Classical and Quantum Contents of Solvable Game Theory on Hilbert Space

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A simple and full Hilbert space formulation of the game theory is presented, which for the two strategy case is completely solvable. The quantum game is interpreted to consist of a family of classical games and the quantum interference invoked by the correlation between the players' strategies. Examination of the Nash equilibria for the Prisoner's Dilemma shows that the quantum solution found previously is optimal in the classical family, while there exists another quantum Nash equilibrium supported by the quantum interference. Novel aspects of the quantum game such as stone-scissor-paper phase game and quantum moderation are also revealed.

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Attempts to extend game-theoretical strategies to quantum domain [1, 2] have caught journalistic attention in academic community and beyond, with their intriguing solution to the classical problem of the Prisoner's Dilemma [3]. The substance of the emerging quantum game theory, however, is still shrouded in mystery, and in spite of the rapid accumulation of literature [4], we still find *ad hoc* assumptions and arbitrary procedures scattered in the field. Quite naturally, there have been persistent doubts on their generality, finality and ultimately, the true contents of the quantum game as compared to the classical game [5, 6]. For the quantum treatment of game strategies to become a *theory*, a simple and workable framework, clarification of its relation to conventional game theories and preferably, analytic solutions illuminating its structure are highly desired.

In this article, we attempt to answer this call with a full Hilbert space formulation of the game theory. It is shown that assigning vectors in a Hilbert space to game strategies entails the introduction of an element that provides correlation for the strategies of the individual players. For two strategy games, the correlation is generated by operators that implement swapping and simultaneous renaming of the player's strategies. The quantum game is then split into two parts, one consisting of a family of classical games and the other representing the genuine quantum ingredient of the game. The game, as a whole, is solvable. We illustrate our formalism by symmetric games with numerical examples on Prisoner's Dilemma and discuss the classical and quantum contents appearing in the Nash equilibria. We also point out the existence of such curious phenomena as the stone-scissor-paper game found for phase variables of the strategy, and the quantum moderation which occurs for fluctuating correlations.

To present our scheme of quantum game, we first consider Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ in which the strategies of the two players A and B are represented by vectors $|\alpha\rangle_A \in \mathcal{H}_A$ and $|\beta\rangle_B \in \mathcal{H}_B$. The entire space of strategy of the game is then given by the direct product $\mathcal{H} = \mathcal{H}_A \times \mathcal{H}_B$. A vector in $\mathcal{H}$ represents a *joint strategy* of the two players and can be written as

$$|\alpha, \beta; \gamma\rangle = J(\gamma)\, |\alpha\rangle_A\, |\beta\rangle_B\,, \tag{1}$$

where the unitary operator $J(\gamma)$ provides quantum correlation (*e.g.,* entanglement) for the separable states $|\alpha\rangle_A\, |\beta\rangle_B$. Note that $J(\gamma)$ is independent of the players' choice and is determined by a third party, which is hereafter referred to as the *coordinator*.

Once the joint strategy is specified with $J(\gamma)$, the players are to receive the payoffs, which are furnished by the expectation values of self-adjoint operators A and B:

$$\begin{aligned} \Pi_A(\alpha, \beta; \gamma) &= \langle \alpha, \beta; \gamma | A | \alpha, \beta; \gamma\rangle\,, \\ \Pi_B(\alpha, \beta; \gamma) &= \langle \alpha, \beta; \gamma | B | \alpha, \beta; \gamma\rangle\,. \end{aligned} \tag{2}$$

Each of the players then tries to optimize their strategy to gain the maximal payoff, and our question is to find, if any, a stable strategy vector which corresponds to the quantum version of the Nash equilibrium. Namely, we seek a point $(\alpha, \beta) = (\alpha^\star, \beta^\star)$ in the strategy space at which the payoffs separately attain the maxima as

$$\delta_\alpha \Pi_A(\alpha, \beta^\star; \gamma)|_{\alpha^\star} = 0, \quad \delta_\beta \Pi_B(\alpha^\star, \beta; \gamma)|_{\beta^\star} = 0, \tag{3}$$

under arbitrary variations in α and β.

