

# Supersymmetric Quantum Mechanics under Point Singularities

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**Abstract.** We provide a systematic study on the possibility of supersymmetry (SUSY) for one dimensional quantum mechanical systems consisting of a pair of lines  $\mathbb{R}$  or intervals  $[-l, l]$  each having a point singularity. We consider the most general singularities and walls (boundaries) at  $x = \pm l$  admitted quantum mechanically, using a  $U(2)$  family of parameters to specify one singularity and similarly a  $U(1)$  family of parameters to specify one wall. With these parameter freedoms, we find that for a certain subfamily the line systems acquire an  $N = 1$  SUSY which can be enhanced to  $N = 4$  if the parameters are further tuned, and that these SUSY are generically broken except for a special case. The interval systems, on the other hand, can accommodate  $N = 2$  or  $N = 4$  SUSY, broken or unbroken, and exhibit a rich variety of (degenerate) spectra. Our SUSY systems include the familiar SUSY systems with the Dirac  $\delta(x)$ -potential, and hence are extensions of the known SUSY quantum mechanics to those with general point singularities and walls. The self-adjointness of the supercharge in relation to the self-adjointness of the Hamiltonian is also discussed.

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## 1. Introduction

A point singularity (or interaction) appears in various different contexts in physics. It may, for instance, appear as a point defect or a junction of two layers of materials, or may be considered as a localized limit of a finite range potential in general. A point singularity is usually modelled by the Dirac  $\delta(\mathbf{x})$ -potential, which offers exact solutions to a number of problems of interest both classically and quantum mechanically. However, in quantum mechanics a point singularity is far from unique — in one dimension, for example, the Dirac  $\delta(x)$  is just one of the  $U(2)$  family of point singularities allowed quantum mechanically [1, 2, 3]. In fact, recent investigations have shown that these point singularities can give rise to unexpectedly interesting phenomena which are not available under the Dirac  $\delta(x)$ -potential. These include duality in spectra, anholonomy (Berry phase) and scale anomaly [4, 5], which normally occur in more complicated systems or quantum field theory.

The first of these, the duality, implies that the spectra of two distinct point singularities may coincide if they are related under some discrete transformations and, in particular, become completely degenerate if the point singularity is self-dual, *i.e.*, invariant under the discrete transformations. The presence of degeneracy, and also the graded structure which is naturally equipped with the system, alluded us to examine the possibility of supersymmetry (SUSY) with self-dual point singularities. This has indeed been confirmed in our previous paper [6], where we have found a number of novel  $N = 1$  and  $N = 2$  SUSY systems on a line  $\mathbb{R}$  or interval  $[-l, l]$  with the family of  $U(2)$  point singularities (under the walls at  $x = \pm l$  allowing for the general boundary condition for the interval case). More recently, SUSY on a circle with two point singularities has also been studied in [7].

Meanwhile, SUSY quantum mechanics (and its extensions) has been studied intensively over the years, initially to provide SUSY breaking mechanisms in field theory and lately to establish schemes to accommodate known solvable models or generate novel ones (see, *e.g.*, [8, 9, 10]). However, for some reason the investigation of SUSY quantum mechanics under point singularities has evaded from the studies, and we know little about it except that a suitable pair<sup>1</sup> of the Dirac  $\delta(x)$ -potentials can be made into a Witten model [14, 15] and realizes an  $N = 2$  SUSY. The aim of this paper is to present a systematic study of SUSY under point singularities, and thereby report that point singularities admit a variety of novel SUSY systems including the known one. The systems we consider are those consisting of a pair of lines  $\mathbb{R}$  or intervals  $[-l, l]$  each having a point singularity,

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<sup>1</sup> There are works [11, 12, 13] on SUSY quantum mechanics for a pair of interval systems one of which has the Dirac  $\delta(x)$ -potential while the other has a ‘partner potential’ with a finite support. In contrast, here we will consider a pair of line/interval systems possessing singularities without such potentials.

where the pair provides a graded structure as the known Dirac  $\delta(x)$  system has. The two point singularities can in general be different and hence our total family of singularities are given by  $U(2) \times U(2)$ . We find that the line systems with point singularities belonging to a certain subfamily generically possess an  $N = 1$  SUSY, broken and unbroken, and that the SUSY can be promoted to  $N = 4$  for a further restricted parameter subfamily. Similarly, for the interval systems, in addition to the general  $U(2) \times U(2)$  point singularities we take account of the most general boundary conditions for the two sets of walls at  $x = \pm l$  represented by  $[U(1)]^4$ . We then find that an  $N = 2$  or  $N = 4$  (broken or unbroken) SUSY appears for a certain subfamily of the combined parameter family characterizing the point singularities and the walls of the system. All of these SUSY systems are classified into a number of different types, and their SUSY and spectral properties are summarized in Appendix B.

The plan of the paper is as follows. In section 2, we discuss the line systems, where we provide our criterion for SUSY and thereby find SUSY systems ((A1) – (B2)). Based on the result of this section, the interval systems are then studied in section 3, where we find quite a few distinct SUSY systems ((a1) – (d6)). Section 4 is devoted to the question of the self-adjointness of the supercharge. Our conclusion and discussions are given in section 5. Appendix A contains computations to supplement our argument in section 4, and Appendix B furnishes the summary table for the various SUSY systems mentioned above.

## 2. Two lines with point singularity

In this section, we investigate the possibility of supersymmetry in a quantum system consisting of two lines each possessing a singularity at  $x = 0$ . The Hilbert space of our system is thus given by  $\mathcal{H} = L^2(\mathbb{R}\setminus\{0\}) \oplus L^2(\mathbb{R}\setminus\{0\}) \simeq L^2(\mathbb{R}\setminus\{0\}) \otimes \mathbb{C}^2$ . Except for the singular point which is now removed on each of the lines, the system is assumed to be free and hence its Hamiltonian reads  $H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \otimes I_2$ , where  $I_2$  denotes the  $2 \times 2$  unit matrix. If we split the space in two at the singularity and thereby regard  $L^2(\mathbb{R}\setminus\{0\}) \simeq L^2(\mathbb{R}^+) \otimes \mathbb{C}^2$ , we can double the graded structure and identify  $\mathcal{H}$  with  $L^2(\mathbb{R}^+) \otimes \mathbb{C}^4$  (see Fig.1). Let  $\psi_1(x)$  and  $\psi_2(x)$  be the wave functions on the two lines, respectively, representing a state in the Hilbert space. According to the above identification, the state may equally be represented by  $\Psi_i(x) = (\psi_i^+(x), \psi_i^-(x))^T$  for  $i = 1, 2$ , where we have defined  $\psi_i^+(x) = \psi_i(x)$  for  $x > 0$  and  $\psi_i^-(x) = \psi_i(x)$  for  $x < 0$ . Combining these, we can express a state in the Hilbert space  $\mathcal{H}$  by the four-components wave function,

$$\Psi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix}, \quad x > 0, \quad (2.1)$$

on which our Hamiltonian takes the form,

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \otimes I, \quad \text{where } I = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}. \quad (2.2)$$

The first question we need to address is to find a proper domain on which the Hamiltonian (2.2) becomes self-adjoint. This is ensured by requiring the probability current  $j_i^\pm(x) = \frac{i\hbar}{2m} [(\psi_i^\pm)^\dagger (\psi_i^\pm)' - (\psi_i^\pm)'^\dagger \psi_i^\pm](x)$  be continuous over the two lines (the dash denotes the derivative  $\psi' = \frac{d}{dx} \psi$ ). If the two singular points,  $x = +0$  on line 1 and  $x = -0$  on line 2, were connected at one point, then the continuity condition at the point would be (for brevity we hereafter denote  $x = 0$  for  $x = +0$ )

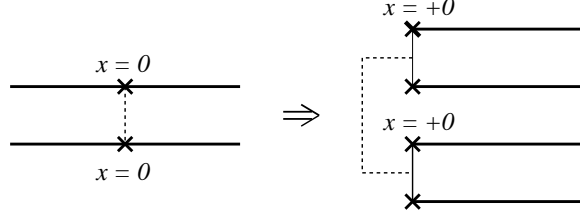
$$0 = \sum_{a=\pm} j_1^a(0) + \sum_{a=\pm} j_2^a(0) = \frac{i\hbar}{2m} (\Psi^\dagger \Psi' - (\Psi')^\dagger \Psi)(0). \quad (2.3)$$

Introducing an arbitrary real constant  $L_0 \neq 0$  with dimension of length, we find that the condition (2.3) is equal to  $|\Psi(0) + iL_0 \Psi'(0)| = |\Psi(0) - iL_0 \Psi'(0)|$ . This in turn can be written as [16, 17]

$$(U - I)\Psi(0) + iL_0(U + I)\Psi'(0) = 0, \quad (2.4)$$

with a matrix  $U \in U(4)$ , or equivalently,

$$\Psi^{(-)}(0) = U \Psi^{(+)}(0), \quad (2.5)$$



**Figure 1.** A system of a pair of two lines each having a singularity at  $x = 0$  may be identified with two systems of a pair of two half lines where the probability flow is allowed to pass between the two systems through  $x = 0$ .

in terms of  $\Psi^{(\pm)} = \Psi \pm iL_0\Psi'$ . The matrix  $U$ , which is called ‘characteristic matrix’ since it characterizes the nature of singularity, specifies a self-adjoint domain  $\mathcal{D}_U(H) \subset \mathcal{H}$  of the Hamiltonian (2.2) by means of the boundary conditions at the two singular points connected. Of course, in our system the two points are disconnected, and hence the actual continuity condition is

$$\sum_{a=\pm} j_1^a(0) = 0 = \sum_{a=\pm} j_2^a(0). \quad (2.6)$$

Correspondingly, our unitary matrix  $U$  must be specialized to  $U \in U(2) \times U(2) \subset U(4)$ , namely,

$$U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}, \quad U_1, U_2 \in U(2). \quad (2.7)$$

Despite the specialization, we prefer to work in the four-components description which is more convenient on account of the fact that all of the components of  $\Psi(x)$  may be interchanged under SUSY transformations.

