Spectral Properties on a Circle with a Singularity

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Abstract. We investigate the spectral and symmetry properties of a quantum particle moving on a circle with a pointlike singularity (or point interaction). We find that, within the $U(2)$ family of the quantum mechanically allowed distinct singularities, a $U(1)$ equivalence (of duality-type) exists, and accordingly the space of distinct spectra is $U(1) \times [SU(2)/U(1)]$, topologically a filled torus. We explore the relationship of special subfamilies of the $U(2)$ family to corresponding symmetries, and identify the singularities that admit an $N = 2$ supersymmetry. Subfamilies that are distinguished in the spectral properties or the WKB exactness are also pointed out. The spectral and symmetry properties are also studied in the context of the circle with two singularities, which provides a useful scheme to discuss the symmetry properties on a general basis.

KEYWORDS: point interaction, point singularity, quantum mechanics, circle, spectrum, symmetries, supersymmetry, WKB exactness

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1. Introduction

Systems with point singularities (*i.e.,* contact interactions or reflecting boundaries) provide an important class of solvable models in quantum mechanics, allowing for useful applications in various fields in physics, such as in nuclear or condensed matter physics. In spite of being relatively simple, these systems exhibit a variety of exotic features, including renormalization, Landau poles, anomalous symmetry breaking, duality, supersymmetry, and spectral anholonomy [1]. In parallel, the recent developments of nanotechnology have enabled us to manufacture nanoscale quantum devices, some of which find an appropriate description in terms of contact interactions. The system of a line with a point singularity can be realized by a quantum well in one dimension and provides the simplest example for such systems. The line system with a singularity has lately been studied intensively, physical/mathematical aspects of the system are well-known by now [2–8].

In the manufacturing process, one may connect the two ends of the line, or just insets an antidot in a larger quantum dot to make a loop. The resultant circle system with point singularities is expected to be quite similar to the line system in many respects. Indeed, the two systems share the same four-parameter $U(2)$ family for the family of possible point singularities and the same boundary condition induced by the singularity [8]. However, one crucial difference between them is the different *topology*, that is, the circle system obviously allows a circular flow of the probability current while the line system does not. Moreover, the circle system can trap a magnetic flux inside the loop, and hence is sensitive to the applied magnetic field responsible for Aharonov-Bohm-type effects. The actual construction of a quantum ring, as a realization of the circle system, has been done recently [9–11], and the effect of a magnetic flux on the energy spectrum has been measured experimentally.

In this paper, we explore the spectral and symmetry properties of the circle system with one singularity taking all possible types of quantum singularities into consideration. One of our aims is to specify the spectral space of the system, which is important to determine the range of the energy spectra that we can obtain by possibly tuning the parameters of the singularity. We shall learn that the spectral space is of three dimensions and given by $\Sigma_{\mathrm{circle}} \cong S^1 \times D^2$. This may be contrasted to the spectral space of the line system $\Sigma_{\mathrm{line}} \cong (S^1 \times S^1)/\mathbb{Z}_2$ which is two dimensional [12], and this shows that the circle system can accommodate more variety in the spectrum than the line system. Behind this lies the difference in the (generalized) symmetry structures of the two systems, and we shall see that this difference also leads to the distinct features in the invariant subfamilies of the $U(2)$ defined from the symmetries. We also discuss other subfamilies that are distinguished in the spectral properties or the WKB exactness.

The plan of this paper is as follows. After the Introduction, we first present the spectral space of the circle system in sect.2 by inspecting the spectral conditions. Then we discuss in sect.3 the spectral preserving symmetries and related invariant subfamilies and, in sect.4, the possibility of supersymmetry on the circle. Other subfamilies which are distinguished in the spectral properties and the WKB exactness will be mentioned in sect.5. In sect.6, we provide a generalized framework accommodating two singularities, where the symmetries and the spectral properties obtained in the earlier sections can be confirmed in a more transparent manner. Finally, sect.7 is devoted to the summary and discussions. Appendix A provides the detailed discussion for the determination of the spectral space $\Sigma_{\rm circle}$, while Appendix B supplements our argument on the scale invariant subfamily in sect.5.

2. Spectral space and degeneracy

Before analyzing the spectral properties of the circle system with a point singularity, we first recall how the system is defined in quantum mechanics [8]. The crucial ingredient in the definition lies in the fact that the effect of the singularity is expressed through a certain boundary condition for the wave functions admitted quantum mechanically. Namely, the possible form of the boundary condition is determined by demanding that, under the presence of the singularity, the Hamiltonian $H = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ of the system be self-adjoint on the space of the wave functions obeying the condition. In terms of a unitary matrix $U \in U(2)$, called the *characteristic matrix*, the boundary condition that meets the demand is found to be

$$(U - I_2)\Psi + iL_0\,(U + I_2)\Psi' = 0\,, \qquad \Psi := \begin{pmatrix} \psi(+0) \\ \psi(l-0) \end{pmatrix}, \quad \Psi' := \begin{pmatrix} \psi'(+0) \\ -\psi'(l-0) \end{pmatrix}, \tag{2.1}$$

where I_2 is the two-by-two identity matrix, L_0 is an auxiliary nonzero constant possessing the dimension of length [8, 1] and $\psi' = \mathrm{d}\psi/\mathrm{d}x$. An alternative expression is provided by

$$U\Psi^{(+)} = \Psi^{(-)}\,, \qquad \Psi^{(\pm)} := \Psi \pm iL_0\Psi', \tag{2.2}$$

which we shall use later. The characteristic matrix $U \in U(2) \cong U(1) \times SU(2)$ can be conveniently parametrized as

$$U = e^{i\xi}\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} = e^{i\xi}\begin{pmatrix} \alpha_{\mathrm{R}} + i\alpha_{\mathrm{I}} & \beta_{\mathrm{R}} + i\beta_{\mathrm{I}} \\ -\beta_{\mathrm{R}} + i\beta_{\mathrm{I}} & \alpha_{\mathrm{R}} - i\alpha_{\mathrm{I}} \end{pmatrix}, \tag{2.3}$$

with $\xi \in [0, \pi)$ and $\alpha, \beta \in \mathbb{C}$ satisfying

$$|\alpha|^2 + |\beta|^2 = \alpha_{\mathrm{R}}^2 + \alpha_{\mathrm{I}}^2 + \beta_{\mathrm{R}}^2 + \beta_{\mathrm{I}}^2 = 1\,. \tag{2.4}$$

Thus the self-adjoint Hamiltonian is characterized by the matrix U and we denote it by H_U. In mathematical terms, the domain of the Hamiltonian is given by

$$\mathcal{D}(H_U) = \{\psi \in \mathcal{H} \,|\, \psi,\, \psi' \in \mathrm{AC}(0,l),\; U\Psi^{(+)} = \Psi^{(-)}\}, \tag{2.5}$$

where $\mathcal{H} = L^2(0,l)$ is the the Hilbert space consisting of square integrable functions on the interval $(0,l)$ and $\mathrm{AC}(0,l)$ is the space of absolutely continuous functions on it.

The boundary condition (2.1) is exactly the same as the one for the line system with a point singularity, which can be realized under some singular potential at $x = 0$ [13–18] generalizing the well-known Dirac delta interaction. Such a potential can be explicitly constructed, *e.g.,* as a sequence of Dirac deltas with appropriately chosen, diverging strengths determined by three of the four parameters of U. The remaining parameter, $\arg\beta$, which represents the phase jump of the wave function at $x = 0$, can be realized as a vector potential. On a line, this vector potential is unphysical and can be gauged away [12], since the relative phase of the two opposite sides of the singularity cannot be measured there. However, on a circle this is no longer true, that is, the phase can be measured by interference of states because the two sides of the singularity are connected. Indeed, the phase jump expresses the magnetic flux that penetrates through the circle and is by all means physical.