A symmetric quantum game is defined by requiring that the strategy spaces of the two players are the same in dimensionality, $\dim \mathcal{H}_A = \dim \mathcal{H}_B = n$, and that the payoffs are symmetric for two players. The latter condition is expressed as

$$\Pi_A(\alpha, \beta; \gamma) = \Pi_B(\beta, \alpha; \gamma) \tag{4}$$

in terms of identically labeled strategies for both players

$$|\alpha\rangle_A = \sum_i \alpha_i\, |i\rangle_A\,, \quad |\beta\rangle_B = \sum_i \beta_i\, |i\rangle_B\,, \tag{5}$$

with complex numbers α_i, β_i normalized as $\sum_i |\alpha_i|^2 = \sum_i |\beta_i|^2 = 1$. Here we have used a common orthonormal basis for both of the players, namely, a set of strategies of the two players which are in one-to-one correspondence

$|i\rangle_A \leftrightarrow |i\rangle_B$ for $i = 0, 1, \ldots, n-1$. Introducing the swap operator S by

$$S\,|i,j\rangle = |j,i\rangle \tag{6}$$

for the states $|i,j\rangle = |i\rangle_A\,|j\rangle_B$, we have $S\,|\alpha,\beta\rangle = |\beta,\alpha\rangle$ for general separable states $|\alpha,\beta\rangle = |\alpha\rangle_A\,|\beta\rangle_B$. For our convenience, we introduce two more operators C and T defined by

$$C\,|i,j\rangle = |\bar{i},\bar{j}\rangle\,, \quad T\,|i,j\rangle = |\bar{j},\bar{i}\rangle\,, \tag{7}$$

where the bar represents the complimentary choice; $\bar{i} = (n-1)-i$. The operator C is the simultaneous renaming (conversion) of strategy for two players, and T is the combination $T = CS$. These operators $\{S, C, T\}$ commute among themselves and satisfy $S^2 = C^2 = T^2 = I$, $T = SC$, $S = CT$ and $C = TS$. With the identity I, they form the dihedral group D_2.

In terms of the correlated payoff operators,

$$\mathcal{A}(\gamma) = J^\dagger(\gamma) A J(\gamma), \quad \mathcal{B}(\gamma) = J^\dagger(\gamma) B J(\gamma), \tag{8}$$

we have $\Pi_A(\alpha,\beta;\gamma) = \langle\alpha,\beta|\,\mathcal{A}(\gamma)\,|\alpha,\beta\rangle$. It is convenient to choose the unitary operator $J(\gamma)$ such that $\mathcal{A}(0)$ is diagonal in the product basis $|i,j\rangle$. The game is then symmetric if $\mathcal{B}(0) = S\mathcal{A}(0)S$, in which case $\mathcal{B}(0)$ is diagonalized simultaneously with the eigenvalues swapped,

$$\begin{aligned}\langle i',j'|\,\mathcal{A}(0)\,|i,j\rangle &= A_{ij}\delta_{i'i}\delta_{j'j},\\ \langle i',j'|\,\mathcal{B}(0)\,|i,j\rangle &= A_{ji}\delta_{i'i}\delta_{j'j}.\end{aligned} \tag{9}$$

Observe that $\Pi_A(\alpha,\beta;0) = \Pi_B(\beta,\alpha;0) = \sum_{i,j} x_i A_{ij} y_j$ with $x_i = |\alpha_i|^2$, $y_j = |\beta_j|^2$ being the probability of choosing the strategies $|i\rangle_A$, $|j\rangle_B$. This means that, at $\gamma = 0$, our quantum game reduces to the classical game with the payoff matrix A_{ij} under mixed strategies.

