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Duality Symmetry and Plane Waves in Non-Commutative Electrodynamics

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Abstract. We generalise the electric-magnetic duality in standard Maxwell theory to its non-commutative version. Both space-space and space-time non-commutativity are necessary. The duality symmetry is then extended to a general class of non-commutative gauge theories that goes beyond non-commutative electrodynamics. As an application of this symmetry, plane wave solutions are analysed. Dispersion relations following from these solutions show that, in the presence of space-time non-commutativity, non-commutative electrodynamics admits two waves with distinct polarisations propagating at different velocities in the same direction.

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1. Introduction

The old idea [1] that spatial coordinates need not commute has recently undergone a revival due to its appearance in the context of string theory [2]. This has also led to the study of gauge field theories defined on non-commutative spaces; these are referred as non-commutative gauge theories. Some points of difference, leading to a modification or generalisation of the results of conventional gauge theories have been noted. In particular, in the non-commutative Maxwell theory, modified dispersion relations [3, 4], violation of the superposition principle [5], non-commuting electric fields [6], *etc.*, have been reported.

The objective of this paper is to generalise the well known electric-magnetic duality in Maxwell's theory to its non-commutative counterpart. The theory is defined in a non-commutative space-time x^μ satisfying the algebra $[x^\mu, x^\nu] = i\theta^{\mu\nu}$, where $\theta^{\mu\nu}$ is a real skew 2-index object. To make a transparent connection of our new found duality symmetry with the conventional one, it is useful to express the non-commutative Maxwell theory in terms of its commutative equivalent by using the Seiberg-Witten (SW) map [2]. Although we perform the analysis up to the first order in the non-commutative parameter θ, our results are expected to hold under more general conditions. In fact, the duality symmetry persists (at least to the first order in θ) for an arbitrary Lagrangian whose structure is dictated by symmetry arguments and not restricted by the SW map. Also, we show how the discrete duality symmetry is realised as a continuous $SO(2)$ symmetry, exactly as happens in the usual Maxwell theory.

Keeping both space-space θ^{ij} and space-time θ^{0i} non-commutativity is essential for our analysis. Indeed, when we use the duality symmetry to construct new solutions, say by taking the plane wave solutions with nonvanishing θ^{ij} but vanishing θ^{0i} [3], these are converted to other plane wave solutions, now with nonvanishing θ^{0i} and vanishing θ^{ij}. This is also consistent with the known results, discussed both in the Lagrangian [7] and Hamiltonian formulations [8], that if the original theory is non-commutative Maxwell theory with only spatial non-commutativity, the dual theory is a non-commutative gauge theory with just space-time non-commutativity.

Our analysis of course goes beyond since we give the plane wave solutions for non-commutativity involving both space and time. Modified dispersion relations are obtained, which generalise previous findings [3] given only for space-space non-commutativity. In particular, we find that if space-time non-commutativity is allowed, there are two waves with distinct polarisations that propagate at different velocities.

2. Duality in the non-commutative Maxwell theory

The non-commutative generalisation of usual Maxwell Lagrange density involves the star product of the non-commutative field strength $\widehat{F}_{\mu\nu}$ obtained from the potential $\widehat{A}_\mu$ as

$$\widehat{F}^{\mu\nu} = \partial^\mu \widehat{A}^\nu - \partial^\nu \widehat{A}^\mu - ig(\widehat{A}^\mu \star \widehat{A}^\nu - \widehat{A}_\nu \star \widehat{A}_\mu), \tag{2.1}$$

where $g = \frac{e}{\hbar c}$ and the star product is defined by $(f \star g)(x) = e^{\frac{i}{2}\theta^{\alpha\beta}\partial_\alpha\partial'_\beta} f(x)g(x')\big|_{x=x'}$. The non-commutative version of the free Maxwell Lagrange density is then given by

$$\mathcal{L} = -\frac{1}{4}\widehat{F}_{\mu\nu} \star \widehat{F}^{\mu\nu}. \tag{2.2}$$

This Lagrange density may be expressed in terms of the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ defined from the ordinary vector potential A_μ by exploiting the SW map,[1]

$$\begin{aligned} \widehat{A}_\mu &= A_\mu - \frac{1}{2}\theta^{\alpha\beta} A_\alpha(\partial_\beta A_\mu + F_{\beta\mu}), \\ \widehat{F}_{\mu\nu} &= F_{\mu\nu} + \theta^{\alpha\beta} F_{\alpha\nu} F_{\mu\beta} - \theta^{\alpha\beta} A_\alpha \partial_\beta F_{\mu\nu}, \end{aligned} \tag{2.3}$$

where we have absorbed g into $\theta^{\alpha\beta}$ by scaling. The outcome is

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2, \tag{2.4}$$

where $\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is the ordinary Maxwell Lagrangian and

$$\mathcal{L}_1 = \frac{1}{8}\theta^{\alpha\beta} F_{\alpha\beta} F_{\mu\nu} F^{\mu\nu}, \qquad \mathcal{L}_2 = -\frac{1}{2}\theta^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} F^{\mu\nu}, \tag{2.5}$$

are the additional terms manifesting the effects of non-commutativity. This was worked out in [9] and used for a detailed Lagrangian [3] as well as Hamiltonian [6] analysis. Using the electric field $\boldsymbol{E}$ and the magnetic induction field $\boldsymbol{B}$ defined by

$$E^i = -F^{0i}, \qquad B^i = -\frac{1}{2}\epsilon^{ijk} F_{jk}, \tag{2.6}$$

and introducing

$$\varepsilon^i = \theta^{0i}, \qquad \theta^i = \frac{1}{2}\epsilon^{ijk}\theta_{jk}, \tag{2.7}$$

we can rewrite the Lagrangian (2.4) as

$$\mathcal{L} = \frac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{B}^2) - \frac{1}{2}(\boldsymbol{\theta}\cdot\boldsymbol{B} - \boldsymbol{\varepsilon}\cdot\boldsymbol{E})(\boldsymbol{E}^2 - \boldsymbol{B}^2) + (\boldsymbol{\theta}\cdot\boldsymbol{E} + \boldsymbol{\varepsilon}\cdot\boldsymbol{B})(\boldsymbol{E}\cdot\boldsymbol{B}). \tag{2.8}$$