Now, to study the spectral property of the circle system, we first consider the positive spectrum provided by the energy eigenfunctions of the form

$$\varphi_k(x) = A_k\, e^{ikx} + B_k\, e^{-ikx}\,, \qquad k > 0\,. \tag{2.6}$$

For such a wave function, the boundary vectors are

$$\Psi = \tau_k \begin{pmatrix} A_k \\ B_k \end{pmatrix}, \qquad\qquad \Psi' = ik\sigma_3\tau_k\sigma_3 \begin{pmatrix} A_k \\ B_k \end{pmatrix}, \tag{2.7}$$

where σ_k, $k = 1, 2, 3$ denote the Pauli matrices and

$$\tau_k := \begin{pmatrix} 1 & 1 \\ e^{ikl} & e^{-ikl} \end{pmatrix}. \tag{2.8}$$

Then, the connection condition (2.1) reads

$$[(U - I_2)\tau_k - kL_0(U + I_2)\sigma_3\tau_k\sigma_3] \begin{pmatrix} A_k \\ B_k \end{pmatrix} = 0\,, \tag{2.9}$$

or, explicitly,

$$\begin{pmatrix} \alpha K_- + (\beta e^{ikl} - e^{-i\xi})K_+ & \alpha K_+ + (\beta e^{-ikl} - e^{-i\xi})K_- \\ \alpha^* e^{ikl} K_+ - (\beta^* + e^{-i\xi} e^{ikl})K_- & \alpha^* e^{-ikl} K_- - (\beta^* + e^{-i\xi} e^{-ikl})K_+ \end{pmatrix} \begin{pmatrix} A_k \\ B_k \end{pmatrix} = 0, \tag{2.10}$$

with $K_\pm := 1 \pm kL_0$. To have a nontrivial solution for the coefficients A_k, B_k, the determinant of the matrix of the lhs of (2.10) must be zero. This gives the condition

$$[\beta_{\rm I} + \sin\xi\, \cos kl] + \left[(\cos\xi - \alpha_{\rm R}) + (\cos\xi + \alpha_{\rm R})\,(kL_0)^2\right] \frac{\sin kl}{2kL_0} = 0 \tag{2.11}$$

for the wave number k. The positive spectrum is an infinite discrete series, in which, for large k, the difference between subsequent levels is getting closer and approaches π/l (see Appendix A). For the negative spectrum one only needs to replace $ik \to \kappa$ in the formulas above to obtain the corresponding condition,

$$[\beta_{\rm I} + \sin\xi\, \cosh\kappa l] + \left[(\cos\xi - \alpha_{\rm R}) - (\cos\xi + \alpha_{\rm R})\,(\kappa L_0)^2\right]\, \frac{\sinh\kappa l}{2\kappa L_0} = 0\,. \qquad (2.12)$$

From (2.12) one finds that at most two negative energy states can exist. Similarly, for a possible zero energy state, the $k \to 0$ limit can be used to obtain the corresponding condition

$$[\beta_{\rm I} + \sin\xi] + [\cos\xi - \alpha_{\rm R}]\,\frac{l}{2L_0} = 0\,. \qquad (2.13)$$

At this point we note that, despite that system is free on $0 < x < l$, negative energy states may appear because the point singularity can act as an attractive potential, as in the special case of a Dirac delta potential with negative strength. A reflecting wall, which also admits potential-type realizations [19], allows maximally one negative energy state [19], while our circle system allows maximally two as in the line system [12]. We also note that, similarly to the line system, for some special U such as $U = -\sigma_1$ the ground state may be doubly degenerate. This is not in conflict with the well-known property of nondegeneracy in energy levels of one dimensional quantum mechanics, because the premises used to prove the property do not hold here. Indeed, in such a finite system neither the wave function nor its derivative at infinity are required to vanish, and even at the singularity this is not required unless our boundary condition happens to impose it.

In fact, one can determine when an energy eigenstate (ground state or higher) becomes doubly degenerate[1] as follows. Observe first that, for states with positive energy $E > 0$, degeneracy occurs when all the four elements of the matrix in (2.10) are zero. From this one derives

$$\alpha_{\rm I} = \beta_{\rm R} = 0\,, \qquad\qquad \beta_{\rm I} \neq 0\,, \qquad (2.14)$$

and, further, the conditions for the energy eigenvalue

$$\beta_{\rm I}\cos kl = -\sin\xi\,, \qquad \beta_{\rm I}\, kL_0\sin kl = -(\cos\xi - \alpha_{\rm R})\,, \qquad \beta_{\rm I}\sin kl = -(\cos\xi + \alpha_{\rm R})\, kL_0 \qquad (2.15)$$

[1] Degeneracy higher than two does not arise since the energy eigenvalue equation is a second order differential equation.

in addition to (2.11). Since (2.14) implies $\alpha_{\rm R}^2 + \beta_{\rm I}^2 = 1$, from (2.15) we find

$$k^2 L_0^2(\cos\xi + \alpha_{\rm R}) = \cos\xi - \alpha_{\rm R}\,. \tag{2.16}$$

On can obtain the conditions for states with $E = 0$ and $E < 0$ analogously, and the result is that, in both cases, one has (2.14) and

$$\xi = \operatorname{arccot}\frac{l}{2L_0} \tag{2.17}$$

together with (2.15) with $k = 0$ for $E = 0$, or (2.15) with $k \to -i\kappa$ for $E < 0$. One then finds that degeneracy of an $E \le 0$ eigenvalue excludes any other degeneracies, and that, unless $\cos\xi = -\alpha_{\rm R}$, (2.16) can hold for only one k. If $\cos\xi = -\alpha_{\rm R}$, one has $\cos\xi = \alpha_{\rm R}$ (see (2.16) and its $E \le 0$ variants) and, consequently, $\xi = \pi/2$, $\alpha_{\rm R} = 0$ and $\beta_{\rm I} = \pm 1$. This shows that in the $SU(2)$ family there are only two types of singularities specified by $U = \pm\sigma_1$ that admit double degeneracy with more then one energy levels. Actually, for the cases $U = \pm\sigma_1$ all the positive energies prove to be doublets. Further, the case $U = \sigma_1$ possesses a singlet zero energy state as the ground state while $U = -\sigma_1$ does not have any nonpositive energies. This (almost) entire degeneracy of energy levels suggests that the system may be bestowed supersymmetry, which we shall confirm later.

Now we come to the point to discuss the spectral space of the circle system with a point singularity, that is, we determine the entirety of distinct spectra that can arise on the circle under the $U(2)$ family of point interactions. From the spectral conditions (2.11)–(2.13) we can see immediately that the spectrum depends at most on the three parameters, ξ, $\alpha_{\rm R}$ and $\beta_{\rm I}$, of the four of $U \in U(2)$, even though the eigenstates depend on all of the four parameters in a nontrivial way [see (2.10)]. We have also seen that the conditions for an energy to be degenerate depend only on the same three parameters. The question is thus whether these three parameters, ξ, $\alpha_{\rm R}$ and $\beta_{\rm I}$, really index different spectra. This can be answered affirmatively by a detailed examination on the possible spectra and their connection with the set of parameters. In fact, our argument presented in Appendix A shows that the spectrum of a circle system uniquely determines the parameters ξ, $\alpha_{\rm R}$ and $\beta_{\rm I}$ and, consequently, the spectral space $\Sigma_{\rm circle} := \{\mathrm{Spec}(H_U)\,|\,U \in U(2)\}$ is given by

$$\Sigma_{\rm circle} = \left\{(\xi, \alpha_{\rm R}, \beta_{\rm I}) \in \mathbb{R}^2 \;\middle|\; \xi \in [0, \pi),\ \alpha_{\rm R}^2 + \beta_{\rm I}^2 \le 1\right\} \cong S^1 \times D^2, \tag{2.18}$$

which is topologically a filled torus. The disc D^2 part of $\Sigma_{\rm circle}$ may equally be realized by $SU(2)/U(1) \cong S^3/S^1$, where the $SU(2) \cong S^3$ given by

$$\{(\alpha_{\rm R}, \alpha_{\rm I}, \beta_{\rm R}, \beta_{\rm I}) \in \mathbb{R}^4 \mid \alpha_{\rm R}^2 + \alpha_{\rm I}^2 + \beta_{\rm R}^2 + \beta_{\rm I}^2 = 1\} \tag{2.19}$$

is factorized by the phase of $\alpha_{\rm I} + i\beta_{\rm R}$ which forms the $U(1) \cong S^1$. We shall encounter this latter identification later, when we discuss a circle with two singularities. It is interesting to compare the spectral space (2.18) with that of the line system which is two dimensional and is given by the Möbius strip with boundary, $\Sigma_{\rm line} \cong (S^1 \times S^1)/\mathbb{Z}_2$ [12].