Our next task is to seek SUSY possible under the Hamiltonian (2.2) with domain specified by (2.5). We suppose that the supercharge  $Q$  be a self-adjoint operator (which will be examined later) and of the form,

$$Q = -i\lambda \frac{d}{dx} \otimes \Gamma + \mu \otimes \Omega, \quad (2.8)$$

where we set  $\lambda = \hbar/(2\sqrt{m})$  and  $\mu$  is a real constant.  $\Gamma$  and  $\Omega$  are Hermitian  $4 \times 4$  matrices and assumed to satisfy the conditions,

$$\Gamma^2 = I, \quad \Omega^2 = I, \quad \{\Gamma, \Omega\} = 0, \quad (2.9)$$

which lead to  $2Q^2 = H + \mu^2$  (more precisely,  $\mu^2$  should be written as  $\mu^2 \cdot \text{id}_{\mathcal{H}}$  where  $\text{id}_{\mathcal{H}}$  denotes the identity operator in the Hilbert space  $\mathcal{H}$ ) with the Hamiltonian  $H$  in (2.2).

The extra term  $\mu^2$  can then be absorbed into  $H$  by redefining  $H + \mu^2 \rightarrow H$ , *i.e.*, by the constant energy shift by  $\mu^2$ , to realize the standard SUSY relation

$$2Q^2 = H. \quad (2.10)$$

Note that this redefinition does not alter the domain  $\mathcal{D}_U(H)$  because it does not affect our argument of the probability conservation.

Our aim now is to find a SUSY invariant pair  $(U, Q)$  in the sense that the SUSY transformation generated by the supercharge  $Q$  in (2.8) preserves the domain of each energy eigenstate  $\Phi$  with<sup>2</sup>

$$H \Phi(x) = E \Phi(x), \quad (2.11)$$

that is, we say that the pair  $(U, Q)$  is ‘SUSY invariant’

$$\text{if } \Phi \in \mathcal{D}_U(H) \quad \text{then } Q \Phi \in \mathcal{D}_U(H), \quad (2.12)$$

for  $\Phi$  satisfying (2.11). Our task to find such a pair may considerably be simplified if we recall that any  $U(4)$  matrix  $U$  can be decomposed as  $U = V^{-1}DV$  with an  $SU(4)$  matrix  $V$  and a diagonal matrix,

$$D = \text{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}), \quad \theta_k \in [0, 2\pi), \quad k = 1, \dots, 4. \quad (2.13)$$

With this decomposition, we see from the boundary conditions (2.4) that if  $\Psi(x) \in \mathcal{D}_U(H)$  then  $W\Psi(x) \in \mathcal{D}_{WUW^{-1}}(H)$  for any  $W \in SU(4)$ . Hence, if a pair  $(U, Q)$  satisfies the SUSY invariant condition (2.12), then  $(WUW^{-1}, WQW^{-1})$  also satisfies the condition (note that  $WQW^{-1}$  is again of the form (2.8)). Choosing in particular  $W = V$ , we can obtain a pair  $(D, VQV^{-1})$ . This implies that, if a pair  $(D, Q)$  is a solution, so is  $(U, V^{-1}QV)$ . Thus our aim is achieved if we obtain a solution for the diagonal case  $(D, Q)$  first, and then transform it to  $(U, V^{-1}QV)$  with

$$V = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}, \quad V_1, V_2 \in SU(2), \quad (2.14)$$

for which  $U = V^{-1}DV$  has the block diagonal form (2.7). We also note that the order of the factors  $e^{i\theta_i}$  in  $D$  in (2.13) is unimportant, because any  $D$  can be put into  $D = S^{-1}\bar{D}S$  with some exchange matrix  $S$  so that  $\bar{D}$  has a desired order of the factors. The exchange matrix  $S$  may be absorbed into  $V$  by redefining  $V \rightarrow S^{-1}V$ , if  $S$  is block diagonal as in

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<sup>2</sup> The equation does not necessarily imply degeneracy in the energy level, because some of the components in the vector eigenstate  $\Phi$  may vanish.

$V$  in (2.14). If not, we need to keep  $S$  as an additional element in the decomposition of  $U$  when we use the ordered  $\bar{D}$ . We record, however, only two cases which we use later, namely,  $S = X$  or  $Y$  where

$$X = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2.15)$$

In short, the characteristic matrix  $U$  may be decomposed as

$$U = V^{-1}DV, \quad D = S^{-1}\bar{D}S, \quad (2.16)$$

with  $\bar{D}$  having diagonal factors in a desired order, using an appropriate exchange matrix  $S$  which may (or may not) be absorbed in  $V$  depending on the order one wants. When we use  $\bar{D}$  with  $S$  and obtain a solution for the SUSY pair  $(\bar{D}, Q)$ , we find the general solution by transforming it to  $(U, V^{-1}S^{-1}QSV)$  as before.

We now consider the SUSY transformation generated by the supercharge  $Q$  in (2.8) for states obeying the boundary condition (2.4) for diagonal  $U = D$ . Under the SUSY transformation, the eigenstate  $\Phi$  and its derivative are transformed into

$$\begin{aligned} (Q\Phi)(x) &= -i\lambda\Gamma\Phi'(x) + \mu\Omega\Phi(x), \\ (Q\Phi')(x) &= -i\lambda\Gamma\Phi''(x) + \mu\Omega\Phi'(x) = iE\lambda\Gamma\Phi(x) + \mu\Omega\Phi'(x). \end{aligned} \quad (2.17)$$

If the transformed state  $Q\Phi$  is to satisfy the original boundary condition (2.4), we need

$$\begin{aligned} (D - I)\Gamma(\Phi^{(+)}(0) - \Phi^{(-)}(0)) - 2\mu L_0(D\Omega\Phi^{(+)}(0) - \Omega\Phi^{(-)}(0)) \\ + EL_0^2(D + I)\Gamma(\Phi^{(+)}(0) + \Phi^{(-)}(0)) = 0. \end{aligned} \quad (2.18)$$

Using (2.5), the condition (2.18) becomes

$$(\lambda(D - I)\Gamma(D - I) + 2\mu L_0[D, \Omega] - EL_0^2(D + I)\Gamma(D + I))\Phi^{(+)}(0) = 0. \quad (2.19)$$

At this point it is important to recognize that the original condition (2.5) provides relations among the components between  $\Psi^{(+)}(0)$  and  $\Psi^{(-)}(0)$ , but not among those within  $\Psi^{(+)}(0)$  or  $\Psi^{(-)}(0)$ . It follows that, if the condition (2.19) is identical to (2.5), then the coefficient matrix for  $\Phi^{(+)}(0)$  in (2.19) must vanish. Furthermore, since the original condition (2.4) is energy independent, we require that the equality (2.19) holds independently of  $E$ . Thus the condition (2.19) actually implies

$$(D + I)\Gamma(D + I) = 0, \quad (2.20)$$

and

$$\lambda(D - I)\Gamma(D - I) + 2\mu L_0[D + I, \Omega] = 0, \quad (2.21)$$

where for our later convenience we have replaced  $D$  with  $D + I$  in the second term.

From (2.20) one immediately sees that at least one element of the diagonal matrix  $D$  must be  $-1$  (otherwise  $D + I$  has an inverse and hence we obtain  $\Gamma = 0$  in contradiction to (2.9)). Let  $n$  be the number of  $e^{i\theta_k} = -1$  elements among the four in (2.13). With the help of the exchange matrix  $S$ , we may choose the ordered diagonal  $\bar{D}$  in (2.16) such that those  $-1$  elements are arranged in the lower right corner, *i.e.*,  $e^{i\theta_{4-n+1}} = \dots = e^{i\theta_4} = -1$ . Under this arrangement, the submatrix of  $\bar{D} + I$  given by its upper left  $(4 - n) \times (4 - n)$  block has an inverse, and hence one observes in (2.20) that all the elements in the corresponding block in  $\Gamma$  must vanish. On the other hand, since the submatrix of  $\bar{D} + I$  given by its lower right  $n \times n$  block vanishes identically, one finds from (2.21) that the elements in the corresponding block in  $\Gamma$  vanish, too. Combining these, one learns that  $\Gamma$  has nonvanishing elements only in the blocks other than these two. Since such a  $\Gamma$  has  $\det \Gamma \neq 0$  (which is required for  $\Gamma^2 = I$  in (2.9)) only if  $n = 2$ , a SUSY invariant pair can be found only for the case,

$$\bar{D} = \begin{pmatrix} T & 0 \\ 0 & -I_2 \end{pmatrix}, \quad T = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix}, \quad \theta_1, \theta_2 \neq \pi. \quad (2.22)$$

Here,  $\Gamma$  takes the form,

$$\Gamma = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}, \quad (2.23)$$

where the  $2 \times 2$  matrix  $A$  is seen to be unitary,  $A \in U(2)$ , to ensure the condition  $\Gamma^2 = I$ .

