in \mathscr{T}$. In the latter case, all the constituent states of $|\psi\rangle$ are strictly equipollent.

Proof. Both of the statements follow immediately from Corollary 1 by applying it to all transpositions. ∎

We note that, in terms of spaces, statement (ii) implies

$$\mathscr{S} = \mathscr{S} \cap (\mathscr{T} \cup \mathscr{G}) = \mathscr{D} \cup (\mathscr{G} \cap \mathscr{S}). \tag{4.10}$$

We also mention that statement (ii) allows us to characterize concisely the space $\mathscr{D}$ of fully separable symmetric states by means of unitary operators. To see this, we first consider unitary operators $U(\theta^i)$ implementing $SU(n)$ transformations on the basis in $\mathscr{H}_i$, where $\theta^i = (\theta^i_1, \theta^i_2, \ldots, \theta^i_m) \in \Theta$ ($m = \dim \mathfrak{su}(n) = n^2 - 1$) are the set of parameters specifying the transformations with Θ being the parameter space covering the entire $SU(n)$. The unitary operator implementing the $SU(n)$ transformations for all the bases in the constituent Hilbert spaces $\mathscr{H}_i$ can then be given by the product $\otimes_{i=1}^{N} U(\theta^i)$. Since statement (ii) assures that all constituent states in $\mathscr{D}$ are strictly equipollent, we can use the same unitary operator for all constituent spaces $\mathscr{H}_i$ with common θ to generate the constituent states $|\psi_i\rangle_i$ from, say, the state $|0\rangle_i$ as $|\psi_i\rangle_i = U(\theta)|0\rangle_i$. If this is done, then we have

Corollary 2 *The space $\mathscr{D}$ of fully separable symmetric states is characterized by*

$$\mathscr{D} = \left\{ |\psi\rangle \,\middle|\, |\psi\rangle = \bigotimes_{i=1}^{N} U(\theta)|0\rangle_i,\, \theta \in \Theta \right\}. \tag{4.11}$$

Note that the foregoing argument for $\mathscr{D}$ is meaningful only if we employ a fixed set of bases in all constituent spaces $\mathscr{H}_i$, since otherwise we can always perform unitary transformations in each of the spaces so that any state $|\psi\rangle \in \mathscr{T}$ has the form in (4.11). The characterization of $\mathscr{D}$ by (4.11) has been obtained earlier in [34] for $n = 2$.

Our arguments so far can be extended to the braid group B_N whose generators σ_i, $1 \leq i \leq N - 1$, satisfy

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for} \quad i = 1, 2, \ldots, N - 2,$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for} \quad |i - j| \geq 2. \tag{4.12}$$

Indeed, if a state $|\psi\rangle$ follows some one dimensional representations of B_N, we have

$$\pi_\sigma |\psi\rangle = e^{i\chi}|\psi\rangle, \quad \forall \sigma \in B_N, \tag{4.13}$$

for which our two lemmas and proposition hold without modification. This implies that states obeying intermediate symmetries (*i.e.*, $e^{i\chi} \neq \pm 1$), which correspond to states realized by anyons in particle systems, must always be globally entangled.

V. FULL SEPARABILITY AND SYMMETRY

In this section, we focus on symmetric states and discuss their separability further. For this, we first introduce the total symmetrization operator [35],

$$T := \frac{1}{N!} \sum_{\sigma \in S_N} \pi_\sigma. \tag{5.1}$$

Since $\pi_\sigma T = T$ for all $\sigma \in S_N$, we see that for any $|\psi\rangle \in \mathscr{H}$ the totally symmetrized state $T|\psi\rangle$ is invariant under the action of the symmetric group,

$$\pi_\sigma T|\psi\rangle = T|\psi\rangle, \quad \forall \sigma \in S_N. \tag{5.2}$$

Note that $T|\psi\rangle \in \mathscr{S}$, and conversely, if $|\psi\rangle \in \mathscr{S}$ then $|\psi\rangle = T|\psi\rangle$. This implies that the space $\mathscr{S}$ of symmetric states (4.6) can also be written as $\mathscr{S} = \{ |\psi\rangle \mid |\psi\rangle = T|\phi\rangle, \exists |\phi\rangle \in \mathscr{H} \}$.