3. Generalized symmetries, symmetries and invariant subfamilies

It has been known that, for the line system with a point singularity, (generalized) symmetries play an important role in classifying the singularities [1]. We shall see that this is also the case for the circle system here. But, first, let us recall what we mean by a (generalized) symmetry. Given a system with a singularity characterized by U, we call a unitary or antiunitary transformation $\mathcal{V}$ of the wave functions, $\psi \overset{\mathcal{V}}{\longmapsto} \tilde{\psi} = \mathcal{V}\psi$, *symmetry* if it commutes with the differential operator H and further if $\tilde{\psi}$ also satisfies the boundary condition (2.1) that ψ fulfills, *i.e.*, if it commutes with H including the domain, $[\mathcal{V}, H_U] = 0$. Conversely, given a transformation $\mathcal{V}$, one may find, among the entire family $\mathcal{F} = U(2)$ of singularities, the subfamily $\mathcal{F}_{\mathcal{V}} \subset \mathcal{F}$ which is a set of U for which $\mathcal{V}$ is a symmetry. Even if $\mathcal{V}$ which commutes with the differential operator H is not a symmetry for $U \in \mathcal{F}$, it may still induce a map $U \overset{\mathcal{V}}{\longmapsto} U_{\mathcal{V}} \in \mathcal{F}$. This motivates us to define *generalized symmetry* as transformations that (are unitary or antiunitary and commute with H and) map any $U \in \mathcal{F}$ to another, generally different $U_{\mathcal{V}} \in \mathcal{F}$. Since $\mathcal{V}$ commutes with H, the generalized symmetry assures that the two systems, one characterized by U and the other by $U_{\mathcal{V}}$, share the same spectrum.

Before investigating various symmetries and generalized symmetries arising for the circle systems, we mention a formula valid for a certain important class of transformations and becomes convenient in the subsequent discussions. Suppose that a transformation $\mathcal{W}$ of the wave functions, $\psi \overset{\mathcal{W}}{\longmapsto} \tilde{\psi} = \mathcal{W}\psi$, commutes with H and induces transformations on the boundary vectors in (2.1) as

$$\Psi \overset{\mathcal{W}}{\longmapsto} \tilde{\Psi} = M\Psi\,, \qquad\qquad \Psi' \overset{\mathcal{W}}{\longmapsto} \tilde{\Psi}' = N\Psi' \tag{3.1}$$

with some two-by-two matrices M and N. Then, in terms of $\Psi^{(+)}$ and $\Psi^{(-)} = U\Psi^{(+)}$ defined in (2.2) we have

$$\tilde{\Psi}^{(\pm)} = M\Psi \pm iL_0 N\Psi' = \frac{1}{2}\left[M(I_2+U) \pm N(I_2-U)\right]\Psi^{(+)}\,, \tag{3.2}$$

and hence

$$\tilde{\Psi}^{(-)} = \left[M(I_2+U) - N(I_2-U)\right]\left[M(I_2+U) + N(I_2-U)\right]^{-1}\tilde{\Psi}^{(+)} \tag{3.3}$$

as long as the inverse matrix in question exists. We thus see that if

$$U_{\mathcal{W}} := [M(I_2+U) - N(I_2-U)]\,[M(I_2+U) + N(I_2-U)]^{-1} \tag{3.4}$$

is unitary and hence belongs to $U(2)$, then $\mathcal{W}$ is a generalized symmetry. In particular, when $M = N \in U(2)$, which we will meet frequently below, (3.4) reduces to

$$U_{\mathcal{W}} = MUM^{-1}\,. \tag{3.5}$$

Since this $U_{\mathcal{W}}$ belongs to $U(2)$, such a $\mathcal{W}$ commuting with H is a generalized symmetry. If, in addition, U commutes with M, then one has $U_{\mathcal{W}} = U$ and hence such $\mathcal{W}$ is a symmetry.

Specializing to the circle system, the first example of symmetry transformations we wish to mention is the parity (or space reflection), $\mathcal{P}$, defined as

$$\psi(x) \overset{\mathcal{P}}{\longmapsto} (\mathcal{P}\psi)(x) = \psi(l-x)\,. \tag{3.6}$$

It clearly commutes with the Hamiltonian, and its action on the boundary vectors [see (2.1) and (2.2)] is found readily to be of the form (3.1) with $M = N = \sigma_1$ and, hence, the parity $\mathcal{P}$ is a generalized symmetry. Indeed, $U \overset{\mathcal{P}}{\longmapsto} U_{\mathcal{P}} = \sigma_1 U \sigma_1$ implies

$$\xi \overset{\mathcal{P}}{\longmapsto} \xi\,, \qquad \alpha \overset{\mathcal{P}}{\longmapsto} \alpha^*\,, \qquad \beta \overset{\mathcal{P}}{\longmapsto} -\beta^*\,, \tag{3.7}$$

and thus the spectral parameters ξ, $\alpha_{\rm R}$ and $\beta_{\rm I}$ remain the same, as required. Since $\sigma_1^2 = I_2$, the parity $\mathcal{P}$ induces *duality* in spectrum in the family $\mathcal{F}$ of singularities on a circle. Note that for systems with U satisfying $[U,\sigma_1] = 0$, the parity $\mathcal{P}$ is a symmetry, and that such a U has such parameters ξ, α, β that $\alpha_{\rm I} = 0$ and $\beta_{\rm R} = 0$. The set of those U form the parity invariant subfamily $\mathcal{F}_{\mathcal{P}}$ which, in view of (2.4), reads

$$\mathcal{F}_{\mathcal{P}} \cong S^1 \times S^1 \subset \mathcal{F}. \tag{3.8}$$

We may consider a one-parameter family ($U(1)$ group) of generalized symmetries constructed from the parity $\mathcal{P}$ used as an infinitesimal generator,

$$\mathcal{P}_\vartheta := e^{-i\frac{\vartheta}{2}\mathcal{P}} = \cos\frac{\vartheta}{2} I_2 - i\sin\frac{\vartheta}{2}\mathcal{P}\,, \qquad \vartheta \in [0, 2\pi)\,. \tag{3.9}$$

These transformations also commute with H and act on the boundary vectors as (3.1) with $M = N = e^{-i\frac{\vartheta}{2}\sigma_1}$, and are thus generalized symmetries. Their physical effect is incorporated through the transformations of the $U(2)$ parameters: ξ, $\alpha_{\rm R}$ and $\beta_{\rm I}$ are kept invariant, while a rotation is induced among $\beta_{\rm R}$ and $\alpha_{\rm I}$ as

$$\beta_{\rm R} + i\alpha_{\rm I} \overset{\mathcal{P}_\vartheta}{\longmapsto} e^{i\vartheta}(\beta_{\rm R} + i\alpha_{\rm I})\,. \tag{3.10}$$

This means that $\mathcal{P}_\vartheta$ furnishes a rotation among the spectrally identical point interaction systems in the parameter space, and that systems that are invariant under $\mathcal{P}_\vartheta$ are those with $\beta_{\rm R} = \alpha_{\rm I} = 0$, which, to no surprise, is the parity invariant subfamily $\mathcal{F}_{\mathcal{P}}$. Now the point is that, because of this $U(1)$ group of generalized symmetries within the family $\mathcal{F} = U(2)$, the spectral space is found to be the coset,

$$\Sigma_{\rm circle} = U(2)/U(1) = U(1) \times [SU(2)/U(1)], \tag{3.11}$$

which is precisely the result (2.19).