For  $\mu \neq 0$  we further need to determine  $\Omega$  and for this we first set

$$\Omega = \begin{pmatrix} B & C \\ C^\dagger & F \end{pmatrix}, \quad B = B^\dagger, \quad F = F^\dagger. \quad (2.24)$$

Then (2.21) requires that

$$[T, B] = 0, \quad C = iKA, \quad (2.25)$$

where we have used the real diagonal matrix,

$$K = \frac{\lambda}{i\mu L_0}(T + I_2)^{-1}(T - I_2) = \frac{\lambda}{\mu} \begin{pmatrix} 1/L(\theta_1) & 0 \\ 0 & 1/L(\theta_2) \end{pmatrix}, \quad (2.26)$$

which is given in terms of the two scale parameters [5] defined by

$$L(\theta_i) = L_0 \cot \frac{\theta_i}{2}, \quad i = 1, 2, \quad (2.27)$$



which are nonvanishing  $L(\theta_i) \neq 0$  because  $\theta_i \neq \pi$  (see (2.22)). On the other hand,  $\{\Gamma, \Omega\} = 0$  in (2.9) is ensured if

$$F = -A^\dagger B A. \quad (2.28)$$

For the remaining condition  $\Omega^2 = I$  in (2.9) to hold, in addition to what we already have, we need only

$$B^2 = I_2 - K^2. \quad (2.29)$$

We therefore arrive at the general solution of SUSY invariant systems, that is, a pair  $(D = S\bar{D}S^{-1}, Q)$  is SUSY invariant if  $\bar{D}$  has the form (2.22) and  $Q = q(A, \mu; K)$  with

$$q(A, \mu; K) = -i\lambda \frac{d}{dx} \otimes \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix} + \mu \otimes \begin{pmatrix} B & iKA \\ -iA^\dagger K & -A^\dagger B A \end{pmatrix}, \quad (2.30)$$

where  $A \in U(2)$ ,  $K$  is given by (2.26) and  $B$  is subject to the conditions,

$$B = B^\dagger, \quad B^2 = I_2 - K^2, \quad [T, B] = 0, \quad (2.31)$$

which are met by

$$B = b_1 \sigma_+ + b_2 \sigma_-, \quad \sigma_\pm = \frac{I_2 \pm \sigma_3}{2}, \quad b_i^2 = 1 - \left[ \frac{\lambda}{\mu L(\theta_i)} \right]^2, \quad (2.32)$$

where  $\sigma_i$ ,  $i = 1, 2, 3$ , are the Pauli matrices. Besides, if  $\theta_1 = \theta = \theta_2$ , we have the additional solution,

$$B = \sum_{i=1}^3 b_i \sigma_i, \quad \sum_{i=1}^3 b_i^2 = 1 - \left[ \frac{\lambda}{\mu L(\theta)} \right]^2. \quad (2.33)$$

One can decouple the freedom of  $A$  in  $\Gamma$  and  $\Omega$  by casting them into

$$\Gamma = \Sigma^{-1} \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \Sigma = \Sigma^{-1} (I_2 \otimes \sigma_1) \Sigma, \quad (2.34)$$

and

$$\Omega = \Sigma^{-1} \begin{pmatrix} B & iK \\ -iK & -B \end{pmatrix} \Sigma = \Sigma^{-1} (K \otimes \sigma_2 + B \otimes \sigma_3) \Sigma, \quad (2.35)$$

by introducing

$$\Sigma = \begin{pmatrix} I_2 & 0 \\ 0 & A \end{pmatrix}. \quad (2.36)$$

In fact,  $\Sigma$  is a special type of the unitary matrix  $V$  in (2.14) which leaves  $\bar{D}$  in (2.22) invariant,  $\bar{D} \rightarrow \Sigma^{-1} \bar{D} \Sigma = \bar{D}$ , and this invariance provides the freedom  $A$  in the choice of the supercharge  $Q$ .

Given a  $D$  (and hence a  $K$ ), there are at most four ‘independent’ supercharges among those  $Q = q(A, \mu; K)$  in (2.30) for  $A \in U(2)$ , and a set of independent supercharges may be furnished by

$$Q_k = q(i\sigma_k, \mu; K), \quad k = 1, 2, 3, \quad Q_4 = q(I_2, \mu; K). \quad (2.37)$$

Because of the  $\mu$ -term in the supercharges, however, the standard orthogonal SUSY algebra,

$$\{Q_i, Q_j\} = H \delta_{ij}, \quad (2.38)$$

for all  $i, j = 1, \dots, 4$ , cannot be realized unless  $B = 0$ , that is,

$$\theta_1 = \theta = \theta_2 \quad \text{or} \quad \theta_1 = \theta = 2\pi - \theta_2, \quad \mu^2 = \left[ \frac{\lambda}{L(\theta)} \right]^2. \quad (2.39)$$

Thus, if we say that a system has an  $N = n$  SUSY if it has  $n$  supercharges satisfying (2.38), then we see that our system can possess an  $N = 4$  SUSY if the two angles  $\theta_1$  and  $\theta_2$  in the characteristic matrix  $U$  are related by (2.39) and further if we specify the parameter  $\mu$  in the supercharge  $Q$  by (2.39). Otherwise, the system possesses only an  $N = 1$  SUSY. Note that the set of supercharges satisfying the orthogonal relation (2.38) is not unique, since it can always be transformed by  $Q_k \rightarrow \Sigma^{-1} Q_k \Sigma$  with  $\Sigma$  in the form (2.36) leaving the relation intact.

If  $\mu = 0$ , on the other hand, one finds from (2.20) and (2.21) that  $T = I_2$ . This suggests that the SUSY system with a point singularity possessing the standard SUSY algebra without the  $\mu$ -term in (2.38) is basically unique — a SUSY is realized only if the characteristic matrix  $U$  has the diagonal part  $D = \text{diag}(1, 1, -1, -1)$  modulo the possible exchanges of the  $\pm 1$  elements. Systems on a circle with different combinations of these special types of point singularities have been studied in detail in [7].

For illustration, we mention a simple but generic  $N = 1$  case obtained by the diagonal  $U = \bar{D}$  in (2.22). The boundary condition (2.4) then reads

$$\psi_1^+(0) + L(\theta_1)\psi_1^{+\prime}(0) = 0, \quad \psi_1^-(0) + L(\theta_2)\psi_1^{-\prime}(0) = 0, \quad \psi_2^+(0) = \psi_2^-(0) = 0. \quad (2.40)$$

Apart from the travelling wave eigenstates which are four-fold degenerate, one finds the two bound states,

$$\Phi^{(1)}(x) = \begin{pmatrix} e^{-x/L(\theta_1)} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Phi^{(2)}(x) = \begin{pmatrix} 0 \\ e^{-x/L(\theta_2)} \\ 0 \\ 0 \end{pmatrix}, \quad (2.41)$$

which are allowed for  $L(\theta_1) > 0$  and  $L(\theta_2) > 0$ , respectively.

To sum up, we have found that a system of two lines whose singularity is characterized by  $U$  admits an  $N = 1$  SUSY if the ordered diagonal matrix  $\bar{D}$  in the decomposition (2.16) has the form (2.22). The supercharge is provided by

$$Q = V^{-1}S^{-1}q(A, \mu; K)SV, \quad (2.42)$$

with  $q(A, \mu; K)$  given by (2.30). The SUSY can be enhanced to  $N = 4$  if  $B = 0$ , that is, if the conditions (2.39) are fulfilled. These SUSY systems exhibit distinct features depending on the choice of the matrix  $B$  and the angles  $\theta_1$  and  $\theta_2$ . In fact, the parameter dependence of the features can be seen just by observing the example mentioned above, because the energy spectrum depends only on the spectral parameters on account of the conjugations on  $U$  which preserve the spectrum. We classify the SUSY systems into the four types:

- (A1)  $B \neq 0, \quad \theta_1 \neq \theta_2,$
- (A2)  $B \neq 0, \quad \theta_1 = \theta_2,$
- (B1)  $B = 0, \quad L(\theta_1) > 0 \quad \text{or} \quad L(\theta_2) > 0,$
- (B2)  $B = 0, \quad L(\theta_1) \leq 0 \quad \text{and} \quad L(\theta_2) \leq 0.$

In type (A1) and (A2) systems, the bound states (2.41) are mapped into themselves under the SUSY transformation generated by the supercharge (2.30) with the diagonal  $B$  given in (2.32). For type (A2) where the two angles  $\theta_1$  and  $\theta_2$  coincide, one may also choose (2.33) with off-diagonal elements for  $B$  so that the SUSY transformation induces the exchange between  $\Phi^{(1)}$  and  $\Phi^{(2)}$ . In both of these types for which  $B \neq 0$ , therefore, the SUSY is broken. In contrast, the ground state in type (B1), which admits at least one bound state, is annihilated under the SUSY transformation, and hence the SUSY is good (*i.e.*, unbroken). Type (B2) possesses only degenerate positive energy states which are related by the SUSY transformation, and the SUSY is broken. The spectral and SUSY properties for types (A1) – (B2) are listed in the table in Appendix B.