If we let $\mathscr{S}^\perp$ be the space of states which are orthogonal to all symmetric states, we can show

Proposition 2 *A fully separable state cannot be contained in $\mathscr{S}^{\perp}$, i.e.,*

$$|\psi\rangle \in \mathscr{T} \quad \Rightarrow \quad |\psi\rangle \notin \mathscr{S}^{\perp}. \tag{5.3}$$

Proof. To the contrary of the statement, assume that there exists a fully separable state $|\psi\rangle \in \mathscr{S}^{\perp}$. Then $\langle\phi|\psi\rangle = 0$ for all $|\phi\rangle \in \mathscr{S}$. As before, we expand $|\psi\rangle$ as

$$|\psi\rangle = \bigotimes_{i=1}^{N} \left(\sum_{a_i=0}^{n-1} c_{a_i}^{i} |a_i\rangle_i \right), \qquad c_{a_i}^{i} \in \mathbb{C}. \tag{5.4}$$

Choose first $|\phi_0\rangle = |0\rangle^{\otimes N} \in \mathscr{S}$ where $|0\rangle^{\otimes N} := \otimes_{i=1}^{N}|0\rangle_i$ for which $\langle\phi_0|\psi\rangle = 0$. This is equivalent to

$$\prod_{i=1}^{N} c_0^{i} = 0, \tag{5.5}$$

which shows that at least one of the elements of $\{c_0^{i} \,|\, 1 \le i \le N\}$ is zero. For convenience, we renumber the indices $i = 1, \ldots, N$ of the constituents $\mathscr{H}_i$ so that

$$c_0^{i} = 0, \quad 1 \le i \le n_0 \tag{5.6}$$

holds for some $1 \le n_0 \le N$. Next we choose

$$|\phi_1\rangle = T\left(|1\rangle_1 \otimes \cdots \otimes |1\rangle_{n_0} \otimes |0\rangle_{n_0+1} \otimes \cdots \otimes |0\rangle_N\right) \in \mathscr{S}. \tag{5.7}$$

From $\langle\phi_1|\psi\rangle = 0$ and (5.6), we obtain

$$\prod_{i=1}^{n_0} c_1^{i} = 0. \tag{5.8}$$

Since $n_0 \ge 1$, at least one of the elements of $\{c_1^{i} \,|\, 1 \le i \le n_0\}$ is zero. We renumber the indices again so that

$$c_1^{i} = 0, \quad 1 \le i \le n_1 \tag{5.9}$$

holds for some $1 \le n_1 \le n_0$. Clearly, this process can be repeated until we arrive at

$$c_{a_1}^{1} = 0, \quad 0 \le a_1 \le n - 1. \tag{5.10}$$

This shows that $|\psi\rangle = 0$, invalidating the assumption made at the beginning. ∎

We remark that the contraposition of Proposition 2 is

$$|\psi\rangle \in \mathscr{S}^{\perp} \quad \Rightarrow \quad |\psi\rangle \notin \mathscr{T}. \tag{5.11}$$

This shows that any state orthogonal to $\mathscr{S}$ cannot be fully separable, which is an extension of the previous result [34] for general n.

In passing, we note that since T is a projection operator $T^2 = T$, it can be used for the direct sum decomposition of the Hilbert space $\mathscr{H} = \mathscr{S} \oplus \mathscr{S}^{\perp}$. Namely, for any $|\psi\rangle \in \mathscr{H}$, we have

$$|\psi\rangle = |\xi\rangle + |\varphi\rangle, \qquad |\xi\rangle \in \mathscr{S}, \quad |\varphi\rangle \in \mathscr{S}^{\perp}, \tag{5.12}$$

with

$$|\xi\rangle = T|\psi\rangle, \qquad |\varphi\rangle = (1 - T)|\psi\rangle. \tag{5.13}$$

We also add that, if we have local unitary transformations $U(\theta^i)$ in $\mathscr{H}_i$ with common parameters $\theta^i = \theta \in \Theta$ for all i and consider the combined unitary transformation $V(\theta) := U(\theta) \otimes \cdots \otimes U(\theta)$ in $\mathscr{H}$, the two operators V and T commute,

$$VT = TV. \tag{5.14}$$

We then show

Proposition 3 *If $|\psi\rangle$ is a fully separable state whose projection $T|\psi\rangle$ on $\mathscr{S}$ is still fully separable, then $|\psi\rangle$ is symmetric, i.e.,*

$$|\psi\rangle \in \mathscr{T} \quad \text{and} \quad T|\psi\rangle \in \mathscr{T} \quad \Rightarrow \quad |\psi\rangle \in \mathscr{S}. \tag{5.15}$$