Another important discrete transformation worth mentioning is the time reflection,

$$\psi \overset{\mathcal{T}}{\longmapsto} \mathcal{T}\psi = \psi^*, \tag{3.12}$$

which leaves H invariant. It transforms the boundary vectors as

$$\Psi \overset{\mathcal{T}}{\longmapsto} \Psi^*\,, \qquad \Psi' \overset{\mathcal{T}}{\longmapsto} \Psi'^*\,, \qquad \Psi^{(\pm)} \overset{\mathcal{T}}{\longmapsto} \Psi^{(\mp)\,*}\,, \tag{3.13}$$

and, consequently, maps the characteristic matrix to its transposed, $U \overset{\mathcal{T}}{\longmapsto} U_{\mathcal{T}} = U^T \in SU(2)$. This shows that the time reflection $\mathcal{T}$ is a generalized symmetry, although it does not belong to the class mentioned in (3.1), being actually antiunitary. In terms of the parameters, we find

$$\xi \overset{\mathcal{T}}{\longmapsto} \xi\,, \qquad \alpha \overset{\mathcal{T}}{\longmapsto} \alpha\,, \qquad \beta \overset{\mathcal{T}}{\longmapsto} -\beta^*\,, \tag{3.14}$$

and hence the spectrum is preserved. Clearly, $\mathcal{T}$ is a duality and the time reversal invariant subfamily $\mathcal{F}_{\mathcal{T}}$ consists of those U with $U = U^T$, *i.e.*, with $\beta_{\rm R} = 0$, and hence

$$\mathcal{F}_{\mathcal{T}} \cong S^1 \times S^2 \subset \mathcal{F}. \tag{3.15}$$

We also mention that the two duality transformations, $\mathcal{P}$ and $\mathcal{T}$, can be combined to give the space-time reflection operator $\mathcal{PT}$. On U it acts as $U \overset{\mathcal{PT}}{\longmapsto} U_{\mathcal{PT}} = \sigma_1 U^T \sigma_1$ and hence

$$\xi \overset{\mathcal{PT}}{\longmapsto} \xi\,, \qquad \alpha \overset{\mathcal{PT}}{\longmapsto} \alpha^*\,, \qquad \beta \overset{\mathcal{PT}}{\longmapsto} \beta\,. \tag{3.16}$$

The subfamily $\mathcal{F}_{\mathcal{PT}}$ of $\mathcal{PT}$-invariant U is determined by $\alpha_{\rm I} = 0$, and hence

$$\mathcal{F}_{\mathcal{PT}} \cong S^1 \times S^2 \subset \mathcal{F}. \tag{3.17}$$

Clearly, from neither $\mathcal{F}_{\mathcal{P}}$ nor $\mathcal{F}_{\mathcal{PT}}$ one can define a one-parameter family of generalized symmetries analogous to $\mathcal{P}_\vartheta$.

We remark at this point that, in the line system, in addition to the parity $\mathcal{P}$ we have two other duality transformations, $\mathcal{Q}$ and $\mathcal{R}$, so that the three $\{\mathcal{P}, \mathcal{Q}, \mathcal{R}\}$ form an $su(2)$ algebra [1]. These $su(2)$ elements can then be used as generators to form a three-parameter ($SU(2)$ group of) family of generalized symmetries on the line [12], as we did above for the parity $\mathcal{P}$ only. This in turn implies that, generically, the set of systems sharing the same spectrum as a given system is larger (because the generalized symmetries are larger in dimension) on a line, and hence the spectral space $\Sigma_{\rm line}$ is smaller. Heuristically speaking, the nontrivial topology, the finite geometry and the corresponding extra length scale cause more variety of spectra on a circle, where $\mathcal{Q}$ and $\mathcal{R}$ are ill-defined.

4. Supersymmetry

We have encountered in sect.2 two cases in the $\mathcal{F} = U(2)$ family where all the positive energy states are doubly degenerate. These cases are characterized by $U = \pm\sigma_1$, and we examine now if these can be interpreted as supersymmetric.

One might think that this is trivial, since the Hamiltonian is the same differential operator as of the free system, and hence the supercharges,

$$Q_1 := \frac{\hbar}{2i\sqrt{m}}\frac{\mathrm{d}}{\mathrm{d}x}, \qquad Q_2 := i\mathcal{P}Q_1, \tag{4.1}$$

will clearly fulfill the algebraic relation of supersymmetry

$$\{Q_i, Q_j\} = H_U\,\delta_{ij}, \qquad i, j = 1, 2. \tag{4.2}$$

However, the point is that the algebra (4.2) should hold also in the sense of domains, not just in the differential operator relation, and it is a nontrivial question if the domains $\mathcal{D}(Q_i)$ of the supercharges Q_i for $i = 1, 2$ can be given so that they can meet this demand.

To see that this is indeed the case, we first note that, for the two cases in question, $U = \varepsilon\sigma_1$ with $\varepsilon = \pm 1$, the domains (2.5) of the Hamiltonian read

$$\mathcal{D}(H_{\varepsilon\sigma_1}) = \big\{\psi \in \mathcal{H} \,\big|\, \psi, \psi' \in \mathrm{AC}(0,l),\ \psi(l) = \varepsilon\psi(0), \psi'(l) = \varepsilon\psi'(0)\big\}. \tag{4.3}$$

Now, if we provide the domains $\mathcal{D}(Q_i)$ as

$$\mathcal{D}(Q_i) = \big\{\psi \in \mathcal{H} \,\big|\, \psi \in \mathrm{AC}(0,l), \psi(l) = \varepsilon\psi(0)\big\}, \tag{4.4}$$

we can readily confirm that Q_i are self-adjoint on these domains [20]. Moreover, by using the formulae

$$\mathcal{D}(A + B) = \mathcal{D}(A) \cap \mathcal{D}(B), \qquad \mathcal{D}(AB) = \{\psi \in \mathcal{D}(B) \,|\, B\psi \in \mathcal{D}(A)\}, \tag{4.5}$$

for the domains of the sum and the product of any two linear operators A and B, we see immediately that the domain of the lhs of (4.2) coincides with the domain (4.3). We therefore conclude that the systems $U = \pm\sigma_1$ indeed possess an $N = 2$ supersymmetry.

Note that for $U = \sigma_1$ the ground state is unique and hence the supersymmetry is unbroken (or 'good'), whereas for $U = -\sigma_1$ the ground state is doubly degenerate and supersymmetry is broken.[2] Due to the topology of the circle, the possibility of supersymmetry is limited compared to the line system where a richer variety of supersymmetric systems have been be found [22, 23], under a slightly generalized supercharges.

[2] Incidentally, we point out that here the Witten parity operator [21] is played by the parity $\mathcal{P}$.

5. More subfamilies and the WKB exactness

We have seen in sect.3 that generalized symmetries can be used to define various subfamilies, such as $\mathcal{F}_{\mathcal{T}}$, $\mathcal{F}_{\mathcal{P}}$, $\mathcal{F}_{\mathcal{PT}}$ as the set of the singularities for which the respective generalized symmetry is actually a symmetry. There are, however, some other subfamilies which are defined without using the generalized symmetries and admit salient properties in the spectrum and the WKB exactness. In this section we discuss these properties in some detail, providing a fuller account of our earlier result in Ref.[8] (from which we adopt the notations for the subfamilies).

We begin our discussion with the *separated* subfamily, $\mathcal{F}_1 \subset \mathcal{F} = U(2)$, which is the set of singularities that prohibit the probability current $j(x) = -\frac{i\hbar}{2m}\left((\psi^*)'\psi - \psi^*\psi'\right)(x)$ from flowing through the singularity, $j(l-0) = j(+0) = 0$. This condition is fulfilled by diagonal U, *i.e.,* by those with $\beta = 0$, and the boundary conditions (2.1) split into two separate ones,

$$\psi(l-0) + L_1\,\psi'(l-0) = 0\,, \qquad \psi(+0) + L_2\,\psi'(+0) = 0\,, \tag{5.1}$$

for the left and right sides of the singularity respectively, where $L_{1,2} = L_0 \cot\frac{\xi \pm \arccos\alpha_{\mathrm{R}}}{2}$. Obviously, the cutoff of physical contact at $x = 0$ allows us to regard such a system effectively an interval $(0,l)$, in other words, a box or infinite potential well system. Among this subfamily $\mathcal{F}_1$ are four special cases $(L_1, L_2) = (0,0)$, (∞,∞), $(0,\infty)$, $(\infty,0)$, in which the theory is explicitly solvable [8]. For example, the Feynman kernel is found to be

$$K(b,T;a,0) = \sqrt{\frac{m}{2\pi i\hbar T}}\sum_{n=-\infty}^{\infty}\epsilon^n\left(e^{\frac{i}{\hbar}\frac{m}{2T}\{(b-a)+2nl\}^2} \mp e^{\frac{i}{\hbar}\frac{m}{2T}\{(b+a)+2nl\}^2}\right)\,, \tag{5.2}$$

where the '$-$'-sign is for $L_1 = 0$ and the '$+$'-sign is for $L_1 = \infty$, and ϵ is 1 for $(L_1, L_2) = (0,0)$ and (∞,∞), and is -1 for the two other cases. This propagator is WKB-exact in the sense that it is a sum of free WKB amplitudes contributed by all possible classical paths that lead from $(a,0)$ to (b,T), including those that perform bouncing motion, hitting the left wall n times and the right one n or $n\pm1$ times (depending on the initial direction of the particle). Even the appearing ±1 factors allow a WKB interpretation since one can observe that any -1 factor belongs to a bouncing on a reflecting wall with $L = 0$ and the 1 ones to bouncing on a wall with $L = \infty$, in view of the fact [19] that, based on some appropriate realizing potential sequences for a reflecting wall, an $L = 0$ wall picks up a WKB factor -1, while an $L = \infty$ wall has the WKB factor 1.