[1] In this paper we adopt the convention that when the non-commutative parameter $\theta^{\mu\nu}$ is involved we keep terms only up to first order in the parameter.

If we further define the electric displacement field $\boldsymbol{D}$ and the magnetic field $\boldsymbol{H}$ by

$$D^i = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_i)}, \qquad H^i = \frac{1}{2}\epsilon^{ijk}\frac{\partial \mathcal{L}}{\partial(\partial_j A_k)}, \tag{2.9}$$

which read

$$\begin{aligned}
\boldsymbol{D} &= \boldsymbol{E} - (\boldsymbol{\theta}\cdot\boldsymbol{B} - \boldsymbol{\varepsilon}\cdot\boldsymbol{E})\boldsymbol{E} + (\boldsymbol{\theta}\cdot\boldsymbol{E} + \boldsymbol{\varepsilon}\cdot\boldsymbol{B})\boldsymbol{B} + (\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\theta} + \frac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{B}^2)\boldsymbol{\varepsilon}, \\
\boldsymbol{H} &= \boldsymbol{B} - (\boldsymbol{\theta}\cdot\boldsymbol{B} - \boldsymbol{\varepsilon}\cdot\boldsymbol{E})\boldsymbol{B} - (\boldsymbol{\theta}\cdot\boldsymbol{E} + \boldsymbol{\varepsilon}\cdot\boldsymbol{B})\boldsymbol{E} - (\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\varepsilon} + \frac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{B}^2)\boldsymbol{\theta},
\end{aligned} \tag{2.10}$$

then the field equations take the Maxwell form:

$$\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{B} + \nabla\times\boldsymbol{E} = 0, \qquad \nabla\cdot\boldsymbol{B} = 0, \tag{2.11}$$

and

$$\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{D} - \nabla\times\boldsymbol{H} = 0, \qquad \nabla\cdot\boldsymbol{D} = 0. \tag{2.12}$$

Note that (2.11) are just the Bianchi identities while (2.12) are the Lagrange equations derived from (2.4).

Because of the presence of both the $\boldsymbol{\varepsilon}$ parameter and $\boldsymbol{\theta}$, we find that $\boldsymbol{D}$ and $\boldsymbol{H}$ in (2.10) share almost the same structure, and this observation leads us to define the duality transformation D by

$$\boldsymbol{E} \to -\boldsymbol{H}, \qquad \boldsymbol{B} \to \boldsymbol{D}, \qquad \boldsymbol{\varepsilon} \to -\boldsymbol{\theta}, \qquad \boldsymbol{\theta} \to \boldsymbol{\varepsilon}. \tag{2.13}$$

Obviously, the first two in (2.13) render the equations (2.11) into (2.12). The last two are needed to ensure the converse, that is, to change (2.12) back into (2.11) so that the entire equations (2.11) and (2.12) are unchanged as a set. To see how this is done, we express $\boldsymbol{E}$ and $\boldsymbol{B}$ in terms of $\boldsymbol{D}$ and $\boldsymbol{H}$ by obtaining the inverse relations for (2.10). Up to $\mathcal{O}(\theta^{\mu\nu})$, they are found to be

$$\begin{aligned}
\boldsymbol{E} &= \boldsymbol{D} - (\boldsymbol{\varepsilon}\cdot\boldsymbol{D} - \boldsymbol{\theta}\cdot\boldsymbol{H})\boldsymbol{D} - (\boldsymbol{\varepsilon}\cdot\boldsymbol{H} + \boldsymbol{\theta}\cdot\boldsymbol{D})\boldsymbol{H} - (\boldsymbol{H}\cdot\boldsymbol{D})\boldsymbol{\theta} + \frac{1}{2}(\boldsymbol{H}^2 - \boldsymbol{D}^2)\boldsymbol{\varepsilon}, \\
\boldsymbol{B} &= \boldsymbol{H} - (\boldsymbol{\varepsilon}\cdot\boldsymbol{D} - \boldsymbol{\theta}\cdot\boldsymbol{H})\boldsymbol{H} + (\boldsymbol{\varepsilon}\cdot\boldsymbol{H} + \boldsymbol{\theta}\cdot\boldsymbol{D})\boldsymbol{D} + (\boldsymbol{H}\cdot\boldsymbol{D})\boldsymbol{\varepsilon} + \frac{1}{2}(\boldsymbol{H}^2 - \boldsymbol{D}^2)\boldsymbol{\theta}.
\end{aligned} \tag{2.14}$$

It is then easy to see that the duality transformation (2.13) induces $\boldsymbol{D} \to -\boldsymbol{B}$, $\boldsymbol{H} \to \boldsymbol{E}$, and hence (2.12) is converted to (2.11) as claimed.