Now we discuss how the previous works [18, 12] on SUSY interval systems with the Dirac  $\delta(x)$ -potentials fit in our general scheme. In order to realize the boundary condition that arises under the Dirac  $\delta(x)$ -potentials with different coupling constants on the two lines, we consider the general ordered diagonal matrix  $\bar{D}$  in (2.22) and choose the conjugation matrix  $V$  and the exchange matrix  $S$  by  $V_1 = V_2 = e^{i\frac{\pi}{4}\sigma_2}$  in (2.14) and  $S = Y$  given in (2.15), respectively. Then we find that the boundary condition (2.4) becomes

$$\psi_i^+(0) + \frac{L(\theta_i)}{2} \left( \psi_i^{+'}(0) + \psi_i^{-'}(0) \right) = 0, \quad \psi_i^+(0) = \psi_i^-(0), \quad i = 1, 2. \quad (2.43)$$

These are indeed the conditions that we find under the potentials  $V(x) = g_i \delta(x)$  with strengths  $g_i = \frac{2\lambda^2}{L(\theta_i)}$ , and hence we see that the pair of line systems having the Dirac  $\delta(x)$ -potentials has an  $N = 1$  SUSY for arbitrary strengths  $g_1$  and  $g_2$ . In particular, if the two strengths are related by  $g_1 = \pm g_2$ , which occur when the angles  $\theta_i$  fulfill (2.39), then one can enhance the SUSY to  $N = 4$  by choosing  $\mu$  as in (2.39). The case discussed in [18, 12, 9] corresponds to  $g_1 = -g_2$ , where the pair of systems is regarded as a Witten model with  $N = 2$ . Our analysis shows, however, that the number of SUSY can be doubled if one takes into the exchange parity operations between the half lines,  $x > 0$  and  $x < 0$ .

### 3. Two intervals with point singularity

In this section, we study systems consisting of two intervals, each given by  $[-l, l]$  with a singular point at  $x = 0$ . As before, we split each of the intervals into two and thereby regard our Hilbert space  $\mathcal{H}$  as  $L^2([0, l]) \otimes \mathbb{C}^4$  (see Fig.2). Our Hamiltonian remains to be the one in (2.2) shifted by  $\mu^2$  so that (2.10) holds. Because of the walls at  $x = l$  we now have, we need to impose, in addition to the boundary condition (2.4) at  $x = 0$ , an extra boundary condition at  $x = l$ . Let  $U \in U(4)$  be the characteristic matrix for the condition at  $x = 0$  given in the block diagonal form (2.7), and similarly  $U_l \in U(4)$  be the characteristic matrix for the condition at  $x = l$ . Since the probability current must vanish at  $x = l$  separately on the branches  $\pm$ , we require

$$j_1^\pm(l) = 0 = j_2^\pm(l). \quad (3.1)$$

Comparing with the previous case (2.6), we realize that the characteristic matrix at the wall is diagonal, *i.e.*,  $U_l = D_l$  where  $D_l$  is of the form (2.13) with  $\theta_i$  being replaced by the corresponding parameter  $\theta_i^l \in [0, 2\pi)$ . With such  $D_l$ , the boundary condition at  $x = l$  is provided by

$$(D_l - 1)\Psi(l) + iL_0(D_l + I)\Psi'(l) = 0. \quad (3.2)$$

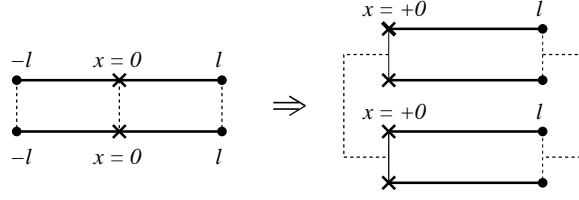
Thus the two interval systems we are considering are characterized by the matrix  $U_{\text{tot}} = U \times D_l \in U(2) \times U(1) \times U(1)$ .

We now find systems that accommodate SUSY. More explicitly, we seek a pair  $(U_{\text{tot}}, Q)$  with  $U_{\text{tot}} = U \times D_l$  and some supercharge  $Q$  for which the boundary conditions both at  $x = 0$  and  $x = l$  are compatible with the SUSY transformations generated by  $Q$ . To this end, we first recall that, if the characteristic matrix  $U$  is decomposed as (2.16), the supercharge  $Q$  compatible with the boundary condition at  $x = 0$  is given by (2.42). Thus, what remains to be seen is under what conditions this supercharge  $Q$  is simultaneously compatible with the boundary condition at  $x = l$ . As we did at the point  $x = 0$  for the diagonal  $D$ , at the point  $x = l$  we also put

$$D_l = S_l^{-1} \bar{D}_l S_l, \quad \bar{D}_l = \text{diag}(e^{i\theta_1^l}, e^{i\theta_2^l}, -1, -1), \quad (3.3)$$

with some appropriate exchange matrix  $S_l$ . Then, by an analogous argument we see that the supercharge compatible with the boundary condition at  $x = l$  is given by  $S_l^{-1} q(A_l, \mu_l; K_l) S_l$ . From the two compatibility conditions at  $x = 0$  and  $x = l$ , one has

$$V^{-1} S^{-1} q(A, \mu; K) S V = Q = S_l^{-1} q(A_l, \mu_l; K_l) S_l, \quad (3.4)$$



**Figure 2.** A system of a pair of two intervals  $[-l, l]$  each having a singularity at  $x = 0$  may be identified with two systems of a pair of two half intervals  $(0, l]$  where the probability flow is allowed to pass between the two systems through  $x = 0$ . The flow is not allowed at the other ends  $x = l$ .

which implies

$$S_l V^{-1} S^{-1} \Gamma S V S_l^{-1} = \Gamma_l, \quad (3.5)$$

and

$$\mu S_l V^{-1} S^{-1} \Omega S V S_l^{-1} = \mu_l \Omega_l. \quad (3.6)$$

The total number of independent supercharges satisfying (3.5) and (3.6) gives the number of the SUSY that the system possesses. This will be seen by the number of the free parameters in  $A$ , and if we have the full, four parameters, *i.e.*, if  $N = 4$ , we may choose  $A = i\sigma_k$  and  $A = I_2$  in (3.4) to furnish a basis of supercharges  $Q_i$ ,  $i = 1, \dots, 4$ , as in (2.37) fulfilling the standard SUSY algebra (2.38).

Because of the exchange matrices  $S$  and  $S_l$ , our analysis becomes involved compared to the previous case. There are, however, two discrete operations which can be used to simplify our arguments. These are the parity,

$$\mathcal{P} : \quad \Psi(x) \longrightarrow (\mathcal{P}\Psi)(x) := \Psi(l - x), \quad (3.7)$$

and the interchange,

$$\mathcal{I}_X : \quad \Psi(x) \longrightarrow (\mathcal{I}_X \Psi)(x) := X\Psi(x), \quad (3.8)$$

where  $X$  is defined in (2.15). If systems that are connected by these discrete operations are regarded to be essentially identical, there remain only the following four types of combinations for the diagonal matrices  $(D, D_l)$ :

- (a)  $D = \text{diag}(e^{i\theta_1}, e^{i\theta_2}, -1, -1), \quad D_l = \text{diag}(e^{i\theta_1^l}, e^{i\theta_2^l}, -1, -1),$
- (b)  $D = \text{diag}(-1, -1, e^{i\theta_1}, e^{i\theta_2}), \quad D_l = \text{diag}(e^{i\theta_1^l}, e^{i\theta_2^l}, -1, -1),$
- (c)  $D = \text{diag}(e^{i\theta_1}, -1, e^{i\theta_2}, -1), \quad D_l = \text{diag}(e^{i\theta_1^l}, e^{i\theta_2^l}, -1, -1),$
- (d)  $D = \text{diag}(e^{i\theta_1}, -1, e^{i\theta_2}, -1), \quad D_l = \text{diag}(e^{i\theta_1^l}, -1, e^{i\theta_2^l}, -1).$

To proceed, except when we examine the implication of the general  $B \neq 0$  case, we shall restrict our analysis to the case  $B = 0$ , that is, we assume that at  $x = 0$  the two angle parameters  $\theta_1$  and  $\theta_2$  satisfy (2.39) and the constant  $\mu$  is chosen to be  $\mu^2 = [\lambda/L(\theta)]^2$ , and that similar conditions hold also at  $x = l$  (for which the angle  $\theta_i^l$  will be used for  $\theta_i$  in (2.39)). This restriction simplifies our argument considerably and ensures that all the allowed (at most four) supercharges satisfy the standard orthogonal SUSY algebra (2.38). Since (3.6) implies  $\mu_l^2 = \mu^2$ , we then have  $|L(\theta_1)| = |L(\theta_2)| = |L(\theta_1^l)| = |L(\theta_2^l)|$ . We thus have only one free scale parameter in all the boundary conditions, and we shall specify it by the angle  $\theta_1 = \theta$ . With respect to the scale  $L(\theta)$  set by  $\theta$ , we introduce the sign functions to the remaining three scale parameters,

$$s_2 = \frac{L(\theta_2)}{L(\theta)}, \quad s_1^l = \frac{L(\theta_1^l)}{L(\theta)}, \quad s_2^l = \frac{L(\theta_2^l)}{L(\theta)}, \quad (3.9)$$

which take either 1 or  $-1$ . Among the possible choices for the signs of  $\mu = \pm\lambda/L(\theta)$  and  $\mu_l = \pm\lambda/L(\theta^l)$  (the choice does not affect the supercharge  $Q$ ), for definiteness we choose  $\mu = \mu_l = +\lambda/L(\theta)$ . With this choice the  $K$  matrix at  $x = 0$  and the corresponding  $K_l$  at  $x = l$  defined similarly as (2.26) become

$$K = \begin{pmatrix} 1 & 0 \\ 0 & s_2 \end{pmatrix}, \quad K_l = \begin{pmatrix} s_1^l & 0 \\ 0 & s_2^l \end{pmatrix}, \quad (3.10)$$

that is, they are proportional to either  $I_2$  or  $\sigma_3$ . In order to solve (3.5) and (3.6), we parametrize  $A$  and  $V_i$ ,  $i = 1, 2$  as

$$A = e^{i\xi} e^{i\frac{\alpha}{2}\sigma_3} e^{i\beta\sigma_2} e^{i\frac{\omega}{2}\sigma_3}, \quad V_i = e^{i\delta_i\sigma_2} e^{i\frac{\tau_i}{2}\sigma_3}, \quad (3.11)$$

with  $\xi, \alpha, \omega, \tau_i \in [0, 2\pi)$  and  $\beta, \delta_i \in [0, \pi)$ , and consider the four types of the combinations, separately.