The second subfamily we mention is the *scale independent* subfamily $\mathcal{F}_2$ consisting of systems for which the coefficients A, B in the eigenfunctions [cf. (2.6)] are k-independent. This happens for the characteristic matrices U with $\xi = \frac{\pi}{2}$ and $\alpha_{\rm R} = 0$ [which form a sphere $S^2 \subset \mathcal{F}$], and for $U = \pm I_2$ [two isolated points in $\mathcal{F}$] (see Appendix B). These are the cases where the boundary conditions do not mix the boundary values of ψ with values of ψ'. More explicitly, in these cases L_1 and L_2 are zero and hence the scale constant L_0 does not appear in the boundary conditions, leaving l as the only scale parameter. One may therefore expect that, in the limit $l \to \infty$, the system becomes a scale invariant 'point interaction on a line' system. Indeed, it has been known [1] that, on the line, systems belonging to the subfamily $\mathcal{F}_2$ are those which are invariant under the dilatation symmetry $(\mathcal{W}_\lambda \psi)(x) = \lambda^{\frac{1}{2}} \psi(\lambda x)$. As for $\mathcal{F}_1$, the systems belonging to $\mathcal{F}_2$ can be solved [8],[3] and the Feynman kernel can be obtained explicitly [8]. For a generic $U \in \mathcal{F}_2$, using the notations

$$C_\pm = \frac{(1+\alpha_{\rm I}) + (\beta_{\rm I} - i\beta_{\rm R})e^{\pm i\theta}}{2\sqrt{(1+\alpha_{\rm I})(1-\beta_{\rm I}^2)}}\,, \qquad \theta = \arg(\beta)\,, \tag{5.3}$$

one finds

$$K(b,T;a,0) = \sqrt{\frac{m}{2\pi i\hbar T}} \sum_{n=-\infty}^{\infty} \left\{ M_n e^{\frac{i}{\hbar}\frac{m}{2T}\{(b-a)+nl\}^2} - N_n e^{\frac{i}{\hbar}\frac{m}{2T}\{(b+a)+nl\}^2} \right\}, \tag{5.4}$$

with

$$M_n = |C_+|^2 e^{-i\theta n} + |C_-|^2 e^{i\theta n}, \qquad N_n = C_-^* C_+ e^{-i\theta n} + C_+^* C_- e^{i\theta n} \tag{5.5}$$

Unlike in the previous subfamily $\mathcal{F}_1$, however, the factors M_n, N_n do not admit a semiclassical interpretation, as one can readily confirm by using (5.5) and (5.3) together with $|\alpha|^2 + |\beta|^2 = 1$ that, *e.g.,* $|M_n| < 1$ for generic n. The situation is similar to the half line systems with a wall that have a finite L, for which the bounce factors are not phase factors [19]. Consequently, we can apply the result found there, that is, such bounce factors cannot be given a semiclassical realization. Hence, generically, the WKB exactness is not perfect in the subfamily $\mathcal{F}_2$.

[3] The intersection of the subfamilies $\mathcal{F}_1$ and $\mathcal{F}_2$ consists of the two special cases $U = \pm\sigma_3$, which are the box systems $(L_1, L_2) = (0, \infty)$, $(\infty, 0)$. Two other important special cases in $\mathcal{F}_2$ are $U = \pm\sigma_1$, which have already been discussed in sect.3. Note that the energy eigenfunctions of $U = \sigma_1$ provide just the basis $\{\cos nx, \sin nx\}$ that is used in the classic Fourier expansion.

However, there is a subfamily within the family $\mathcal{F}_2$ where the WKB exactness holds perfectly. It is the *smooth* subfamily $\mathcal{F}_3$, containing the cases in $\mathcal{F}_2$ with $\alpha_{\rm I} = 0$. $\mathcal{F}_3$ is a one-parameter U(1) subfamily, parametrized solely by the θ of above. The boundary conditions here read

$$\psi(0) = e^{i\theta}\psi(l)\,, \qquad \psi'(0) = e^{i\theta}\psi'(l)\,, \tag{5.6}$$

which are nothing but the boundary conditions for the smooth circle [24], *i.e.,* for the circle with no singularity. As mentioned in sect.2, the phase parameter θ is regarded as the flux of a magnetic field penetrating through the circle. In this subfamily, the propagator (5.4) simplifies to the well-known kernel of the smooth circle

$$K(b,T;a,0) = \sqrt{\frac{m}{2\pi i\hbar T}} \sum_{n=-\infty}^{\infty} e^{i\theta n} e^{\frac{i}{\hbar}\frac{m}{2T}[(b-a)+nl]^2}\,, \tag{5.7}$$

which is readily seen to be WKB exact — the nth term belongs to a classical path on which the particle takes n turns before reaching the point b, without acquiring any additional action contribution each time when it crosses the point $x = l \equiv 0$.

Another subfamily worth mentioning is the *isospectral* subfamily $\mathcal{F}_4$, comprising those U with $\xi = 0$ and $\beta_{\rm I} = 0$. These systems are peculiar in that they possess the same positive energy spectrum, $k = n\pi/l \quad (n = 1, 2, \ldots)$, independently of U, although the possible zero or negative energy is U dependent. This subfamily admits a generalization to the *semi-isospectral* subfamily $\mathcal{F}_5$, characterized by the condition $\sin\xi = \pm\beta_{\rm I}$, where the positive spectrum consists of two infinite sequences, one that is equidistant and U-independent and another one that is U-dependent and given by transcendental roots of (2.11).

6. A circle with two singularities

The spectral properties and the (generalized) symmetries of a circle system with a singularity may also be studied by considering it as a special case of a system with two singularities. We shall see in this section that the latter, extended system has a graded structure which offers a convenient framework to discuss the spectrum preserving generalized symmetries on a direct basis, which is also useful to study the spectral properties for the system with one singularity.

Let us place an extra singularity at $x = l/2$ on the circle, and thereby consider the spectral properties of the modified system. This modification allows us to regard the system as a pair of two subsystems, one given by the interval $[0, l/2)$ and the other by $[l/2, l)$ connected appropriately at the ends. Given a state $\psi(x)$ on the circle, we define the corresponding state $\Phi(x)$ by

$$\Phi(x) = \begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix} := \begin{pmatrix} \psi(x) \\ \psi(l-x) \end{pmatrix}, \qquad 0 \le x < l/2\,, \tag{6.1}$$

on which our Hamiltonian acts by

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} \otimes I_2\,. \tag{6.2}$$

Now, let U_1, $U_2 \in U(2)$ be the two characteristic matrices specifying the two singularities at $x = 0$ and $x = l/2$, respectively. Then the connections conditions read

$$\begin{aligned} (U_1 - I_2)\Phi(0) + iL_0(U_1 + I_2)\Phi'(0) = 0, \\ (U_2 - I_2)\Phi(l/2) + iL_0(U_2 + I_2)\Phi'(l/2) = 0. \end{aligned} \tag{6.3}$$