At this point we point out that the standard Maxwell electric-magnetic duality $\boldsymbol{E} \to -\boldsymbol{B}$, $\boldsymbol{B} \to \boldsymbol{E}$ is recovered for vanishing non-commutativity $\theta^{\mu\nu} = 0$ in (2.13). As is known, this discrete symmetry gets lifted to a continuous $SO(2)$ symmetry $\mathrm{D}(\alpha)$ with $\alpha \in [0, 2\pi)$ by the following transformation,

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{B} \end{pmatrix} \to \begin{pmatrix} \boldsymbol{E}_{\mathrm{D}} \\ \boldsymbol{B}_{\mathrm{D}} \end{pmatrix} = R(\alpha) \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{B} \end{pmatrix}, \qquad \text{where} \quad R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}. \tag{2.15}$$

The standard Maxwell equations, satisfied by the original variables, are now fulfilled by the rotated dual variables, and the discrete electric-magnetic duality is realised by the choice $\alpha = \pi/2$, *i.e.*, by $\mathrm{D}(\frac{\pi}{2})$. The $SO(2)$ dual rotation is actually a canonical transformation, because the Hamiltonian, which is just the norm of the vector in the $\boldsymbol{E} - \boldsymbol{B}$ space, is preserved by the transformation and, likewise, the algebra among the variables (*i.e.*, the canonical structure) is preserved.

Likewise, in the non-commutative case $\theta^{\mu\nu} \neq 0$ we have a generalised $SO(2)$ symmetry transformation in which the duality (2.13) is embedded. Namely, if we introduce the doublets, $(\boldsymbol{E}, \boldsymbol{H})$ and $(\boldsymbol{\theta}, \boldsymbol{\varepsilon})$, and define the dual map $\mathrm{D}(\alpha)$ by

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} \to \begin{pmatrix} \boldsymbol{E}_{\mathrm{D}} \\ \boldsymbol{H}_{\mathrm{D}} \end{pmatrix} = R(\alpha) \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}, \qquad \begin{pmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\theta} \end{pmatrix} \to \begin{pmatrix} \boldsymbol{\varepsilon}_{\mathrm{D}} \\ \boldsymbol{\theta}_{\mathrm{D}} \end{pmatrix} = R(\alpha) \begin{pmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\theta} \end{pmatrix}, \tag{2.16}$$

we find that this induces the same rotation on the doublet $(\boldsymbol{D}, \boldsymbol{B})^t \to (\boldsymbol{D}_{\mathrm{D}}, \boldsymbol{B}_{\mathrm{D}})^t = R(\alpha)\,(\boldsymbol{D}, \boldsymbol{B})^t$. We then confirm readily that the field equations are still satisfied by the dual variables, and that the discrete dual (2.13) is realised by $\mathrm{D}(\frac{\pi}{2})$. Like in the usual Maxwell case, one also sees that the canonical structure is preserved under the rotation. Combined with the invariance of the field equations (which implies that the Hamiltonian in the two sets of variables, original and dual, should agree), one may regard the rotation as a 'canonical transformation', at least heuristically, even though one cannot explicitly demonstrate this directly, due to the presence of higher order time derivatives that arise under the space-time non-commutativity parameter θ^{0i}. For the discrete case $\mathrm{D}(\frac{\pi}{2})$ the duality as a canonical transformation has been discussed in the Lagrangian [7] and Hamiltonian formulations [8]. We shall discuss more about the dual rotation later.

3. Duality with arbitrary coefficients for $\mathcal{L}_1$ and $\mathcal{L}_2$

The duality symmetry can be observed even with arbitrary coefficients for $\mathcal{L}_1$ and $\mathcal{L}_2$ in (2.4). Namely, we consider a system governed by the Lagrangian,[2]

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + a\mathcal{L}_2, \tag{3.1}$$

[2] Since θ can be scaled freely, one can always normalize the coefficient for $\mathcal{L}_1$ (or $\mathcal{L}_2$) even if one starts with arbitrary coefficients for both $\mathcal{L}_1$ and $\mathcal{L}_2$.

with an arbitrary real parameter a. Note that this is the most general Lagrangian which is form invariant under Lorentz transformations and is constructed out of $F^{\mu\nu}$ and $\theta^{\mu\nu}$ up to $\mathcal{O}(\theta^{\mu\nu})$. The non-commutative Maxwell theory (2.4) is recovered by choosing $a = 1$. The field equations (2.11) and (2.12) still hold for any a with $\boldsymbol{D}$ and $\boldsymbol{H}$ defined by (2.9) which are now a-dependent,

$$\begin{aligned}\boldsymbol{D} &= \big[1-(2a-1)(\boldsymbol{\theta}\cdot\boldsymbol{B}-\boldsymbol{\varepsilon}\cdot\boldsymbol{E})\big]\boldsymbol{E}+a(\boldsymbol{\theta}\cdot\boldsymbol{E}+\boldsymbol{\varepsilon}\cdot\boldsymbol{B})\boldsymbol{B}\\ &\quad + a(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\theta}+\frac{1}{2}(2a-1)(\boldsymbol{E}^2-\boldsymbol{B}^2)\boldsymbol{\varepsilon},\\ \boldsymbol{H} &= \big[1-(2a-1)(\boldsymbol{\theta}\cdot\boldsymbol{B}-\boldsymbol{\varepsilon}\cdot\boldsymbol{E})\big]\boldsymbol{B}-a(\boldsymbol{\theta}\cdot\boldsymbol{E}+\boldsymbol{\varepsilon}\cdot\boldsymbol{B})\boldsymbol{E}\\ &\quad - a(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\varepsilon}+\frac{1}{2}(2a-1)\big(\boldsymbol{E}^2-\boldsymbol{B}^2\big)\boldsymbol{\theta}.\end{aligned} \tag{3.2}$$