Now we restrict ourselves to two strategy games $n = 2$. The entire Hilbert space is spanned by $|\alpha,\beta;\gamma\rangle$ with the unitary operator $J(\gamma)$ in the form

$$J(\gamma) = J(0)\, e^{i\gamma_1 S/2} e^{i\gamma_2 T/2}, \tag{10}$$

where $\gamma = (\gamma_1, \gamma_2)$ are real parameters. Note that, on account of the relation $S+T-C = I$ valid for $n = 2$, only two operators are independent in the set $\{S, C, T\}$. For simplicity, we assume that A is diagonalized under the basis $|i,j\rangle$, which implies $J(0) = I$ and $\mathcal{A}(0) = A$. The correlated payoff operator $A(\gamma)$ is split into two terms

$$\mathcal{A}(\gamma) = \mathcal{A}^{\rm pc}(\gamma) + \mathcal{A}^{\rm in}(\gamma) \tag{11}$$

where $\mathcal{A}^{\rm pc}$ is the "pseudo classical" term and $\mathcal{A}^{\rm in}$ is the "interference" term given, respectively, by

$$\begin{aligned}\mathcal{A}^{\rm pc}(\gamma) &= \cos^2\frac{\gamma_1}{2}A + (\cos^2\frac{\gamma_2}{2} - \cos^2\frac{\gamma_1}{2})SAS + \sin^2\frac{\gamma_2}{2}CAC,\\ \mathcal{A}^{\rm in}(\gamma) &= \frac{i}{2}\sin\gamma_1(AS-SA) + \frac{i}{2}\sin\gamma_2(AT-TA).\end{aligned} \tag{12}$$

Correspondingly, the full payoff is also split into two contributions from $\mathcal{A}^{\rm pc}$ and $\mathcal{A}^{\rm in}$ as $\Pi_A = \Pi_A^{\rm pc} + \Pi_A^{\rm in}$. To evaluate the payoff, we may choose both α_0 and β_0 to be real without loss of generality, and adopt the notaions $(\alpha_0, \alpha_1) = (a_0, a_1 e^{i\xi})$ and $(\beta_0, \beta_1) = (b_0, b_1 e^{i\chi})$. The outcome is

$$\begin{aligned}\Pi_A^{\rm pc}(\alpha,\beta;\gamma) &= \sum_{i,j} a_i^2 b_j^2 \mathcal{A}_{ij}^{\rm pc}(\gamma),\\ \Pi_A^{\rm in}(\alpha,\beta;\gamma) &\\ = -a_0 a_1 b_0 b_1 &[G_+(\gamma)\sin(\xi+\chi) + G_-(\gamma)\sin(\xi-\chi)],\end{aligned} \tag{13}$$

with $\mathcal{A}_{ij}^{\rm pc}(\gamma) = \langle i,j|\,\mathcal{A}^{\rm pc}(\gamma)\,|i,j\rangle$ and

$$\begin{aligned}G_+(\gamma) &= (A_{00} - A_{11})\sin\gamma_2,\\ G_-(\gamma) &= (A_{01} - A_{10})\sin\gamma_1.\end{aligned} \tag{14}$$

The split of the payoff shows that the quantum game consists of two ingredients. The first is the pseudo classical ingredient associated with $\Pi_A^{\rm pc}$ in (13), whose form indicates that we are, in effect, simultaneously playing three different classical games, *i.e.,* the original classical game A, the altruistic game SAS [7] and the converted-value game CAC with the mixture specified by given γ_1 and γ_2. Regarding γ as tunable parameters, we see that the quantum game contains a *family* of classical games that includes the original game. The second ingredient of the quantum game is the purely quantum component $\Pi_A^{\rm in}$, which occurs only when both of the two players adopt quantum strategies with $a_0 a_1 b_0 b_1 \neq 0$ and non-vanishing phases ξ and χ. The structure of $\Pi_A^{\rm in}$ suggests that this interference term cannot be simulated by a classical game and hence represents the *bona fide* quantum aspect.

We can find the quantum Nash equilibrium strategy explicitly by considering the condition (3) for the payoff obtained above. It can be readily confirmed that, modulo arbitrary phases, the "edge" strategies $|\alpha^\star,\beta^\star\rangle$ given by

$$|\alpha^\star,\beta^\star\rangle = |0,0\rangle\,, |1,1\rangle\,, |0,1\rangle\,, |1,0\rangle \tag{15}$$

can furnish Nash equilibria, depending on the signs of the functions

$$H_\pm(\gamma) = \tau(A) \pm \left[G'_+(\gamma) + G'_-(\gamma)\right], \tag{16}$$

where we define $\tau(A) = A_{00} - A_{01} - A_{10} + A_{11}$ and also $G'_+(\gamma) = (A_{00} - A_{11})\cos\gamma_2$, $G'_-(\gamma) = (A_{01} - A_{10})\cos\gamma_1$. The precise conditions for the appearance of the equilibria, together with their maximal payoffs $\Pi_A^\star(\gamma) = \Pi_A(\alpha^\star(\gamma),\beta^\star(\gamma);\gamma)$ obtained under variations of γ, are summarized in TABLE I.