Note that the system is characterized by the pair (U_1, U_2), and hence there exists an $\mathcal{F}(U_1, U_2) = U(2) \times U(2)$ family of systems if we allow both elements of the pair to vary in the group $U(2)$. If we freeze one of them, for instance, by putting $U_2 = \sigma_1$ to realize the free connection condition at $x = l/2$, then we recover the original family $\mathcal{F}(U_1, \sigma_1) = U(2)$ of systems with one singularity. An important point to be noted is that, if Φ obeys (6.3), then the new state

$$\tilde{\Phi} := V\Phi, \qquad \text{for} \quad V \in SU(2), \tag{6.4}$$

obeys the connection conditions (6.3) with U_1, U_2 replaced by $\tilde{U}_1$, $\tilde{U}_2$ which are given by the conjugation,

$$\tilde{U}_1 = VU_1V^{-1}, \qquad \tilde{U}_2 = VU_2V^{-1}. \tag{6.5}$$

Observe that for the Hamiltonian (6.2) we have $[V,\, H] = 0$ with $V \in SU(2)$ being regarded as a multiplication operator. This implies that, if Φ [obeying (6.3)] is an eigenstate of H with energy E, so is $\tilde{\Phi} = V\Phi$, that is,

$$\text{if} \quad H\Phi = E\Phi, \qquad \text{then} \quad H\tilde{\Phi} = E\tilde{\Phi} \quad \text{for} \quad \tilde{\Phi} = V\Phi. \tag{6.6}$$

This shows that the transformation $\Phi \mapsto \tilde{\Phi}$ is a generalized symmetry and that there exists an isospectral family of systems which are related by (6.5).

Now we specialize to the case with one singularity at $x = 0$ by choosing $U_2 = \sigma_1$. The isospectral family is then obtained by collecting those pairs $(\tilde{U}_1, \tilde{U}_2)$ in (6.5) for which $\tilde{U}_2 = U_2 = \sigma_1$. Obviously, the solution is given by $V = e^{i\alpha\sigma_1}$, whose totality provides the isospectral group $U(1)$. Namely, if we let $\mathcal{I}(U_1, U_2)$ be the isospectral group for the system specified by the pair (U_1, U_2), then we have $\mathcal{I}(U_1, \sigma_1) = U(1)$. Accordingly, the spectral space for the circle system with two singularities,

$$\Sigma(U_1, U_2) := \mathcal{F}(U_1, U_2)/\mathcal{I}(U_1, U_2) \tag{6.7}$$

is given by

$$\Sigma(U_1, \sigma_1) = U(2)/U(1), \tag{6.8}$$

which is in fact the spectral space Σ_{circle} (3.11).

Another special case arises when the second singularity does not allow the probability current flow through it and is also parity invariant. This system is actually an interval of length l with a singularity at $x = 0$ with walls having an identical boundary condition for the right and the left sides, and this arises when U_2 belongs to the *self-dual* [25, 1] subfamily $\mathcal{F}_{\text{SD}} \subset \mathcal{F}_{\mathcal{P}}$, *i.e.,* taking the form $U_2 = e^{i\theta} I_2\,, \quad \theta \in [0, 2\pi)\,.$ In this case, the isospectral family given by those pairs of $(\tilde{U}_1, \tilde{U}_2)$ for which $\tilde{U}_2 = U_2 = e^{i\theta} I_2$ is just the entire $SU(2)$, that is, $\mathcal{I}(U_1, e^{i\theta} I_2) = SU(2)$. Thus the spectral space is found to be

$$\Sigma(U_1, e^{i\theta} I_2) = U(2)/SU(2) \cong S^1 \times S^1/\mathbb{Z}_2, \tag{6.9}$$

which is just $\Sigma_{\rm line}$ we know of [12].

The spectral space of a system with a generic (but fixed) U_2 is also studied analogously. For this, one just decomposes U_1 and U_2 as

$$U_i = V_i^{-1} D_i V_i, \qquad V_i \in SU(2), \qquad D_i = \begin{pmatrix} e^{i\theta_i^+} & 0 \\ 0 & e^{i\theta_i^-} \end{pmatrix}, \qquad i = 1, 2. \tag{6.10}$$

Then, one finds from the isospectral conjugation that its spectral space is identical to that of the diagonal $U_2 = D_2$ (since it is obtained by choosing $V = V_2$ in (6.6)). For $U_2 = D_2$, the isospectral group is determined by those $V \in SU(2)$ for which $V D_2 V^{-1} = D_2$. If $\theta_2^+ = \theta_2^-$, such D_2 belongs to the self-dual subfamily and the isospectral group $\mathcal{I}(U_1, D_2)$ is given by $SU(2)$ as mentioned above. If $\theta_2^+ \neq \theta_2^-$, the isospectral group $\mathcal{I}(U_1, D_2)$ is the $U(1)$ consisting of the elements $V = e^{i\rho\sigma_3}$. Clearly, the generic $U_2 = V_2^{-1} D_2 V_2$ has the isospectral group $\mathcal{I}(U_1, U_2)$ given by the $U(1)$ consisting of the elements $V = V_2^{-1} e^{i\rho\sigma_3} V_2 = e^{i\rho V_2^{-1}\sigma_3 V_2}$. Thus, for generic $U_2 = V_2^{-1} D_2 V_2$ where V_2 and D_2 are fixed with $\theta_2^+ \neq \theta_2^-$, we find $\mathcal{I}(U_1, V_2^{-1} D_2 V_2) = U(1)$ and

$$\Sigma(U_1, V_2^{-1} D_2 V_2) = U(2)/U(1). \tag{6.11}$$

The foregoing discussions allude us to ask the total spectral space that arises if we are allowed to vary both U_1 and U_2, that is, in circle systems with the total family $\mathcal{F}(U_1, U_2) = U(2) \times U(2)$. Due to the isospectral transformation (6.6), we have the isospectral group, $\mathcal{I}(U_1, U_2) = SU(2)$, and hence the total spectral space is expected to be

$$\Sigma(U_1, U_2) = [U(2) \times U(2)]/SU(2). \tag{6.12}$$

We need, of course, to supplement our argument to prove this by showing that each element of the spectral space leads to a distinct spectrum, as has been done in sect.3 (with Appendix A) for the case of one singularity.

We close our discussion by mentioning briefly the generalizations of the connection formulas to the present case of two singularities. For this we introduce the combined boundary vectors

$$\Psi := \begin{pmatrix} \Phi(0) \\ \Phi(l/2) \end{pmatrix}, \qquad \Psi' := \begin{pmatrix} \Phi'(0) \\ \Phi'(l/2) \end{pmatrix}, \tag{6.13}$$

and the combined characteristic matrix

$$U := \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}, \tag{6.14}$$

and thereby rewrite the connection conditions (6.3) into an analogous form of (2.1) as

$$(U - I_4)\Psi + iL_0(U + I_4)\Psi' = 0, \tag{6.15}$$

where I_4 is the four-by-four unit matrix. For the energy eigenvalue problem, the positive energy eigenfunctions are of the form

$$\Phi(x) = \begin{pmatrix} A_k\, e^{ikx} + B_k\, e^{-ikx} \\ C_k\, e^{ikx} + D_k\, e^{-ikx} \end{pmatrix}, \tag{6.16}$$

with boundary vectors

$$\Psi = T_k \begin{pmatrix} A_k \\ B_k \\ C_k \\ D_k \end{pmatrix}, \qquad \Psi' = T_k\, \Sigma_3 \begin{pmatrix} A_k \\ B_k \\ C_k \\ D_k \end{pmatrix}, \tag{6.17}$$

where

$$T_k = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ e^{ikl/2} & e^{-ikl/2} & 0 & 0 \\ 0 & 0 & e^{ikl/2} & e^{-ikl/2} \end{pmatrix}, \qquad \Sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{6.18}$$

The boundary conditions therefore read

$$\left[(U - I_4)\, T_k - kL_0(U + I_4)\, T_k\, \Sigma_3\right] \begin{pmatrix} A_k \\ B_k \\ C_k \\ D_k \end{pmatrix} = 0\,. \tag{6.19}$$

These formulas are the generalizations of (2.7)–(2.9). The slight difference that only $T_k \Sigma_3$ is needed instead of $\Sigma_3 T_k \Sigma_3$ is a consequence of the convenient definition of the second component of Φ [see (6.1)]. The nonpositive energy eigenfunctions can be obtained from the positive energy ones the same way as in sect.3.