The inverse relations of these are then given by

$$\begin{aligned}\boldsymbol{E} &= \big[1+(2a-1)(\boldsymbol{\theta}\cdot\boldsymbol{H}-\boldsymbol{\varepsilon}\cdot\boldsymbol{D})\big]\boldsymbol{D}-a(\boldsymbol{\theta}\cdot\boldsymbol{D}+\boldsymbol{\varepsilon}\cdot\boldsymbol{H})\boldsymbol{H}\\ &\quad - a(\boldsymbol{D}\cdot\boldsymbol{H})\boldsymbol{\theta}-\frac{1}{2}(2a-1)(\boldsymbol{D}^2-\boldsymbol{H}^2)\boldsymbol{\varepsilon},\\ \boldsymbol{B} &= \big[1+(2a-1)(\boldsymbol{\theta}\cdot\boldsymbol{H}-\boldsymbol{\varepsilon}\cdot\boldsymbol{D})\big]\boldsymbol{H}+a(\boldsymbol{\theta}\cdot\boldsymbol{D}+\boldsymbol{\varepsilon}\cdot\boldsymbol{H})\boldsymbol{D}\\ &\quad + a(\boldsymbol{D}\cdot\boldsymbol{H})\boldsymbol{\varepsilon}-\frac{1}{2}(2a-1)\big(\boldsymbol{D}^2-\boldsymbol{H}^2\big)\boldsymbol{\theta}.\end{aligned} \tag{3.3}$$

As in the non-commutative case, one can observe that the duality transformation (2.13) — which now depends on a — induces $\boldsymbol{D} \to -\boldsymbol{B}$, $\boldsymbol{H} \to \boldsymbol{E}$, and hence it preserves the field equations (2.11) and (2.12) as a set. One therefore sees that, as long as the duality is concerned, the non-commutative Maxwell theory occupies no special position among the other theories defined by (3.1).

Our argument on the duality can be made concise if we use $F^{\mu\nu}$ and $\theta^{\mu\nu}$ and introduce

$$G^{\mu\nu} = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}, \tag{3.4}$$

which is the tensor for $\boldsymbol{D}$ and $\boldsymbol{H}$, *i.e.*,

$$D^i = -G^{0i}, \qquad H^i = -\frac{1}{2}\epsilon^{ijk}G_{jk}. \tag{3.5}$$

In terms of $F^{\mu\nu}$, the tensor $G^{\mu\nu}$ is written as

$$\begin{aligned}G^{\mu\nu} &= F^{\mu\nu}-\frac{1}{4}(\theta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}+2\theta^{\alpha\beta}F_{\alpha\beta}F^{\mu\nu})\\ &\quad + a\Big[(\theta^{\mu\alpha}F^{\nu\beta}-\theta^{\nu\alpha}F^{\mu\beta})F_{\alpha\beta}+\theta_{\alpha\beta}F^{\mu\alpha}F^{\nu\beta}\Big],\end{aligned} \tag{3.6}$$

which admits the inverse,

$$F^{\mu\nu} = G^{\mu\nu} + \frac{1}{4}(\theta^{\mu\nu} G_{\alpha\beta} G^{\alpha\beta} + 2\theta^{\alpha\beta} G_{\alpha\beta} G^{\mu\nu}) \\ - a\Big[(\theta^{\mu\alpha} G^{\nu\beta} - \theta^{\nu\alpha} G^{\mu\beta}) G_{\alpha\beta} + \theta_{\alpha\beta} G^{\mu\alpha} G^{\nu\beta}\Big]. \tag{3.7}$$

In this tensorial notation, the field equations (2.11) and (2.12) take the simple form,

$$\partial_\mu {}^*F^{\mu\nu} = 0, \qquad \partial_\mu G^{\mu\nu} = 0. \tag{3.8}$$

where we have used the dual of the field strength, ${}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$. In terms of the dual tensors ${}^*G^{\mu\nu}$ and ${}^*\theta^{\mu\nu}$ similarly defined, the duality transformation (2.13) takes the simple form,

$$F^{\mu\nu} \to -{}^*G^{\mu\nu}, \qquad \theta^{\mu\nu} \to -{}^*\theta^{\mu\nu}. \tag{3.9}$$

It is then immediate to find that the duality transformation (3.9) induces $G^{\mu\nu} \to -{}^*F^{\mu\nu}$, confirming that the field equations (3.8) are indeed preserved under (3.9). Note that the dual map (3.9) is well-defined on the level of the gauge potential $A^\mu \to A^\mu_{\rm D}(A)$ if and only if A_μ fulfills the field equations, because for this the corresponding dual curvature $F^{\mu\nu} \to F^{\mu\nu}_{\rm D}(A) = \partial^\mu A^\nu_{\rm D} - \partial^\nu A^\mu_{\rm D} = -{}^*G^{\mu\nu}$ must satisfy the Bianchi identity which is nothing but the second equation in (3.8). In other words, the duality provides a map among solutions, not among arbitrary configurations of A^μ.