TABLE I: The quantum Nash equilibria with edge strategies. The conditions for their appearance and their maximal payoffs under variations of γ are shown.

$\lvert\alpha^\star,\beta^\star\rangle$	$\lvert 0,0\rangle$	$\lvert 1,1\rangle$	$\lvert 0,1\rangle$	$\lvert 1,0\rangle$
Condition	$H_+ > 0$	$H_- > 0$	$H_+ < 0$, $H_- < 0$	$H_+ < 0$, $H_- < 0$
$\mathrm{Max}(\Pi_A^\star)$	A_{00}	A_{00}	A_{11}	$A_{01}+A_{10}-A_{11}$
$\mathrm{Max}(\Pi_B^\star)$	A_{00}	A_{00}	$A_{01}+A_{10}-A_{11}$	A_{11}

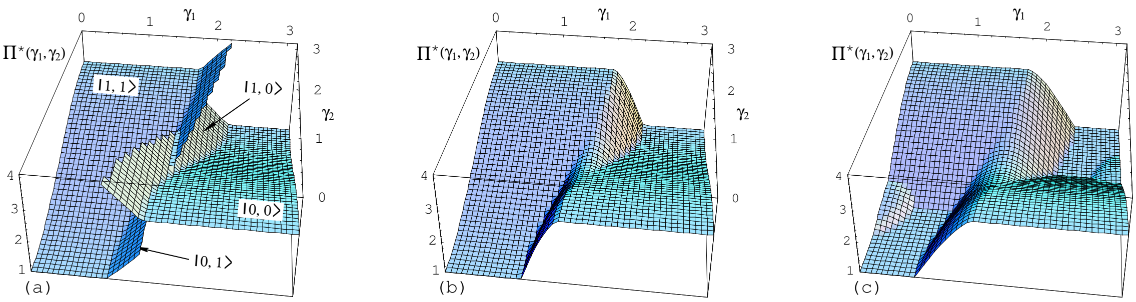

FIG. 1: The quantum Nash equilibrium payoff $\Pi_A^\star(\gamma) = \Pi_A(\alpha^\star, \beta^\star; \gamma)$ (or $\Pi_A(P_A^\star, P_B^\star; \gamma)$ for mixed quantum strategies) as a function of γ. Only the region $\gamma_1, \gamma_2 \in [0, \pi]$ is shown since $\Pi_A^\star(\gamma)$ has the reflection invariance $\Pi_A^\star(2\pi-\gamma_1, \gamma_2) = \Pi_A^\star(\gamma_1, 2\pi-\gamma_2) = \Pi_A^\star(\gamma_1, \gamma_2)$. The extra invariance $\Pi_A^\star(\pi - \gamma_1, \pi - \gamma_2) = \Pi_A^\star(\gamma_1, \gamma_2)$ is also visible. (a) Edge state Nash equilibria for $A_{00} = 3$, $A_{01} = 0$, $A_{10} = 5$ and $A_{11} = 1$. The value $(\gamma_1, \gamma_2) = (0.9272, 0)$ gives the maximum payoff $\Pi_A^\star = 4$ for one of the players. (b) Symmetric mixed quantum Nash equilibrium with the same parameters as (a). The maximum payoff $\Pi_A^\star = 3$ is obtained at $\gamma_2^\star = 0$ and $1.3694 \le \gamma_1^\star < \pi$. (c) Mixed Nash equilibrium for $A_{00} = 3$, $A_{01} = 0$, $A_{10} = 5$ and $A_{11} = 0.2$. The two bumps near the left and right ends are due the pure symmetric Nash equilibria (20).