Proof. Let $|\psi\rangle$ be a fully separable state $|\psi\rangle \in \mathscr{T}$ for which $T|\psi\rangle \in \mathscr{T}$. This means $T|\psi\rangle \in \mathscr{D}$, and hence from Corollary 2 we have $T|\psi\rangle = V(\theta)|0\rangle^{\otimes N}$ for some $\theta \in \Theta$. Since T and V commute as noted in (5.14), it follows that the state

$$|\eta\rangle := V^{-1}(\theta)|\psi\rangle \tag{5.16}$$

generated from $|\psi\rangle$ by applying the inverse operator V^{-1} of V has the projection,

$$T|\eta\rangle = |0\rangle^{\otimes N}. \tag{5.17}$$

We then see that since V^{-1} is a product of local unitary transformations, and since $|\psi\rangle \in \mathscr{T}$, the state $|\eta\rangle$ in (5.16) is also fully separable $|\eta\rangle \in \mathscr{T}$. Now, from $^{N\otimes}\langle 0|T|\eta\rangle = 1$ we may write $|\eta\rangle$ without loss of generality as

$$|\eta\rangle = \bigotimes_{i=1}^{N} \left(|0\rangle_i + \sum_{a=1}^{n-1} c_a^i |a\rangle_i \right). \tag{5.18}$$

The projection of the state then reads

$$
\begin{aligned}
T|\eta\rangle &= |0\rangle^{\otimes N} + h_1 T(|1\rangle_1|0\rangle_2|0\rangle_3 \cdots |0\rangle_{N-1}|0\rangle_N) + h_2 T(|1\rangle_1|1\rangle_2|0\rangle_3 \cdots |0\rangle_{N-1}|0\rangle_N) + \cdots \\
&\quad + h_{N-1} T(|1\rangle_1|1\rangle_2|1\rangle_3 \cdots |1\rangle_{N-1}|0\rangle_N) + h_N T(|1\rangle_1|1\rangle_2|1\rangle_3 \cdots |1\rangle_{N-1}|1\rangle_N) + \cdots ,
\end{aligned} \tag{5.19}
$$

where the coefficients in the expansion (5.19) can be determined from (5.18). For instance, we find

$$h_1 = \sum_{k=1}^{N} c_1^k, \qquad h_2 = \sum_{k,k'=1, k \neq k'}^{N} c_1^k c_1^{k'}, \tag{5.20}$$

and

$$h_{N-1} = \sum_{k=1}^{N} c_1^1 c_1^2 \cdots \widehat{c_1^k} \cdots c_1^N, \qquad h_N = \prod_{i=1}^{N} c_1^i, \tag{5.21}$$

where the hat in $\widehat{c_1^k}$ indicates that the factor c_1^k is missing in the product. To be consistent with (5.17), all of these coefficients must vanish. From $h_N = 0$, we see that at least one of the elements of $\{c_1^i \,|\, 1 \le i \le N\}$ is zero, and we may choose $c_1^1 = 0$ for definiteness. Then from $h_{N-1} = 0$ we find that at least one of the elements of $\{c_1^i \,|\, 2 \le i \le N\}$ is zero, and again we choose $c_1^2 = 0$. This procedure can be repeated until we reach $h_1 = 0$ which yields $c_1^N = 0$ and, consequently, we obtain

$$c_1^i = 0, \quad 1 \le i \le N. \tag{5.22}$$

The conclusion will not be changed even if we choose different factors for the vanishing elements in each of the steps in the procedure. Obviously, similar arguments can be applied for the terms $T(|i\rangle_1|0\rangle_2 \cdots |0\rangle_N)$ up to $T(|i\rangle_1|i\rangle_2 \cdots |i\rangle_N)$ for $i \ge 2$ as well. Combining all the results obtained at the end of these procedures, we find

$$c_a^i = 0, \qquad 1 \le i \le N, \quad 0 \le a \le n-1. \tag{5.23}$$

This shows that $|\eta\rangle = |0\rangle^{\otimes N}$, which in turn indicates that $|\psi\rangle = U|\eta\rangle \in \mathscr{S}$, proving Proposition 3. ∎

We remark that Proposition 3 can equally be expressed as

$$|\psi\rangle \in \mathscr{T} \quad \text{and} \quad |\psi\rangle \notin \mathscr{S} \quad \Rightarrow \quad T|\psi\rangle \notin \mathscr{T}, \tag{5.24}$$

which implies that it is impossible to retain the full separability of a state under total symmetrization unless the original state is symmetric.