More generally, the $SO(2)$ dual rotation $\mathrm{D}(\alpha)$ in (2.16) is implemented by

$$F^{\mu\nu} \to \cos\alpha\, F^{\mu\nu} - \sin\alpha\, {}^*G^{\mu\nu}, \qquad \theta^{\mu\nu} \to \cos\alpha\, \theta^{\mu\nu} - \sin\alpha\, {}^*\theta^{\mu\nu}. \tag{3.10}$$

Note that (3.10) implies $G^{\mu\nu} \to \cos\alpha\, G^{\mu\nu} - \sin\alpha\, {}^*F^{\mu\nu}$, showing that (3.10) preserves the field equations (3.8). Observe that, if we define $K^{\mu\nu}_\pm = (F^{\mu\nu} \pm G^{\mu\nu})/2$, we have $K^{\mu\nu}_\pm \to \cos\alpha\, K^{\mu\nu}_\pm \mp \sin\alpha\, {}^*K^{\mu\nu}_\pm$, and hence the product

$$K^{\mu\nu}_+ K^-_{\mu\nu} = \frac{1}{4}\left(F_{\mu\nu}F^{\mu\nu} - G_{\mu\nu}G^{\mu\nu}\right) = 3(\mathcal{L}_1 + a\mathcal{L}_2) \tag{3.11}$$

is invariant under the rotation (3.10). We thus learn that the rotation affects the Lagrangian (3.1) only in the first term $\mathcal{L}_0$ and yields the change,

$$\mathcal{L} \to \mathcal{L} + \frac{1}{2}\sin^2\alpha\, F_{\mu\nu} G^{\mu\nu} + \frac{1}{2}\cos\alpha \sin\alpha\, F_{\mu\nu}{}^*G^{\mu\nu}. \tag{3.12}$$

Upon using (3.8) we have $\frac{1}{2}F_{\mu\nu}G^{\mu\nu} = \partial_\mu(A_\nu G^{\mu\nu})$ and hence the second term on the rhs is a total divergence. For the last term, we note that since $-{}^*G^{\mu\nu}$ is a curvature, $\frac{1}{4}G_{\mu\nu}{}^*G^{\mu\nu}$, as well as $\frac{1}{4}F_{\mu\nu}{}^*F^{\mu\nu}$, is a total divergence. It follows that their difference $\frac{1}{2}(F_{\mu\nu} - G_{\mu\nu}){}^*F^{\mu\nu}$ (up to $\mathcal{O}(\theta^{\mu\nu})$) is also a total divergence, and so is $\frac{1}{2}F_{\mu\nu}{}^*G^{\mu\nu}$. We therefore find that the Lagrangian is invariant under the rotation (3.10) up to a total divergence, and hence the $SO(2)$ dual rotation $\mathrm{D}(\alpha)$ is a symmetry transformation.

4. Plane wave solutions

We now utilize the discrete duality transformation $\mathrm{D}(\frac{\pi}{2})$ to generate new solutions from known plane wave solutions (more general solutions for $\mathrm{D}(\alpha)$ may be obtained by the rotation once the discrete dual solution is obtained). To this end, we first recall that for

$$\boldsymbol{\varepsilon} = 0, \qquad \boldsymbol{\theta} \neq 0, \tag{4.1}$$

the non-commutative Maxwell theory (2.4) admits the plane wave solution [3],

$$\boldsymbol{E} = \boldsymbol{E}(\omega t - \boldsymbol{k} \cdot \boldsymbol{r}), \qquad \boldsymbol{B} = \boldsymbol{\kappa} \times \boldsymbol{E} + \boldsymbol{b}, \tag{4.2}$$

where $\boldsymbol{\kappa} = c\boldsymbol{k}/\omega$ and $\boldsymbol{b}$ represents a constant background. The constants ω and $k = |\boldsymbol{k}|$ fulfill the dispersion relation,

$$\omega = ck(1 - \boldsymbol{\theta}_T \cdot \boldsymbol{b}_T), \tag{4.3}$$

with $\boldsymbol{\theta}_T$ (and similarly $\boldsymbol{b}_T$) being the transverse part of $\boldsymbol{\theta}$ (and $\boldsymbol{b}$) with respect to the wave vector $\boldsymbol{k}$. No condition arises for the transverse part $\boldsymbol{E}_T$ of the electric field $\boldsymbol{E} = \boldsymbol{E}_T + E_L\hat{\boldsymbol{\kappa}}$ (we put a hat on normalized vectors as $\hat{\boldsymbol{\kappa}} = \boldsymbol{\kappa}/|\boldsymbol{\kappa}|$) but the longitudinal part is subject to the condition,

$$E_L = -(\boldsymbol{b}_T \cdot \boldsymbol{E}_T)\theta_L - (\boldsymbol{\theta}_T \cdot \boldsymbol{E}_T)b_L. \tag{4.4}$$

Thus, the plane wave possesses two degrees of freedom for polarisation as in the standard Maxwell theory but the polarisation is no longer transverse in general. This solution may be characterized by the property that no background field contributes to $\boldsymbol{E}$ and $\boldsymbol{D}$ while it does to $\boldsymbol{H}$ as well as $\boldsymbol{B}$. In fact, from (2.10) combined with (4.2) and (4.4) one finds that $\boldsymbol{H}$ possesses the constant background,

$$\boldsymbol{h} = \big[1 - (\boldsymbol{\theta} \cdot \boldsymbol{b})\big]\boldsymbol{b} - \frac{1}{2}|\boldsymbol{b}|^2\boldsymbol{\theta}. \tag{4.5}$$