The special cases we discussed above may then be explicitly examined by choosing the matrix U appropriately. In particular, for the spectral properties, one can use the isospectral transformation (6.6) to specify U further and thereby simplify the discussion considerably [22, 26]. For instance, for the generic case, (6.6) can be used to diagonalize either U_1 or U_2 in determining the spectrum. This unified matrix form of the boundary conditions furnishes a useful tool for more than two singularities and, generally, for any larger set of boundary conditions, too.

7. Conclusions

We have studied in this paper the spectral properties of a quantum particle on a circle with a pointlike singularity and found that, when considering the whole four-parameter family of possible point singularities, the space of distinct spectra forms a three-dimensional set given by $\Sigma_{\text{circle}} \cong S^1 \times D^2$. This result is contrasted to the case of point singularities on the line, where the spectral space is two dimensional and given by $\Sigma_{\text{line}} \cong (S^1 \times S^1)/\mathbb{Z}_2$. By analyzing the possible generalized symmetries of the circle systems, we have found that the difference between the two spectral spaces can be attributed to the difference in the symmetry structure — the structure of the line systems is somewhat richer than that of the circle system.

Further, we have determined when the circle systems possess degenerate energy levels, and proved that the cases where all the positive energy levels are doubly degenerate actually exhibit an $N = 2$ supersymmetry. The class of supercharges considered is a standard one to realize the supersymmetry algebra, and we have shown that the algebra holds including the domains for the operators involved. The question of domain in the supersymmetry algebra is a nontrivial mathematical problem and has been answered only partially for simple systems such as lines/intervals [22, 26]. On the physical aspects, on the other hand, the dependence of the degeneracy of an energy eigenstate on the parameters characterizing the point singularity opens the possibility that, if a quantum nanodevice realizing a point interaction on a circle can be fabricated with tunable parameters, then experiments demonstrating coherent control of quantum states will be possible to carry out (experiments like, *e.g.,* the ones reported in [27]).

We have also extended earlier results on the connection between symmetries and special subfamilies of the four-parameter full family of point singularities, where the subfamilies are defined with respect to the symmetries such as parity and time reflection. In a few special subfamilies, *i.e.*, separated subfamily or smooth subfamily, there are cases where the WKB exactness is observed in the transition amplitude. This implies that in those cases the amplitude admits the interpretation that it is a sum of contributions of free propagations plus a certain bouncing/penetration effect occurring at the singularity.

Finally, we have put our discussion on the generalized symmetries with one point singularity in the context of two singularities on a circle. There, we have seen that both of the spectral spaces, Σ_{circle} and Σ_{line}, appear as special cases, showing that this provides a scheme to discuss the spectral properties of these systems on a general basis. Obviously, this generalization should be useful to incorporate further singularities allowing for their classification by symmetries. Increasing the number of singularities is not just a matter of mathematical extension, because repeated structures such as lattices do appear in physics even in one dimension, and we hope that our methods and results presented in this paper can be extended to those systems which undoubtedly have richer and more interesting properties.

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Appendix A. The spectral space Σ_{circle}

To prove that a spectrum uniquely determines ξ, α_{R} and β_{I}, first we study the dependence of the spectrum on these parameters. When doing this, it is useful to recall that $\xi \in [0, \pi)$, and $\alpha_{\mathrm{R}}^2 + \beta_{\mathrm{I}}^2 \leq 1$.

To start with, if $\xi = \beta_{\mathrm{I}} = 0$ (let us call this case 'case I') then the positive energies satisfy

$$[(1 - \alpha_{\mathrm{R}}) + (1 + \alpha_{\mathrm{R}})(kL_0)^2] \sin kl = 0 . \tag{A.1}$$

Since $|\alpha_{\mathrm{R}}| \leq 1$, $1 - \alpha_{\mathrm{R}}$ and $1 + \alpha_{\mathrm{R}}$ are non-negative, and only one of them can be zero, therefore $[(1 - \alpha_{\mathrm{R}}) + (1 + \alpha_{\mathrm{R}})(kL_0)^2]$ is always positive. Consequently, the positive energies fulfil $\sin kl = 0$, with the solutions $k_n = \frac{\pi}{l} n$, $n = 1, 2, \ldots$

If at least one of ξ and β_{I} is nonzero then let us first consider the subcase $\alpha_{\mathrm{R}} = -\cos\xi$ (called 'case II'). Here, $\sin\xi \neq 0$, since $\sin\xi = 0 \Rightarrow \xi = 0 \Rightarrow \alpha_{\mathrm{R}} = -1 \Rightarrow \beta_{\mathrm{I}} = 0$, and this case is now excluded. Thus we can write (2.11) in the form

$$\frac{\beta_{\mathrm{I}}}{\sin\xi} + \cos kl + \frac{\cot\xi}{kL_0} \sin kl = 0 . \tag{A.2}$$

On the left hand side, if $k \to \infty$ then the third term tends to zero so $\frac{\beta_{\mathrm{I}}}{\sin\xi} + \cos kl \to 0$, $\cos kl \to -\frac{\beta_{\mathrm{I}}}{\sin\xi}$. This means that the large roots k will get closer and closer to the values of the form $\frac{1}{l}(\arccos \frac{-\beta_{\mathrm{I}}}{\sin\xi} + 2\pi n)$, respectively, $\frac{1}{l}(-\arccos \frac{-\beta_{\mathrm{I}}}{\sin\xi} + 2\pi n)$, in an alternating sequence.

In the remaining case — *i.e.,* when $\alpha_{\mathrm{R}} \neq -\cos\xi$ and at least one of ξ and β_{I} is nonzero 'case III') — (2.11) can be written as

$$\frac{a_1}{kl} + \frac{a_2}{kl} \cos kl + \left[\frac{a_3}{(kl)^2} + 1 \right] \sin kl = 0 , \tag{A.3}$$

with

$$a_1 = \frac{2\beta_{\mathrm{I}}}{\cos\xi + \alpha_{\mathrm{R}}} \frac{l}{L_0} , \qquad a_2 = \frac{2\sin\xi}{\cos\xi + \alpha_{\mathrm{R}}} \frac{l}{L_0} , \qquad a_3 = \frac{\cos\xi - \alpha_{\mathrm{R}}}{\cos\xi + \alpha_{\mathrm{R}}} \left(\frac{l}{L_0} \right)^2 . \tag{A.4}$$

If $k \to \infty$ then the terms proportional to a_1, a_2 and a_3 tend to zero so $\sin kl \to 0$. Now the large roots k are getting closer and closer to the values of the form $\frac{\pi}{l} n$, $n \in \mathbb{Z}$.

Now we will determine more details about the asymptotic behaviour of the roots k. For this reason, we make the following Ansatz:

$$k_n l = \pi n + \varepsilon_n \,, \qquad \text{where} \qquad \varepsilon_n = \frac{c_1}{n} + \frac{c_2}{n^2} + \frac{c_3}{n^3} + \cdots \,. \tag{A.5}$$

The coefficients c_i can be determined iteratively from (A.3). For our purposes the first three coefficients will be the interesting ones. To be up to this order, it is enough to use the formulae

$$\cos k_n l = \cos(\pi n)\cos\varepsilon_n = (-1)^n[1 - \frac{1}{2}\varepsilon_n^2] + \mathcal{O}\left(\varepsilon_n^4\right) \,, \tag{A.6}$$

$$\sin k_n l = \cos(\pi n)\sin\varepsilon_n = (-1)^n[\varepsilon_n - \frac{1}{6}\varepsilon_n^3] + \mathcal{O}\left(\varepsilon_n^5\right) \,, \tag{A.7}$$

$$(k_n l)^2 = \pi^2 n^2 + 2\pi c_1 + \mathcal{O}\left(\frac{1}{n}\right) \,, \tag{A.8}$$

$$\varepsilon_n^2 = \frac{c_1^2}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right) \qquad\qquad \varepsilon_n^3 = \frac{c_1^3}{n^3} + \mathcal{O}\left(\frac{1}{n^4}\right) \,. \tag{A.9}$$

Inserting these into the condition (A.3) multiplied, for convenience, by $(kl)^2$, and grouping the terms as decreasing powers of n, the vanishing of the coefficients of n^1, n^0 and n^{-1} lead to

$$c_1 = -\frac{1}{\pi}[(-1)^n a_1 + a_2] \,, \qquad c_2 = 0 \,, \qquad c_3 = \frac{-c_1}{\pi^2}a_3 + \frac{c_1^2}{6\pi}(c_1 + 3a_2 - 6) \,, \tag{A.10}$$

respectively. We can see that there is a sequence $c_1^{(+)}, c_2^{(+)}, c_3^{(+)}, \ldots$ for even ns, and another sequence $c_1^{(-)}, c_2^{(-)}, c_3^{(-)}, \ldots$ for odd ns. Note that at least one of a_1 and a_2 is nonzero, because $\xi = \beta_{\mathrm{I}} = 0$ is now excluded. Therefore, at least one of $c_1^{(+)}$ and $c_1^{(-)}$ is nonzero. Thus in case III the roots do not *exactly* fulfil $\sin kl = 0$, they are only getting closer and closer to it, $\sin kl$ only *tends to* zero.