VI. CONCLUSION AND DISCUSSIONS

In this paper, we studied some of the basic properties of entanglement of multipartite systems based on exchange symmetry for general constituent number N and also for general level n. The states primarily considered are those which are symmetric or anti-symmetric under permutations of the constituent states comprising the system. We argued that different partitions of a state are mutually compatible (Lemma 1), and that symmetric property under a permutation requires a certain rigid structure for the state imposed by the action of the permutation on the constituent systems (Lemma 2). Based on these observations we arrive at Theorem 1, which states that if a state is symmetric under all transpositions, it is either globally entangled or otherwise it is fully separable, sharing essentially the same constituent states. Further, if the state is anti-symmetric, it is necessarily globally entangled. We also presented a number of propositions revealing the close connection between symmetry and separability, some of which are a generalization of the recent results of Ref.[34] for multipartite n-level systems.

We remark that the statement of Theorem 1 can be extended to states which are anyonic under transpositions, that is, like anti-symmetric states, anyonic states with eigenvalue $\lambda = e^{i\chi} \neq \pm 1$ must always be globally entangled. If our arguments can be generalized to systems with an infinite dimensional Hilbert space $\mathscr{H}$, our results may also be useful for analyzing physical properties of entangled states in, *e.g.*, infinite spin systems or particle systems moving on a plane where anyonic states are permitted. In this respect, it will be of interest to investigate the possible role of entanglement in the manifestation of phase transitions or topological quantizations observed in these systems. In fact, for systems of N-qubits the behavior of bipartite entanglement in symmetric states under phase transitions has been studied in [36, 37, 38].

The intimate link between symmetry and entanglement found in this paper suggests that there can be a novel account of known physical properties from exchange symmetry of entangled states. This line of study will be facilitated if we can extend our arguments to mixed states with exchange symmetry, as has been done for the the case of bipartite systems in [28, 33] based on approaches specific to indistinguishable particles [27]. Finally, we mention that our arguments in this paper are given solely on the basis of states and do not rely on any particular entanglement measures. Thus, our results may also serve as a testing ground to find better entanglement measures in order to characterize the interplay of multipartite constituent states more comprehensively.

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[1] A. Einstein, B. Podolsky and N. Rosen, *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?*, Phys. Rev. **47** (1935) 777-780.

[2] J. S. Bell, *On the Einstein-Podolsky-Rosen Paradox*, Physics **1** (1964) 195-200.

[3] A. Aspect, J. Dalibard and G. Roger, *Experimental Test of Bell's Inequalities Using Time-Varying Analyzers*, Phys. Rev. Lett. **49** (1982) 1804-1807.

[4] Gregor Weihs, Thomas Jennewein, Christoph Simon, Harald Weinfurter, and Anton Zeilinger, *Violation of Bell's Inequality under Strict Einstein Locality Conditions*, Phys. Rev. Lett. **81** (1998) 5039-5043.

[5] M. A. Rowe, D. Kielpinski, V. Meyer, C. A. Sackett, W. M. Itano, C. Monroe and D. J. Wineland, *Experimental violation of a Bell's inequality with efficient detection*, Nature **409** (2001) 791-794.

[6] H. Sakai, T. Saito, T. Ikeda, K. Itoh, T. Kawabata, H. Kuboki, Y. Maeda, N. Matsui, C. Rangacharyulu, M. Sasano, Y. Satou, K. Sekiguchi, K. Suda, A. Tamii, T. Uesaka, and K. Yako, *Spin correlations of strongly interacting massive fermion pairs as a test of Bell's inequality*, Phys. Rev. Lett. **97** (2006) 150405.

[7] A. J. Leggett, *Nonlocal Hidden-Variable Theories and Quantum Mechanics: An Incompatibility Theorem*, Found. Phys. **33** (2003) 1469-1493.

[8] Simon Gröblacher, Tomasz Paterek, Rainer Kaltenbaek, Časlav Brukner, Marek Żukowski, Markus Aspelmeyer and Anton Zeilinger, *An experimental test of non-local realism*, Nature **446** (2007) 871-876.

[9] Asher Peres, *Quantum Theory: Concepts and Methods*, Kluwer, Dordrecht, 1993.

[10] Michael A. Nielsen and Isaac L. Chuang, *Quantum Computaion and Quantum Information*, Cambridge university press, Cambridge, 2000.