### 3.1. Type (a) $D = \text{diag}(e^{i\theta_1}, e^{i\theta_2}, -1, -1)$ , $D_l = \text{diag}(e^{i\theta_1^l}, e^{i\theta_2^l}, -1, -1)$

In this case, the rotation matrix  $V_2$  becomes irrelevant for determining the characteristic matrix  $U$ , and we may choose  $V_2 = I_2$  without loss of generality. We then obtain three distinct solutions for (3.5) and (3.6) (for the detail, see Appendix A.1),

$$\begin{aligned} \text{(a1)} \quad & s_2 = s_1^l = s_2^l = 1, & A_l &= V_1^\dagger A, \\ \text{(a2)} \quad & s_1^l = 1, \quad s_2^l = s_2 = -1, \quad \delta_1 = 0, & A_l &= V_1^\dagger A, \\ \text{(a3)} \quad & s_1^l = s_2 = -1, \quad s_2^l = 1, \quad \delta_1 = \pi/2, & A_l &= V_1^\dagger A. \end{aligned}$$

All the above three solutions admit the four supercharges  $Q_i$  in (2.37) and hence the systems possess an  $N = 4$  SUSY.

In order to see what our SUSY systems are in more detail, let us first consider the type (a1) solution which implies the boundary condition,

$$\begin{aligned}\psi_1^\pm(0) + L(\theta)\psi_1^{\pm'}(0) &= 0, & \psi_1^\pm(l) + L(\theta)\psi_1^{\pm'}(l) &= 0, \\ \psi_2^\pm(0) &= 0, & \psi_2^\pm(l) &= 0.\end{aligned}\tag{3.12}$$

For energy eigenstates of the Hamiltonian  $H$  fulfilling (3.12), we find the series of eigenstates,

$$\Phi_n(x) = \begin{pmatrix} N_1(\sin k_n x - L(\theta)k_n \cos k_n x) \\ N_2(\sin k_n x - L(\theta)k_n \cos k_n x) \\ N_3 \sin k_n x \\ N_4 \sin k_n x \end{pmatrix},\tag{3.13}$$

where  $N_i$ ,  $i = 1, \dots, 4$ , are arbitrary constants subject to the normalization condition  $1 = \|\Phi_n\|^2 = \int_0^\infty dx |\Phi_n(x)|^2$ , and we have introduced

$$k_n = \frac{n\pi}{l}, \quad n = 1, 2, \dots\tag{3.14}$$

Consequently, each of the energy levels is four-fold degenerate. In addition, we obtain the doubly degenerate ground states,

$$\Phi_{\text{grd}}(x) = \begin{pmatrix} \tilde{N}_1 e^{-x/L(\theta)} \\ \tilde{N}_2 e^{-x/L(\theta)} \\ 0 \\ 0 \end{pmatrix},\tag{3.15}$$

with vanishing energy  $E_{\text{grd}} = 0$ . These ground states are annihilated by any of the supercharges  $Q_i$ ,  $i = 1, \dots, 4$ , and hence we see that the  $N = 4$  SUSY of type (a1) is good (unbroken).

For type (a2), the boundary condition becomes

$$\begin{aligned}\psi_1^\pm(0) \pm L(\theta)\psi_1^{\pm'}(0) &= 0, & \psi_1^\pm(l) \pm L(\theta)\psi_1^{\pm'}(l) &= 0, \\ \psi_2^\pm(0) &= 0, & \psi_2^\pm(l) &= 0,\end{aligned}\tag{3.16}$$

which admits the energy eigenstates (with  $k_n$  given in (3.14))

$$\Phi_n(x) = \begin{pmatrix} N_1(\sin k_n x - L(\theta)k_n \cos k_n x) \\ N_2(\sin k_n x + L(\theta)k_n \cos k_n x) \\ N_3 \sin k_n x \\ N_4 \sin k_n x \end{pmatrix},\tag{3.17}$$



which are four-fold degenerate. As before, the ground states are

$$\Phi_{\text{grd}}(x) = \begin{pmatrix} \tilde{N}_1 e^{-x/L(\theta)} \\ \tilde{N}_2 e^{x/L(\theta)} \\ 0 \\ 0 \end{pmatrix}, \quad (3.18)$$

with energy  $E_{\text{grd}} = 0$ . Again, these doubly degenerate ground states are annihilated by the supercharges  $Q_i$ , and hence type (a2) provides an  $N = 4$  good SUSY, too. Type (a3) furnishes essentially the same  $N = 4$  good SUSY system as type (a2), except that the upper two components of all the eigenstates  $\Phi(x)$  are interchanged.

**3.2. Type (b)**  $D = \text{diag}(-1, -1, e^{i\theta_1}, e^{i\theta_2}), \quad D_l = \text{diag}(e^{i\theta_1^l}, e^{i\theta_2^l}, -1, -1)$

This time the rotation matrix  $V_1$  is irrelevant for specifying  $U$  and hence we take  $V_1 = I_2$ . As for type (a), we obtain from (3.5) and (3.6) the following three solutions (see Appendix A.2):

$$\begin{aligned} \text{(b1)} \quad & s_2 = s_1^l = -1, \quad s_2^l = 1, \quad \beta = 0, \quad A_l = A^\dagger V_2, \\ \text{(b2)} \quad & s_2 = s_2^l = -1, \quad s_1^l = 1, \quad \beta = \pi/2, \quad A_l = A^\dagger V_2, \\ \text{(b3)} \quad & s_2 = 1, \quad s_1^l = s_2^l = -1, \quad A_l = A^\dagger V_2. \end{aligned}$$

This time, since  $\beta$  is specified in Type (b1) and (b2), these two types admit only two independent supercharges obtained, for instance, by  $A = i\sigma_3$  and  $A = I_2$ , and hence they possess an  $N = 2$  SUSY. Type (b3), on the other hand, admit all the four supercharges  $Q_i$  and hence has an  $N = 4$  SUSY.

For type (b1), the boundary condition becomes

$$\begin{aligned} \psi_1^\pm(0) = 0, \quad \psi_1^\pm(l) \mp L(\theta)\psi_1^{\pm'}(l) = 0, \quad \psi_2^\pm(l) = 0, \\ e^{\pm i\tau_2}\psi_2^\pm(0) + \tan \delta_2 \psi_2^\mp(0) + L(\theta) \left( e^{\pm i\tau_2}\psi_2^{\pm'}(0) \pm \tan \delta_2 \psi_2^{\mp'}(0) \right) = 0. \end{aligned} \quad (3.19)$$

We then find two distinct series of eigenstates; one given by

$$\Phi_n^{(1)}(x) = \begin{pmatrix} N_1 \sin k_n^- x \\ 0 \\ N_2 \cos \delta_2 \sin k_n^-(x-l) \\ N_2 \sin \delta_2 e^{i\tau_2} \sin k_n^-(x-l) \end{pmatrix}, \quad (3.20)$$

and the other by

$$\Phi_n^{(2)}(x) = \begin{pmatrix} 0 \\ N_3 \sin k_n^+ x \\ -N_4 \sin \delta_2 \sin k_n^+(x-l) \\ N_4 \cos \delta_2 e^{i\tau_2} \sin k_n^+(x-l) \end{pmatrix}, \quad (3.21)$$

where discrete  $k_n^\pm = k_n^\pm(\theta) > 0$ ,  $n = 1, 2, 3 \dots$ , are obtained as the solutions of

$$L(\theta)k_n^\pm \pm \tan(k_n^\pm l) = 0. \quad (3.22)$$

For  $0 < L(\theta) < l$  we have the additional eigenstates,

$$\Phi_{\text{grd}}^{(1)}(x) = \begin{pmatrix} \tilde{N}_1 \sinh \kappa^- x \\ 0 \\ \tilde{N}_2 \cos \delta_2 \sinh \kappa^-(x-l) \\ \tilde{N}_2 \sin \delta_2 e^{i\tau_2} \sinh \kappa^-(x-l) \end{pmatrix}, \quad (3.23)$$

while for  $-l < L(\theta) < 0$  we obtain

$$\Phi_{\text{grd}}^{(2)}(x) = \begin{pmatrix} 0 \\ \tilde{N}_3 \sinh \kappa^+ x \\ -\tilde{N}_4 \sin \delta_2 \sinh \kappa^+(x-l) \\ \tilde{N}_4 \cos \delta_2 e^{i\tau_2} \sinh \kappa^+(x-l) \end{pmatrix}, \quad (3.24)$$

where  $\kappa^\pm = \kappa^\pm(\theta) > 0$  are the solutions of

$$L(\theta)\kappa_n^\pm \pm \tanh(\kappa_n^\pm l) = 0. \quad (3.25)$$

These provide the ground state of the system with energy

$$E_{\text{grd}}^\pm = -\frac{\hbar^2(\kappa^\pm)^2}{2m} + \left(\frac{\lambda}{L(\theta)}\right)^2. \quad (3.26)$$