One can obtain a new plane wave solution from this solution by performing the discrete duality transformation (2.13). Since the duality involves the interchange of the parameters $\boldsymbol{\theta}$ and $\boldsymbol{\varepsilon}$, the new solution is valid for

$$\boldsymbol{\varepsilon} \neq 0, \qquad \boldsymbol{\theta} = 0, \tag{4.6}$$

and takes the form,

$$\boldsymbol{H} = \boldsymbol{H}(\omega t - \boldsymbol{k}\cdot\boldsymbol{r}), \qquad \boldsymbol{D} = -\boldsymbol{\kappa}\times\boldsymbol{H} + \boldsymbol{d}, \tag{4.7}$$

with a constant background $\boldsymbol{d}$. The dispersion relation now reads

$$\omega = ck(1 - \boldsymbol{\varepsilon}_T\cdot\boldsymbol{d}_T). \tag{4.8}$$

Analogously to the electric field $\boldsymbol{E}$ in the previous case, the magnetic field $\boldsymbol{H}$ admits two polarisation degrees of freedom in the transverse part $\boldsymbol{H}_T$ while the longitudinal part is determined as

$$H_L = -(\boldsymbol{d}_T\cdot\boldsymbol{H}_T)\varepsilon_L - (\boldsymbol{\varepsilon}_T\cdot\boldsymbol{H}_T)d_L. \tag{4.9}$$

In this solution, $\boldsymbol{E}$ has a constant background $\boldsymbol{e}$ given by

$$\boldsymbol{e} = \big[1 - (\boldsymbol{\varepsilon}\cdot\boldsymbol{d})\big]\boldsymbol{d} - \frac{1}{2}|\boldsymbol{d}|^2\boldsymbol{\varepsilon}. \tag{4.10}$$

The above argument can be carried over to the general case where both $\boldsymbol{\theta}$ and $\boldsymbol{\varepsilon}$ are nonvanishing and also for theories with an arbitrary coefficient a. As before, if we start with the plane wave ansatz (4.2) for $\boldsymbol{E}$ and $\boldsymbol{B}$, then from the field equations (2.11) and (2.12) we find that the longitudinal part of the electric field is determined as

$$\begin{aligned} E_L = &-a(\boldsymbol{b}_T\cdot\boldsymbol{E}_T)\theta_L - a\big(\boldsymbol{\theta}_T\cdot\boldsymbol{E}_T + \boldsymbol{\varepsilon}_T\cdot(\boldsymbol{\kappa}\times\boldsymbol{E}_T)\big)b_L \\ &+ (2a-1)\big(\boldsymbol{b}_T\cdot(\boldsymbol{\kappa}\times\boldsymbol{E}_T)\big)\varepsilon_L. \end{aligned} \tag{4.11}$$

On the other hand, the transverse part $\boldsymbol{E}_T$ must fulfill the condition,

$$\left(M^{ij} - \lambda\,\delta^{ij}\right)E_T^j = 0, \tag{4.12}$$

with

$$\begin{aligned} M^{ij} &= (a-1)(\theta_T^i b_T^j + \theta_T^j b_T^i) + (2a-1)(\epsilon^{ik}\varepsilon_T^j + \epsilon^{jk}\varepsilon_T^i)b_T^k - a\epsilon^{ik}\varepsilon_T^k b_T^j, \\ \lambda &= 1 - \kappa^2 + 2(2a-1)(\boldsymbol{\theta}_T\cdot\boldsymbol{b}_T), \end{aligned} \tag{4.13}$$

where we have chosen the third spacial ($z-$)axis in the direction of $\hat{\boldsymbol{\kappa}}$ so that i, j run over 1, 2 and used the antisymmetric tensor ϵ^{ij} with $\epsilon^{12} = 1$. From (4.12) it follows that, unless the matrix $(M^{ij} - \lambda\,\delta^{ij})$ vanishes identically, the polarisation in the transverse part $\boldsymbol{E}_T$ is determined by the non-commutative parameters $\boldsymbol{\theta}$, $\boldsymbol{\varepsilon}$ and the coefficient a together with $\boldsymbol{\kappa}$. Explicitly, we find that $\boldsymbol{E}_T$ points to the specific directions,

$$\boldsymbol{E}_T \propto \left[\frac{Y+Z}{\hat{\boldsymbol{b}}_T\cdot(\hat{\boldsymbol{\kappa}}\times\hat{\boldsymbol{\theta}}_T)} + \frac{X\mp\sqrt{A}}{1-\hat{\boldsymbol{b}}_T\cdot\hat{\boldsymbol{\theta}}_T}\right]\hat{\boldsymbol{b}}_T + \left[\frac{Y+Z}{\hat{\boldsymbol{b}}_T\cdot(\hat{\boldsymbol{\kappa}}\times\hat{\boldsymbol{\theta}}_T)} + \frac{X\pm\sqrt{A}}{1-\hat{\boldsymbol{b}}_T\cdot\hat{\boldsymbol{\theta}}_T}\right]\hat{\boldsymbol{\theta}}_T, \tag{4.14}$$

with

$$
\begin{aligned}
A &= 4(a-1)^2(\theta_T b_T)^2 + 4(a-1)(3a-2)(b_T)^2\big(\boldsymbol{\varepsilon}_T \cdot (\hat{\boldsymbol{\kappa}} \times \boldsymbol{\theta}_T)\big) \\
&\qquad + (3a-2)^2(\varepsilon_T b_T)^2 - a^2(\boldsymbol{\varepsilon}_T \cdot \boldsymbol{b}_T)^2, \\
X &= 2(a-1)\theta_T b_T + (3a-2) b_T \big(\boldsymbol{\varepsilon}_T \cdot (\hat{\boldsymbol{\kappa}} \times \hat{\boldsymbol{\theta}}_T)\big), \\
Y &= (3a-2) b_T (\boldsymbol{\varepsilon}_T \cdot \hat{\boldsymbol{\theta}}_T), \\
Z &= a(\boldsymbol{\varepsilon}_T \cdot \boldsymbol{b}_T).
\end{aligned}
\tag{4.15}
$$