[11] Akimasa Miyake, *Classification of multipartite entangled states by multidimensional determinants*, Phys. Rev. **A67** (2003) 012108.

[12] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, *Quantum Entanglement*, quant-ph/0702225.

[13] D. A. Meyer and N. R. Wallach, *Global entanglement in multiparticle systems*, J. Math. Phys. **43** (2002) 4273-4278.

[14] G. K. Brennen, *An observable measure of entanglement for pure states of multi-qubit systems*, Quant. Inf. Comp. **3** (2003) 619-626.

[15] P. J. Love, A. M. van den Brink, A. Y. Smirnov, M. H. S. Amin, M. Grajcar, E. Il'ichev, A. Izmalkov and A. M. Zagoskin, *A characterization of global entanglement*, Quant. Inf. Processing **6** (2007) 187-195.

[16] H. Weyl, *Theory of Groups and Quantum Mechanics*, Methuen, London, 1931.

[17] W. Pauli, *Über das Wasserstoffspektrum vom Standpunkt der neuen Quantenmechanik*, Z. Physik **36** (1926) 336-363.

[18] Leonard I. Schiff, *Quantum Mechanics*, McGraw-Hill, Singapore, 1955.

[19] G. H. Green, *A generalized method of field quantization*, Phys. Rev. **90** (1953) 270-273.

[20] S. Kamefuchi and Y. Takahashi, *A generalization of field quantization and statistics*, Nucl. Phys. **36** (1962) 177-206.

[21] K. Drühl, R. Haag and J. E. Roberts, *On parastatistics*, Commun. Math. Phys. **18** (1970) 204-226.

[22] J. M. Leinaas and J. Myrheim, *On the Theory of Identical Particles*, Nuovo Cimento **37B** (1977) 1-23.

[23] Frank Wilczek, *Quantum Mechanics of Fractional-Spin Particles*, Phys. Rev. Lett. **49** (1982) 957-959.

[24] A. Yu. Kitaev, *Fault-tolerant quantum computation by anyons*, Ann. Phys. **303** (2003) 2-30.

[25] Robert König and Renato Renner, *A de Finetti representation for finite symmetric quantum states*, J. Math. Phys. **46** (2005) 122108.

[26] Renato Renner, *Symmetry of large physical systems implies independence of subsystems*, Nature Physics **3** (2007) 645-649.

[27] John Schliemann, J. Ignacio Cirac, Marek Kuś, Maciej Lewenstein and Daniel Loss, *Quantum correlations in two-fermion systems*, Phys. Rev. **A64** (2001) 022303.

[28] K. Eckert, J. Schliemann, D. Bruß and M. Lewenstein, *Quantum correlations in systems of indistinguishable particles*, Ann. Phys. **299** (2002) 88-127.

[29] G. Ghirardi, L. Marinatto and T. Weber, *Entanglement and Properties of Composite Quantum Systems: a Conceptual and Mathematical Analysis*, J. Stat. Phys. **108** (2002) 49-121.

[30] G. Ghirardi and L. Marinatto, *General criterion for the entanglement of two indistinguishable particles*, Phys. Rev. **A70** (2004) 012109.

[31] G. Ghirardi and L. Marinatto, *Criteria for the entanglement of composite systems of identical particles*, Fortschr. Phys. **52** (2004) 1045-1051.

[32] G. Ghirardi and L. Marinatto, *Identical particles and entanglement*, Optics and Spectroscopy **99** (2005) 386-390.

[33] Jan Naudts and Tobias Verhulst *Ensemble-averaged entanglement of two-particle states in Fock space*, Phys. Rev. **A75** (2007) 062104.

[34] P. Dömötör and M. G. Benedict, *Coherent states and global entanglement in an N qubit system*, Phys. Lett. A **372** (2008) 3792-3795.

[35] Note that the action of the total symmetrization operator T is in general basis-dependent, because the permutations (3.1) are performed among the corresponding basis states chosen in each of $\mathscr{H}_i$.

[36] Vladislav Popkov and Mario Salerno, *Logarithmic divergence of the block entanglement entropy for the ferromagnetic Heisenberg model*, Phys. Rev. **A71** (2005) 012301.

[37] Julien Vidal, *Concurrence in collective models*, Phys. Rev. **A73** (2006) 062318.

[38] Julien Vidal, Sébastien Dusuel and Thomas Barthel, *Entanglement entropy in collective models*, J. Stat. Mech. (2007) P01015.