For  $L(\theta) = l$ , the state  $\Phi_{\text{grd}}^{(1)}$  reduces to

$$\Phi_{\text{grd}}^{(1)}(x) = \begin{pmatrix} \bar{N}_1 x \\ 0 \\ \bar{N}_2 \cos \delta_2 (x-l) \\ \bar{N}_2 \sin \delta_2 e^{i\tau_2} (x-l) \end{pmatrix}, \quad (3.27)$$

while for  $L(\theta) = -l$ ,  $\Phi_{\text{grd}}^{(2)}$  reduces to

$$\Phi_{\text{grd}}^{(2)}(x) = \begin{pmatrix} 0 \\ \bar{N}_3 x \\ -\bar{N}_4 \sin \delta_2 (x-l) \\ \bar{N}_4 \cos \delta_2 e^{i\tau_2} (x-l) \end{pmatrix}. \quad (3.28)$$

These eigenstates are all doubly degenerate irrespective of the energy, and are related by the SUSY transformations generated by the two supercharges mentioned above. We also note that the ground state energy is positive  $E_{\text{grd}}^\pm > 0$  and hence the  $N = 2$  SUSY of the

system is broken. Type (b2) is essentially the same as (b1), and we will omit to give its detail here.

For type (b3), the boundary condition is

$$\begin{aligned}\psi_1^\pm(0) &= 0, & \psi_1^\pm(l) - L(\theta)\psi_1^{\pm'}(l) &= 0, \\ \psi_2^\pm(0) + L(\theta)\psi_2^{\pm'}(0) &= 0, & \psi_2^\pm(l) &= 0.\end{aligned}\tag{3.29}$$

The energy eigenstates are

$$\Phi_n(x) = \begin{pmatrix} N_1 \sin k_n^- x \\ N_2 \sin k_n^- x \\ N_3 \sin k_n^-(x-l) \\ N_4 \sin k_n^-(x-l) \end{pmatrix},\tag{3.30}$$

where  $k_n^- > 0$  are given in (3.22). Besides, for  $0 < L(\theta) < l$ , we have the ground states provided by (3.30) with the replacement  $k_n^- \rightarrow i\kappa^-$ , that is,  $\kappa^- > 0$  is the solution of (3.25). Further, if  $L(\theta) = l$ , the states (3.30) with the sine functions formally replaced by their arguments  $\sin z \rightarrow z$  become eigenstates. All of these eigenstates are four-fold degenerate, and they are related by the SUSY transformations generated by  $Q_i$ . As before, the  $N = 4$  SUSY is broken in the system.

At this point, we briefly mention the  $B \neq 0$  case for which the supercharges (2.30) do not necessarily satisfy the standard orthogonal SUSY algebra (2.38). For simplicity, we only consider the special case where we have  $V_1 = V_2 = I_2$  and choose  $B$  by (2.32). Here we can readily solve (3.5) and (3.6) to obtain, for instance, the solution,

$$\begin{aligned}L(\theta_1^l) &= -L(\theta_1), & L(\theta_2^l) &= -L(\theta_2), & \beta &= \beta_l = 0, \\ \alpha_l + \omega_l &= \alpha + \omega, & \xi_l &= \xi, & \mu_l &= \pm\mu, & b_1^l &= \mp b_1, & b_2^l &= \mp b_2.\end{aligned}\tag{3.31}$$

The corresponding supercharges may then be provided by  $q(I_2, \mu; K)$  and  $q(i\sigma_3, \mu; K)$ , and the boundary condition becomes

$$\begin{aligned}\psi_1^\pm(0) &= 0, & \psi_1^\pm(l) - L(\theta_1)\psi_1^{\pm'}(l) &= 0, \\ \psi_2^\pm(0) + L(\theta_1)\psi_2^{\pm'}(0) &= 0, & \psi_2^\pm(l) &= 0.\end{aligned}\tag{3.32}$$

Because of the two scale parameters  $L(\theta_1)$  and  $L(\theta_2)$  we have now, the energy eigenstates consist of the two series,

$$\Phi_n^{(1)}(x) = \begin{pmatrix} N_1 \sin k_n^-(\theta_1)x \\ 0 \\ N_3 \sin k_n^-(\theta_1)(x-l) \\ 0 \end{pmatrix}, \quad \Phi_n^{(2)}(x) = \begin{pmatrix} 0 \\ N_2 \sin k_n^-(\theta_2)x \\ 0 \\ N_4 \sin k_n^-(\theta_2)(x-l) \end{pmatrix},\tag{3.33}$$

where the discrete  $k_n^-(\theta_i) > 0$ ,  $i = 1, 2$ , are given by (3.22) with  $\theta = \theta_i$ . Similarly, for  $0 < L(\theta_i) < l$  there arise ground states obtained by the replacement  $k_n^- \rightarrow ik_n^-$  in (3.33) with energy  $E_{\text{grd}} = -\hbar^2[\kappa^-(\theta_i)]^2/(2m) + [\lambda/L(\theta_i)]^2 > 0$ . As the scale parameters approach  $L(\theta_i) = l$  these solutions reduce to states analogous to (3.27) or (3.28). All of these states are doubly degenerate and are related by the SUSY transformations generated by the two supercharges, implying that the  $N = 2$  SUSY of the system is broken.

In the examples discussed above, we observe that the spectrum of the system consists of one or two ‘regular’ series of levels specified by  $k_n$  or  $k_n^\pm$  with integers  $n$ , plus a few ‘isolated’ levels some of which may become the ground states. Obviously, these correspond to states with positive and negative energies, respectively, if the constant energy shift in the Hamiltonian is absent. The same spectral composition will be seen in all the examples we shall discuss below.

**3.3. Type (c)**  $D = \text{diag}(e^{i\theta_1}, -1, e^{i\theta_2}, -1)$ ,  $D_l = \text{diag}(e^{i\theta_1^l}, e^{i\theta_2^l}, -1, -1)$

The conditions (3.5) and (3.6) are met again by three distinct types of solutions (see Appendix A.3):

$$\begin{aligned}
\text{(c1)} \quad & s_2 = s_1^l = 1, \quad s_2^l = -1, \quad \delta_1 = 0, \quad \beta = \pi/2, \quad \beta_l = \pi/2 - \delta_2, \quad \omega_l = \tau_2 \pm \pi, \\
& \xi_l = (\alpha - \omega)/2, \quad \alpha_l = 2\xi - \tau_1 \pm \pi \quad \text{or} \quad \xi_l = (\alpha - \omega)/2 + \pi, \quad \alpha_l = 2\xi - \tau_1 \mp \pi, \\
\text{(c2)} \quad & s_2 = s_2^l = 1, \quad s_1^l = -1, \quad \delta_1 = \pi/2, \quad \beta = \pi/2, \quad \beta_l = \delta_2, \quad \omega_l = \tau_2, \\
& \xi_l = (\alpha - \omega)/2, \quad \alpha_l = -2\xi - \tau_1 \quad \text{or} \quad \xi_l = (\alpha - \omega)/2 + \pi, \quad \alpha_l = 2\pi - 2\xi - \tau_1, \\
\text{(c3)} \quad & s_2 = -1, \quad s_1^l = s_2^l = 1, \quad \beta = \pi/2,
\end{aligned}$$

where in (c3) the remaining parameters  $(\xi_l, \alpha_l, \beta_l, \omega_l)$  and  $(\xi, \alpha, \omega, \delta_i, \tau_i)$  are determined from (3.5). All of these types have an  $N = 2$  SUSY, because they possess two independent supercharges. Explicitly, they may be chosen to be the pairs,  $q(e^{i(\delta_2 - \frac{\pi}{2})\sigma_2} e^{i\frac{\tau_2}{2}\sigma_3}, \mu; K)$  and  $q(i\sigma_3 e^{i(\delta_2 - \frac{\pi}{2})\sigma_2} e^{i\frac{\tau_2}{2}\sigma_3}, \mu; K)$  for (c1),  $q(e^{i\delta_2\sigma_2} e^{i\frac{\tau_2}{2}\sigma_3}, \mu; K)$  and  $q(i\sigma_3 e^{i\delta_2\sigma_2} e^{i\frac{\tau_2}{2}\sigma_3}, \mu; K)$  for (c2), and  $V^{-1}Y^{-1}q(i\sigma_1, \mu; K)YV$  and  $V^{-1}Y^{-1}q(i\sigma_2, \mu; K)YV$  for (c3), respectively.