Once $\boldsymbol{E}$ is known, $\boldsymbol{B}$ is determined uniquely from (4.2). Besides $\boldsymbol{B}$, the field $\boldsymbol{H}$ possesses the constant background,

$$
\boldsymbol{h} = \big[1 - (2a-1)(\boldsymbol{\theta} \cdot \boldsymbol{b})\big]\boldsymbol{b} - \frac{1}{2}(2a-1)|\boldsymbol{b}|^2 \boldsymbol{\theta}, \tag{4.16}
$$

which is nonvanishing in the limit $\theta^{\mu\nu} \to 0$, whereas the constant background of $\boldsymbol{D}$ is

$$
\boldsymbol{d} = a(\boldsymbol{\varepsilon} \cdot \boldsymbol{b})\boldsymbol{b} - \frac{1}{2}(2a-1)|\boldsymbol{b}|^2 \boldsymbol{\varepsilon}, \tag{4.17}
$$

which vanishes in the limit.

A nonvanishing $\boldsymbol{E}_T$ requires $\det(M^{ij} - \lambda\,\delta^{ij}) = 0$, and from this one obtains the dispersion relation

$$
\omega = ck\Big[1 - \frac{1}{2}(3a-1)(\boldsymbol{\theta}_T \cdot \boldsymbol{b}_T) - \frac{1}{4}(5a-2)\big(\boldsymbol{\varepsilon}_T \cdot (\hat{\boldsymbol{\kappa}} \times \boldsymbol{b}_T)\big) \pm \frac{1}{4}\sqrt{A}\Big] \tag{4.18}
$$

corresponding to the two expressions for $\boldsymbol{E}_T$ given in (4.14). One then observes that, for $\boldsymbol{\varepsilon}_T \neq 0$, the velocity of the wave depends on the direction it propagates. Observe also that, unless $A = 0$, there are two solutions for (4.18), which implies that the two waves with distinct polarisations propagate at different velocities even in the same direction. Note that, as mentioned earlier, the two polarisations can be in arbitrary directions if $M^{ij} = \lambda\,\delta^{ij}$. Without attempting a general solution for this equation, let us remark that this condition is trivially satisfied for non-commutative electrodynamics ($a = 1$) with $\boldsymbol{\varepsilon} = 0$. In this case both M^{ij} and λ vanish identically so that the solution reduces to the one discussed in [3].

If we implement the discrete duality (2.13) to the above solution, we obtain a new solution with the ansatz (4.7). The longitudinal and transverse parts of $\boldsymbol{H}$ are gained directly from (4.11) and (4.14) by the duality map, but rather than displaying them we shall present the solution for $\boldsymbol{E}$ and $\boldsymbol{B}$ with the help of (3.3). For this we first observe from (4.16) and (4.17) that in the dual solution the electric field $\boldsymbol{E}$ has the constant background,

$$
\boldsymbol{e} = \big[1 - (2a-1)(\boldsymbol{\varepsilon} \cdot \boldsymbol{d})\big]\boldsymbol{d} - \frac{1}{2}(2a-1)|\boldsymbol{d}|^2 \boldsymbol{\varepsilon}, \tag{4.19}
$$

while $\boldsymbol{B}$ has

$$\boldsymbol{b} = a(\boldsymbol{\theta}\cdot\boldsymbol{d})\boldsymbol{d} - \frac{1}{2}(2a-1)|\boldsymbol{d}|^2\boldsymbol{\theta}. \tag{4.20}$$

The oscillatory longitudinal part of $\boldsymbol{B}$ vanishes identically due to (2.11), and the transverse part of $\boldsymbol{B}$ is found to be proportional to

$$\boldsymbol{B}_T \propto \left[\frac{\tilde{Y}+\tilde{Z}}{\hat{\boldsymbol{e}}_T\cdot(\hat{\boldsymbol{\kappa}}\times\hat{\boldsymbol{\varepsilon}}_T)} + \frac{\tilde{X}\mp\sqrt{\tilde{A}}}{1-\hat{\boldsymbol{e}}_T\cdot\hat{\boldsymbol{\varepsilon}}_T}\right]\hat{\boldsymbol{e}}_T + \left[\frac{\tilde{Y}+\tilde{Z}}{\hat{\boldsymbol{e}}_T\cdot(\hat{\boldsymbol{\kappa}}\times\hat{\boldsymbol{\varepsilon}}_T)} + \frac{\tilde{X}\pm\sqrt{\tilde{A}}}{1-\hat{\boldsymbol{e}}_T\cdot\hat{\boldsymbol{\varepsilon}}_T}\right]\hat{\boldsymbol{\varepsilon}}_T, \tag{4.21}$$