For illustration, let us consider the case where the classical game $\gamma = 0$ exhibits the Prisoner's Dilemma, $A_{01} < A_{11} < A_{00} < A_{10}$. Adopting the numerical values used in [2], we observe from FIG. 1(a) that two asymmetric Nash equilibria coexist in the middle strip that separates the two domains where the symmetric Nash equilibria arise. The maximal payoff for player A is achieved by the equilibrium strategy $|1, 0\rangle$ at the optimal choice $(\gamma_1^\star, \gamma_2^\star) = (2 \arcsin \sqrt{\lambda}, 0)$, and also by its symmetric partner $|0, 1\rangle$ at $(\pi - 2 \arcsin \sqrt{\lambda}, \pi)$ where we use

$$\lambda = (A_{11} - A_{01})/(A_{10} - A_{01}). \qquad (17)$$

Interestingly, the maximal payoff is better than that obtained when the players decide to "deny" $|1, 1\rangle$ or "confess" $|0, 0\rangle$. Note that the joint strategies (1) of the players realized at these Nash equilibria are actually entangled due to the correlation factor $J(\gamma)$. Indeed, the entropy of entanglement $S(\rho_{\rm red})$ evaluated for the reduced density operator $\rho_{\rm red}$ [8] of the optimal state reads

$$S(\rho_{\rm red}) = -\lambda \log \lambda - (1 - \lambda) \log(1 - \lambda), \qquad (18)$$

which is nonvanishing since $0 < \lambda < 1$ for the Prisoner's Dilemma. The optimal equilibrium, however, does not provide a desired resolution for the dilemma, because it is achieved at the expense of player B receiving a lower payoff. In fact, in the middle strip, the original Prisoner's Dilemma turns into the Game of Chicken which has its own dilemma of a different kind.

Leaving the numerical example aside for now, we return to the general case, and examine the possibility of a pure Nash equilibrium which is not one of the edge states. The interference term $\Pi_A^{\rm in}$ now comes into play, and applying the condition for phases, $\delta_\xi \Pi_A|_{\xi=\chi=\xi^\star}$, we obtain

$$\cos 2\xi^\star = -G_-(\gamma)/G_+(\gamma). \qquad (19)$$

When $|G_-| \le |G_+|$, there is an equilibrium solution $\xi = \chi = \xi^\star$ with the payoff $\Pi_A^{\rm in} = a_0 a_1 b_1 b_1 \Delta(\gamma)$, where $\Delta(\gamma) = \sqrt{G_+^2 - G_-^2}$. The condition for the amplitudes $\delta_{a_0} \Pi_A = 0$ and $\delta_{b_0} \Pi_B = 0$ then provides, along with the edge state solutions (15), the symmetric solution,

$$a_0^\star = b_0^\star = \left(\frac{H_-(\gamma) - \Delta(\gamma)}{H_+(\gamma) + H_-(\gamma) - 2\Delta(\gamma)} \right)^{1/2}, \qquad (20)$$

which is valid ($0 \le a_0^\star \le 1$) if $(H_+ - \Delta)(H_- - \Delta) \ge 0$. There is no asymmetric pure Nash equilibria apart from the two edge solutions.

When $|G_-| > |G_+|$, there is no dominant strategy for the phases: the player A tries to top player B by choosing a phase which is off by $\pi/2$ to maximize $\Pi_A^{\rm in}$. The player B does the same, and if the game is played repeatedly, the result is a uniform random distribution for both ξ and χ. This is a continuous version of the paper-scissor-stone game for phases, and results in the zero average for the interference term, $\Pi_A^{\rm in} = 0$. Thus, we reach formally the same symmetric solution (20) with $\Delta(\gamma) = 0$. The existence requirement of the solution simplifies to $H_+ H_- \ge 0$ for this case, and we find, from TABLE I, that the equilibrium appears precisely in the region of the asymmetric pure Nash equilibria.