For type (c1), the boundary condition reads

$$\begin{aligned}
& \psi_1^+(0) + L(\theta)\psi_1^{+\prime}(0) = 0, \quad \psi_1^-(0) = 0, \quad \psi_1^\pm(l) \pm L(\theta)\psi_1^{\pm\prime}(l) = 0, \\
& e^{i\tau_2}\psi_2^+(0) + \tan \delta_2\psi_2^-(0) + L(\theta) \left( e^{i\tau_2}\psi_2^{+\prime}(0) + \tan \delta_2\psi_2^{-\prime}(0) \right) = 0, \quad (3.34) \\
& e^{-i\tau_2}\psi_2^-(0) - \tan \delta_2\psi_2^+(0) = 0, \quad \psi_2^\pm(l) = 0,
\end{aligned}$$

which admits two regular series of eigenstates given by

$$\Phi_n^{(1)}(x) = \begin{pmatrix} N_1(\sin k_n x - L(\theta)k_n \cos k_n x) \\ 0 \\ -N_2 \sin \delta_2 \sin k_n(x-l) \\ N_2 \cos \delta_2 e^{i\tau_2} \sin k_n(x-l) \end{pmatrix}, \quad (3.35)$$

and

$$\Phi_n^{(2)}(x) = \begin{pmatrix} 0 \\ N_3 \sin k_n^- x \\ N_4 \cos \delta_2 \sin k_n^-(x-l) \\ N_4 \sin \delta_2 e^{i\tau_2} \sin k_n^-(x-l) \end{pmatrix}. \quad (3.36)$$

For  $0 < L(\theta) < l$ , we have the isolated eigenstate

$$\Phi^{(1)}(x) = \Phi_{\text{grd}}(x) = \begin{pmatrix} \tilde{N}_1 e^{-x/L(\theta)} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (3.37)$$

which is the ground state with  $E_{\text{grd}} = 0$  and

$$\Phi^{(2)}(x) = \begin{pmatrix} 0 \\ \tilde{N}_3 \sinh \kappa^{(2)} x \\ \tilde{N}_4 \cos \delta_2 \sinh \kappa^{(2)}(x-l) \\ \tilde{N}_4 \sin \delta_2 e^{i\tau_2} \sinh \kappa^{(2)}(x-l) \end{pmatrix}, \quad (3.38)$$

with  $E = -\hbar^2(\kappa^-)^2/(2m) + [\lambda/L(\theta)]^2 > 0$ . For  $L(\theta) = l$ , this reduces to

$$\Phi^{(2)}(x) = \begin{pmatrix} 0 \\ \tilde{N}_3 x \\ \tilde{N}_4 \cos \delta_2 (x-l) \\ \tilde{N}_4 \sin \delta_2 e^{i\tau_2} (x-l) \end{pmatrix}. \quad (3.39)$$

These eigenstates may be classified into the two series; one is a ‘good SUSY series’ given by (3.35) and (3.37) and the other is a ‘broken SUSY series’ given by (3.36), (3.38) and (3.39).

Type (c2) provides a system which is analogous to (c1), and we shall not present the content of the system. For type (c3), on the other hand, the boundary condition is given by

$$\begin{aligned} e^{i\tau_1} \psi_1^+(0) + \tan \delta_1 \psi_1^-(0) + L(\theta) \left( e^{i\tau_1} \psi_1^{+\prime}(0) + \tan \delta_1 \psi_1^{-\prime}(0) \right) &= 0, \\ e^{-i\tau_1} \psi_1^-(0) - \tan \delta_1 \psi_1^+(0) &= 0, & \psi_1^\pm(l) + L(\theta) \psi_1^{\pm\prime}(l) &= 0, \\ e^{i\tau_2} \psi_2^+(0) + \tan \delta_2 \psi_2^-(0) - L(\theta) \left( e^{i\tau_2} \psi_2^{+\prime}(0) + \tan \delta_2 \psi_2^{-\prime}(0) \right) &= 0, \\ e^{-i\tau_2} \psi_2^-(0) - \tan \delta_2 \psi_2^+(0) &= 0, & \psi_2^\pm(l) &= 0. \end{aligned} \quad (3.40)$$

The eigenstates are then

$$\Phi_n^{(1)}(x) = \begin{pmatrix} N_1 \cos \delta_1 (\sin k_n x - L(\theta) k_n \cos k_n x) \\ N_1 \sin \delta_1 e^{i\tau_1} (\sin k_n x - L(\theta) k_n \cos k_n x) \\ -N_2 \sin \delta_2 \sin k_n (x - l) \\ N_2 \cos \delta_2 e^{i\tau_2} \sin k_n (x - l) \end{pmatrix}, \quad (3.41)$$

and

$$\Phi_n^{(2)}(x) = \begin{pmatrix} -N_3 \sin \delta_1 \sin k_n^+ x \\ N_3 \cos \delta_1 e^{i\tau_1} \sin k_n^+ x \\ N_4 \cos \delta_2 \sin k_n^+ (x - l) \\ N_4 \sin \delta_2 e^{i\tau_2} \sin k_n^+ (x - l) \end{pmatrix}. \quad (3.42)$$

We have the isolated eigenstate

$$\Phi^{(1)}(x) = \Phi_{\text{grd}}(x) = \begin{pmatrix} \tilde{N}_1 \cos \delta_1 e^{-x/L(\theta)} \\ \tilde{N}_1 \sin \delta_1 e^{i\tau_1} e^{-x/L(\theta)} \\ 0 \\ 0 \end{pmatrix}, \quad (3.43)$$

which is the ground state with  $E_{\text{grd}} = 0$ . Also, for  $-l < L(\theta) < 0$ , we have additionally

$$\Phi^{(2)}(x) = \begin{pmatrix} -\tilde{N}_3 \sin \delta_1 \sinh \kappa^+ x \\ \tilde{N}_3 \cos \delta_1 e^{i\tau_1} \sinh \kappa^+ x \\ \tilde{N}_4 \cos \delta_2 \sinh \kappa^+ (x - l) \\ \tilde{N}_4 \sin \delta_2 e^{i\tau_2} \sinh \kappa^+ (x - l) \end{pmatrix}, \quad (3.44)$$

with  $E = -\hbar^2(\kappa^+)^2/(2m) + [\lambda/L(\theta)]^2 > 0$ . For  $L(\theta) = -l$ , this reduces to a state obtained similarly as (3.39) from (3.38). Again, these eigenstates are classified into two series; one is a good SUSY series given by (3.41) and (3.43) and the other is a broken SUSY series given by (3.42) and (3.44). For the special choice  $\delta_1 = \delta_2 = \pi/4$  and  $\tau_1 = \tau_2 = 0$ , the resultant system (formulated in the original Hilbert space  $\mathcal{H} = L^2([-l, l]) \otimes \mathbb{C}^2$ ) turns out to be the SUSY system obtained under the attractive and repulsive pair of the Dirac  $\delta(x)$ -potentials  $V(x) = \mp \frac{2\lambda^2}{L(\theta)} \delta(x)$ .

**3.4. Type (d)**  $D = \text{diag}(e^{i\theta_1}, -1, e^{i\theta_2}, -1)$ ,  $D_l = \text{diag}(e^{i\theta_1^l}, -1, e^{i\theta_2^l}, -1)$

We here find from (3.5) and (3.6) six distinct types of solutions (see Appendix A.4).

Among them, four are

$$\begin{aligned}
\text{(d1)} \quad & s_1^l = 1, \quad s_2^l = -s_2, \quad \delta_1 = 0, \quad \delta_2 = \pi/2, \quad \beta = \beta_l = 0, \\
& \xi_l = (\alpha + \omega - \tau_1 + \tau_2 \pm \pi/2)/2, \quad \alpha_l + \omega_l = 2\xi - \tau_1 + \tau_2 \mp \pi/2, \\
\text{(d2)} \quad & s_1^l = -1, \quad s_2^l = s_2, \quad \delta_1 = \pi/2, \quad \delta_2 = 0, \quad \beta = \beta_l = 0, \\
& \xi_l = -(\alpha + \omega + \tau_1 + \tau_2 \mp \pi/2)/2, \quad \alpha_l + \omega_l = -2\xi - \tau_1 + \tau_2 \pm \pi/2, \\
\text{(d3)} \quad & s_1^l = 1, \quad s_2 = s_2^l = -1, \quad \delta_1 = \delta_2 \neq 0 \text{ or } \pi/2, \quad \beta = \beta_l = \pi/2, \\
& \xi = 0 \text{ or } \pi, \quad \xi_l = -(\tau_1 + \tau_2)/2 + \xi, \quad \alpha_l - \omega_l = \alpha - \omega, \\
\text{(d4)} \quad & s_1^l = 1, \quad s_2 = s_2^l = -1, \quad \delta_2 = \pi - \delta_1 \neq \pi/2, \quad \beta = \beta_l = \pi/2, \\
& \xi = \pi/2 \text{ or } 3\pi/2, \quad \xi_l = -(\tau_1 + \tau_2)/2, \quad \alpha_l - \omega_l = \alpha - \omega.
\end{aligned}$$

All of these have an  $N = 2$  SUSY, where the supercharges may be given by  $Y^{-1}q(I_2; K)Y$  and  $Y^{-1}q(i\sigma_3; K)Y$  for type (d1) and (d2), or by  $Y^{-1}q(ie^{-i(\tau_1+\tau_2)/2}\sigma_1; K)Y$  and  $Y^{-1}q(ie^{-i(\tau_1+\tau_2)/2}\sigma_2; K)Y$  for type (d3) and (d4). The remaining two are

$$\begin{aligned}
\text{(d5)} \quad & s_1^l = 1, \quad s_2^l = s_2, \quad \delta_1 = \delta_2 = 0, \quad \beta_l = \beta, \quad \xi_l = \xi - (\tau_1 + \tau_2)/2, \\
& \alpha_l = \alpha - (\tau_1 - \tau_2)/2, \quad \omega_l = \omega - (\tau_1 - \tau_2)/2, \\
\text{(d6)} \quad & s_2 = 1, \quad s_1^l = s_2^l = -1, \quad \delta_1 = \delta_2 = \pi/2, \quad \beta_l = \beta, \quad \xi_l = -\xi - (\tau_1 + \tau_2)/2, \\
& \alpha_l = -\omega - (\tau_1 - \tau_2)/2 \pm \pi/2, \quad \omega_l = -\alpha - (\tau_1 - \tau_2)/2 \pm \pi/2,
\end{aligned}$$

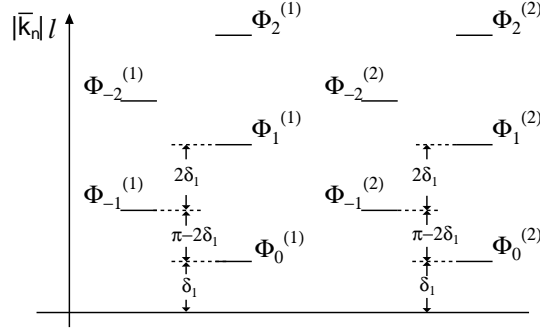
which have four supercharges  $Y^{-1}Q_i Y$ ,  $i = 1, 2, 3, 4$ , and hence possess an  $N = 4$  SUSY.