where

$$\begin{aligned}
\tilde{A} &= 4(a-1)^2(\varepsilon_T e_T)^2 - 4(a-1)(3a-2)(e_T)^2\big(\boldsymbol{\theta}_T\cdot(\hat{\boldsymbol{\kappa}}\times\boldsymbol{\varepsilon}_T)\big) \\
&\qquad + (3a-2)^2(\theta_T e_T)^2 - a^2(\boldsymbol{\theta}_T\cdot\boldsymbol{e}_T)^2, \\
\tilde{X} &= 2(a-1)\varepsilon_T e_T - (3a-2)e_T\big(\boldsymbol{\theta}_T\cdot(\hat{\boldsymbol{\kappa}}\times\hat{\boldsymbol{\varepsilon}}_T)\big), \\
\tilde{Y} &= -(3a-2)e_T(\boldsymbol{\theta}_T\cdot\hat{\boldsymbol{\varepsilon}}_T), \\
\tilde{Z} &= -a(\boldsymbol{\theta}_T\cdot\boldsymbol{e}_T).
\end{aligned} \tag{4.22}$$

In terms of $\boldsymbol{B}$, the transverse part of $\boldsymbol{E}$ is determined from (4.2) and the longitudinal part of $\boldsymbol{E}$ is found to be

$$\begin{aligned}
E_L &= -a(\boldsymbol{e}_T\cdot\boldsymbol{B}_T)\theta_L + (2a-1)\big(\boldsymbol{\varepsilon}_T\cdot(\boldsymbol{\kappa}\times\boldsymbol{B}_T) - \boldsymbol{\theta}_T\cdot\boldsymbol{B}_T\big)e_L \\
&\qquad + (2a-1)\big(\boldsymbol{e}_T\cdot(\boldsymbol{\kappa}\times\boldsymbol{B}_T)\big)\varepsilon_L.
\end{aligned} \tag{4.23}$$

The above nonvanishing $\boldsymbol{B}_T$ is ensured if $\det(\tilde{M}^{ij} - \tilde{\lambda}\,\delta^{ij}) = 0$ where

$$\begin{aligned}
\tilde{M}^{ij} &= (a-1)(\varepsilon_T^i e_T^j + \varepsilon_T^j e_T^i) - (2a-1)(\epsilon^{ik}\theta_T^j + \epsilon^{jk}\theta_T^i)e_T^k + a\epsilon^{ik}\theta_T^k e_T^j, \\
\tilde{\lambda} &= 1 - \kappa^2 + 2(2a-1)(\boldsymbol{\varepsilon}_T\cdot\boldsymbol{e}_T),
\end{aligned} \tag{4.24}$$

and this leads to the dispersion relation,

$$\omega = ck\Big[1 - \frac{1}{2}(3a-1)(\boldsymbol{e}_T\cdot\boldsymbol{\varepsilon}_T) + \frac{1}{4}(5a-2)\big(\boldsymbol{\theta}_T\cdot(\hat{\boldsymbol{\kappa}}\times\boldsymbol{e}_T)\big) \pm \frac{1}{4}\sqrt{\tilde{A}}\Big]. \tag{4.25}$$

Notice that the condition $\tilde{M}^{ij} = \tilde{\lambda}\,\delta^{ij}$, which allows for arbitrary polarisation for $\boldsymbol{B}_T$, is fulfilled by choosing $a = 1$ and $\boldsymbol{\theta} = 0$, *i.e.*, by the non-commutative electrodynamics (2.4) with only *space-time* non-commutativity, where the solution becomes the one with the dispersion relation (4.8) mentioned above. In other words, the non-commutative electrodynamics with only *space-space* non-commutativity $\boldsymbol{\varepsilon} = 0$ does not admit arbitrary polarisation for $\boldsymbol{B}_T$ if the plane wave solution has a constant background for $\boldsymbol{E}$.

We have seen, therefore, that the non-commutative electrodynamics is distinguished in that it admits a plane wave solution with arbitrary polarisation in the transverse part (from which the longitudinal part is determined) if the non-commutative parameter $\theta^{\mu\nu}$ is chosen properly to a given constant background. Otherwise, a severe restriction on the polarisation arises as a consequence of the non-commutativity (or the introduction of the arbitrary coefficient a) for plane wave solutions with a generic constant background. This result is valid no matter how small the constant background and/or the non-commutativity parameters are, and this may be used to provide some stringent constraints on the value of $\boldsymbol{\varepsilon}$ or $\boldsymbol{\theta}$ in laboratory tests. The singularity of the plane wave solution in the limit $\theta^{\mu\nu} \to 0$ suggests, however, that there may be other 'quasi plane wave' solutions which do not assume the plane wave ansatz for $\theta^{\mu\nu} \neq 0$ but nonetheless reduce to the usual plane wave in the limit $\theta^{\mu\nu} \to 0$.

5. Conclusion

We have generalised the notion of electric-magnetic duality to non-commutative gauge theories, expanded as a power series in terms of the non-commutativity parameter. The coefficients of the terms additional to the usual Maxwell piece were completely arbitrary and not constrained by the Seiberg-Witten map. Naturally, non-commutative electrodynamics obtained by an application of this map manifested this duality, since it just meant the fixing of the arbitrary parameters. The explicit map between the electric and magnetic variables was provided. Our results were compatible with previous analysis on S-duality in non-commutative theories [7, 8, 10]. It was also feasible to represent the duality transformation as an $SO(2)$ continuous symmetry transformation, thereby highlighting its possible connection with canonical transformations. As an application, we worked out the plane wave solutions under various conditions. Dispersion relations obtained from these solutions led to certain intriguing possibilities; in particular, the propagation of two waves, with distinct polarisations, at different velocities in the same direction.

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