In the possession of this collected knowledge, we can turn to the inverse problem we wish to solve, *i.e.,* to identify the parameters ξ, α_{R} and β_{I} from a given spectrum.

If all the positive energies satisfy $\sin kl = 0$ *exactly* then we can know that we are in case I. This determines ξ and β_{I} (namely, $\xi = \beta_{\mathrm{I}} = 0$) but α_{R} is yet unknown. Let us see whether the possible zero and negative energies determine α_{R}. The condition for a zero energy state (2.13) reads in this case simply $\alpha_{\mathrm{R}} = 1$. Therefore, if the spectrum contains a zero energy state then $\alpha_{\mathrm{R}} = 1$. If not, then let us see the possibility for negative energies: (2.12) is now

$$[(1 - \alpha_{\mathrm{R}}) - (1 + \alpha_{\mathrm{R}})(\kappa L_0)^2]\sinh \kappa l = 0 \,, \tag{A.11}$$

which gives that there exists one negative energy state with

$$\kappa = \frac{1}{L_0}\sqrt{\frac{1-\alpha_{\rm R}}{1+\alpha_{\rm R}}} \tag{A.12}$$

if $\alpha_{\rm R} \neq -1$ and no negative energy state if $\alpha_{\rm R} = -1$. Consequently, from the absence of negative energy states we learn $\alpha_{\rm R} = -1$, and from one negative energy state with κ we can identify $\alpha_{\rm R}$ as

$$\alpha_{\rm R} = \frac{1-(\kappa L_0)^2}{1+(\kappa L_0)^2}\,. \tag{A.13}$$

If we see that $\cos kl$ tends to a definite value as k increases then we know that we face at case II [since in case I $\cos kl$ oscillates between 1 and -1, and in case III $\cos(k_{n=2j}l) \to 1$ and $\cos(k_{n=2j+1}l) \to -1$]. From $\lim(\cos kl)$ we obtain $\beta_{\rm I}/\sin\xi$, and then, from (A.2), using any root k from the known spectrum for which $\sin kl \neq 0$, we determine $\cot\xi$, which uniquely tells ξ.

In the end, if we find that the positive spectrum is such that the values of $\sin kl$ tend to zero but are not exactly zero then we know we are in case III. For large enough ks we can determine which integer n belongs to a k (by rounding kl/π to the nearest integer). Then, we can identify the coefficients $c_1^{(+)}$ and $c_3^{(+)}$ as

$$c_1^{(+)} = \lim_{\substack{n\to\infty \\ n\,\text{even}}} n(k_n l - \pi n)\,, \qquad c_3^{(+)} = \lim_{\substack{n\to\infty, \\ n\,\text{even}}} n^3\left[k_n l - \pi n - \frac{c_1^{(+)}}{n}\right], \tag{A.14}$$

and $c_1^{(-)}$ and $c_3^{(-)}$ in a similar way. From $c_1^{(+)}$ and $c_1^{(-)}$ we can obtain a_1 and a_2 [cf. (A.10)] as

$$a_1 = -\frac{\pi}{2}\left[c_1^{(+)} - c_1^{(-)}\right], \qquad a_2 = -\frac{\pi}{2}\left[c_1^{(+)} + c_1^{(-)}\right], \tag{A.15}$$

and then, corresponding to that which of $c_1^{(+)}$ and $c_1^{(-)}$ is nonzero — we know that at least one of them is nonzero —, a_3 can be determined from $c_3^{(+)}$ or $c_3^{(-)}$, respectively [cf. (A.10)].

From a_1, a_2 and a_3 the parameters ξ, $\alpha_{\rm R}$ and $\beta_{\rm I}$ are calculated as follows [all steps will be based on (A.4)]. If $a_3 = -(l/L_0)^2$ then $\cos\xi = 0 \quad\Rightarrow\quad \xi = \pi/2$, and $\alpha_{\rm R} = 2/[(L_0/l)a_2]$ and $\beta_{\rm I} = a_1/a_2$. If $a_3 \neq -(l/L_0)^2$ then, observing that

$$\frac{(L_0/l)\,a_2}{1+(L_0/l)^2 a_3} = \tan\xi, \tag{A.16}$$

ξ is determined uniquely. Then, we have

$$\alpha_{\mathrm{R}} = \frac{1-(L_0/l)^2 a_3}{1+(L_0/l)^2 a_3}\cos\xi\,, \qquad\qquad \beta_{\mathrm{I}} = \frac{a_1}{a_2}\sin\xi\,. \tag{A.17}$$

We can summarize the above considerations with that the spectrum of a circle system uniquely determines its parameters ξ, α_{R} and β_{I}.

Appendix B. The scale independent boundary conditions

On dimensional grounds, the coefficients A, B in the eigenfunctions (2.6) will be k-independent if L_0 actually drops out from the boundary conditions expressed by (2.1). This happens if both lines of the matrix equation (2.1) — or, two appropriate linear combinations of them — contain only one of the two boundary value vectors Ψ, Ψ'.

First, suppose that neither of the rows of the matrices $U-I_2$ and $U+I_2$ are identically zero. Then an appropriate linear combination of the two lines of (2.1) is needed to drop Ψ out from, say, the first line. In this case, any other linear combination will leave some Ψ in the second line so the goal of another linear combination will be to drop Ψ' out from the second line. This is possible only if both matrices $U-I_2$ and $U+I_2$ are such that their first row is a multiple of their second row. Then we have

$$\det(U-I_2) = \det(U+I_2) = 0\,, \tag{B.1}$$

which tells that the two eigenvalues of U are ± 1. Therefore, $U = P_+ - P_-$, where P_+ is the projector projecting onto the eigensubspace of U corresponding to the eigenvalue 1, and P_- projecting onto the other eigensubspace. From this we see that U is self-adjoint, and this property leads to the requirements $\xi = \frac{\pi}{2}$, $\alpha_{\mathrm{R}} = 0$.

Second, if some of the rows of $U-I_2$ and $U+I_2$ are identically zero then first we observe that this can happen to at most two of the four rows in question: Otherwise at least one of the matrices $U-I_2$, $U+I_2$ would be zero, but then the other one should be $\pm 2I_2$, which has only nonzero rows. Now, if two rows of the four are zero then it is easy to see that one of these rows must be an upper row and the other a lower row (the difference of $U+I_2$ and $U-I_2$ is $2I_2$, which makes the other cases impossible). This means four possibilities, the matrices $U = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$, which give the two isolated scale independent systems $U = \pm I_2$ (the two other cases, $U = \pm\sigma_3$, are included in the Us with $\xi = \frac{\pi}{2}$, $\alpha_{\mathrm{R}} = 0$). At last, if one of the four rows is zero — say, a row of $U - I_2$ —, then one of the two lines of (2.1) is already Ψ independent. The other line will then necessarily contain Ψ so, to make it Ψ' independent, a suitable multiple of the Ψ independent line has to be added to it. This means that the two rows of $U + I_2$ has to be each other's multiple, consequently, $\det(U + I_2) = 0$. However, $U - I_2$ has a zero row so $\det(U - I_2) = 0$, too. Thus we arrive again back to (B.1), and hence to $\xi = \frac{\pi}{2}$, $\alpha_{\mathrm{R}} = 0$.

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