The foregoing argument is made formal by considering the *mixed* quantum strategies specified by probability distributions $P_A(\alpha)$, $P_B(\beta)$ over the players' actions normalized under suitable measures $d\alpha$, $d\beta$. The distribution-averaged payoff can be defined as $\Pi_X(P_A, P_B; \gamma) = \int d\alpha d\beta P_A(\alpha) \Pi_X(\alpha, \beta; \gamma) P_B(\beta)$ for $X = A, B$. The players seek a distribution $P_A = P_B = P^\star$ which simultaneously maximize $\Pi_A(P_A, P_B; \gamma)$ and $\Pi_B(P_A, P_B; \gamma)$. Such a distribution furnishes a mixed quantum Nash equilibrium, extending the concept of the pure quantum Nash equilibrium specified by single values of α and β. Note, however, that the latter is already probabilistic in terms of classical strategies, possessing a classical mixed strategy game as a subset. The former, on the other hand, is probabilistic in terms of quantum strategies, and is realized by an ensemble of quantum systems.

In FIG. 1(b), the mixed quantum Nash equilibrium payoff for the Prisoner's Dilemma of FIG. 1(a) is shown as a function of γ. In the middle strip, asymmetric equilibria, which are known to be dynamically unstable [9], are now replaced by the mixed Nash equilibrium given by (20) with $\Delta(\gamma) = 0$. The global maximum of the payoff $\Pi^\star_A(\gamma^\star) = A_{00}$ is attained along the line $\gamma^\star_2 = 0$, $2 \arcsin \sqrt{\eta} \le \gamma^\star_1 \le \pi$ with $\eta = (A_{10}-A_{00})/(A_{10}-A_{01})$. We mention that the quantum Nash equilibrium found in [2] corresponds to $(\gamma_1, \gamma_2) = (\pi/2, 0)$ in our scheme. Unlike the optimal point $|1, 0\rangle$, the joint strategy state of this equilibrium remains to be $|0, 0\rangle$ and hence is *not* entangled. In fact, the entanglement of the Nash equilibrium is irrelevant in this example, since the interference term vanishes for all parameter values, leaving only classically interpretable terms.

The truly quantum characteristics of the game manifests itself when the non-edge solution (20) appears. We show an example of such cases in FIG. 1(c) which is obtained by modifying the payoff parameters slightly from the previous ones. Here, the Nash equilibria (20) contributes to the increase of the payoff as seen by the convex structures at the two ends. Examination of the solution to other types of quantum games [10, 11] would naturally be the next task.

There are several different interpretations possible for the role of the coordinator $J(\gamma)$. The first is that it furnishes the unitary family of payoff operators $\mathcal{A}(\gamma)$ from a given classical payoff matrix A_{ij}, and therefore acts independently from the two players. The second is that it acts as a collaborator to the players and serves to maximize the payoff at the Nash equilibria by tuning γ as $\delta_\gamma \Pi_A|_{\gamma^\star} = 0$. In the above numerical examples, we have started from the first interpretation and tacitly moved to the second. Yet another interpretation of the coordinator is that it generates quantum fluctuations for the payoffs by randomizing the parameter γ. In this case, the game is effectively given as an average over the fluctuations, and may be studied by integrating out the parameters γ_1 and γ_2. At the level of the payoff operator, the outcome is expressed as

$$\int d\gamma_1 d\gamma_2 \, \mathcal{A}(\gamma) = \frac{1}{2} A + \frac{1}{2} CAC, \tag{21}$$

which implies that the quantum fluctuations yield *quantum moderation* to the game by washing out the individual's preference.

We conclude this article with remarks on the contents of the quantum game theory. The quantum game, which in our scheme is formulated for the full Hilbert space strategies, extends the classical game by enlarging the repertoire of strategies with extra parameters γ of the coordinator and the phases ξ, χ in the strategy vectors. The extension is non-calssical, but it is possible to interpret the quantum game as that consisting of a family of classical games $\mathcal{A}^{\rm pc}(\gamma)$, supplemented by a genuine quantum effect $\mathcal{A}^{\rm in}(\gamma)$. The pseudo classical family furnishes the link between quantum strategies and the games in classical settings. Our numerical examples show that the quantum Nash equilibrium found earlier for solving the Prisoner's Dilemma can be understood as an equilibrium in the classical family, for which no quantum effect is invoked. The non-edge pure quantum Nash equilibrium found in this work provides the first quantum Nash equilibrium that has no counterpart in classical games. For the comprehensive classification of the quantum games in our scheme, the full analysis of the pseudo classical family is indispensable. This should also pave the way to the extension for games with more than two strategies and two players.

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