For types (d1) and (d2), one observes that under the SUSY transformations the eigenstates exchange either the upper two or the lower two components. This implies that these systems are essentially the sum of two disconnected single lines with a point singularity, and hence reduce to the systems considered earlier in [6]. Type (d3), on the other hand, provides a novel SUSY system, and we here mention only the simple case  $\theta = 0$ . The boundary condition then becomes

$$\begin{aligned}
e^{-i\tau_i}\psi_i^-(0) - \tan \delta_1 \psi_i^+(0) &= 0, & e^{i\tau_i}\psi_i^{+\prime}(0) + \tan \delta_1 \psi_i^{-\prime}(0) &= 0, \\
\psi_i^{+\prime}(l) &= 0, & \psi_i^-(l) &= 0, \quad i = 1, 2,
\end{aligned} \tag{3.45}$$

and the eigenstates are

$$\Phi_n^{(1)}(x) = N_1 \begin{pmatrix} -\cos \bar{k}_n(x-l) \\ e^{i\tau_1} \sin \bar{k}_n(x-l) \\ 0 \\ 0 \end{pmatrix}, \quad \Phi_n^{(2)}(x) = N_2 \begin{pmatrix} 0 \\ 0 \\ -\cos \bar{k}_n(x-l) \\ e^{i\tau_2} \sin \bar{k}_n(x-l) \end{pmatrix}, \tag{3.46}$$



**Figure 3.** Energy levels in the  $N = 2$  SUSY system of type (d3) for  $\theta = 0$ . All the levels are doubly degenerate unless  $\delta_1 = 0$  or  $\pi/2$ .

where  $\bar{k}_n = k_n + \delta_1/l$  for  $n \in \mathbb{Z}$ . Unless  $\delta_1 = 0$  or  $\pi/2$ , all eigenstates are doubly degenerate and related under SUSY transformations generated by the two supercharges, implying that the  $N = 2$  SUSY is broken. For  $\delta_1 = 0$ , except for the ground states which are doubly degenerate, all excited states are four-fold degenerate, and for  $\delta_1 = \pi/2$ , all eigenstates are four-fold degenerate (see Fig.3).

Type (d4) has the boundary condition

$$\begin{aligned} \psi_1^+(0) + L(\theta)\psi_1^{+'}(0) = 0, \quad \psi_1^+(l) + L(\theta)\psi_1^{+'}(l) = 0, \quad \psi_1^-(0) = \psi_1^-(l) = 0, \\ s_2\psi_2^+(0) + L(\theta)\psi_2^{+'}(0) = 0, \quad s_2\psi_2^+(l) + L(\theta)\psi_2^{+'}(l) = 0, \quad \psi_2^-(0) = \psi_2^-(l) = 0, \end{aligned} \quad (3.47)$$

and the regular series of eigenstates are

$$\Phi_n(x) = \begin{pmatrix} N_1(\sin k_n x - L(\theta)k_n \cos k_n x) \\ N_2 \sin k_n x \\ N_3(\sin k_n x - s_2 L(\theta)k_n \cos k_n x) \\ N_4 \sin k_n x \end{pmatrix}, \quad (3.48)$$

The ground states are then found to be

$$\Phi_{\text{grd}}(x) = \begin{pmatrix} \tilde{N}_1 e^{-x/L(\theta)} \\ 0 \\ \tilde{N}_2 e^{-s_2 x/L(\theta)} \\ 0 \end{pmatrix}, \quad (3.49)$$

with  $E_{\text{grd}} = 0$ . The energy levels in the regular series are four-fold degenerate, while the ground states are doubly degenerate. The ground states are annihilated by the four supercharges and, hence, the  $N = 4$  SUSY is good. When  $\theta = 0$ , the system becomes equivalent to one given by  $\delta_1 = 0$  in (3.46).



For type (d6), the boundary condition reads

$$\psi_i^+(0) = 0, \quad \psi_i^-(0) + L(\theta)\psi_i^{-\prime}(0) = 0, \quad \psi_i^+(l) - L(\theta)\psi_i^{+\prime}(l) = 0, \quad \psi_i^-(l) = 0, \quad (3.50)$$

for  $i = 1, 2$ , and the regular eigenstates are

$$\Phi_n(x) = \begin{pmatrix} N_1 \sin k_n^- x \\ N_2 \sin k_n^- (x - l) \\ N_3 \sin k_n^- x \\ N_4 \sin k_n^- (x - l) \end{pmatrix}. \quad (3.51)$$

As before, for  $0 < L(\theta) < l$ , isolated degenerate eigenstates are obtained by  $k_n^- \rightarrow i\kappa^-$  in (3.51) with  $E > 0$ . All eigenstates, including the ground states, are four-fold degenerate and related by the SUSY transformations, and hence the  $N = 4$  SUSY is broken. For  $\theta = 0$ , the system coincides with one given by  $\delta_1 = \pi/2$  in (3.46). These types of systems (a1) – (d6) discussed above are listed in Appendix B.

Finally, we mention that a yet further extension of SUSY systems from those considered in the present paper may be realized for restricted systems by incorporating the possibility of separate flips of components of states. Namely, for a state  $\Psi(x) = (\psi_1^+(x), \psi_1^-(x), \psi_2^+(x), \psi_2^-(x))^T$  we may consider a number of discrete transformations which flip each of the four components, separately. One of them is defined by

$$\mathcal{F}_1^+ : \quad \Psi(x) \longrightarrow (\mathcal{F}_1^+ \Psi)(x) = \begin{pmatrix} \psi_1^+(l - x) \\ \psi_1^-(x) \\ \psi_2^+(x) \\ \psi_2^-(x) \end{pmatrix}. \quad (3.52)$$

This and other similarly defined flip operators,  $\mathcal{F}_1^-$ ,  $\mathcal{F}_2^+$  and  $\mathcal{F}_2^-$ , are well-defined for systems for which no probability current flow is allowed at the singularities. Such systems occur when the characteristic matrix  $U$  is diagonal  $U = D$ , namely, when  $U_{\text{tot}} = D \times D_l \in [U(1)]^4$  for which the four operators, which fulfill  $[\mathcal{F}_i^\pm]^2 = \text{id}_{\mathcal{H}}$ , induce the transformations  $U_{\text{tot}} \rightarrow U_{\text{tot}}^{\mathcal{F}_i^\pm}$  where  $U_{\text{tot}}^{\mathcal{F}_i^\pm}$  is given by the exchange of the corresponding diagonal components between  $D$  and  $D_l$ . Now, if the pair  $(U_{\text{tot}}, Q)$  is a SUSY system, then clearly the new pair  $(U_{\text{tot}}^{\mathcal{F}_i^\pm}, \mathcal{F}_i^\pm Q [\mathcal{F}_i^\pm]^{-1})$  provides also a SUSY system. For example, if we implement this extension to type (b3), we find three novel SUSY systems characterized by

$$\begin{aligned} D^{(1)} &= \text{diag}(e^{i\theta}, -1, e^{i\theta}, e^{i\theta}), & D_l^{(1)} &= \text{diag}(-1, e^{-i\theta}, -1, -1), \\ D^{(2)} &= \text{diag}(e^{i\theta}, e^{i\theta}, e^{i\theta}, e^{i\theta}), & D_l^{(2)} &= \text{diag}(-1, -1, -1, -1), \\ D^{(3)} &= \text{diag}(e^{i\theta}, e^{i\theta}, -1, e^{i\theta}), & D_l^{(3)} &= \text{diag}(-1, -1, e^{-i\theta}, -1), \end{aligned} \quad (3.53)$$

where we have used  $\mathcal{F}_1^+$  for the first, and the combinations  $\mathcal{F}_1^+ \mathcal{F}_1^-$  and  $\mathcal{F}_1^+ \mathcal{F}_1^- \mathcal{F}_2^+$  for the second and the third, respectively. The spectral properties of these are, of course, the same as the original systems.

#### 4. Self-adjointness of the supercharge

In this section we wish to address the question of the self-adjointness of the supercharge  $Q = q(A, \mu; K)$  in (2.30) for systems of two intervals discussed in section 3. If  $Q$  is a self-adjoint operator, then for any state  $\Psi(x)$  belonging to its domain  $\Psi \in \mathcal{D}(Q) \subset \mathcal{H} \simeq L^2((0, l]) \otimes \mathbb{C}^4$  we have

$$\int_0^l \Psi^\dagger(x) (Q\Psi)(x) dx = \int_0^l (Q\Psi)^\dagger(x) \Psi(x) dx, \quad (4.1)$$

and also  $\mathcal{D}(Q^\dagger) = \mathcal{D}(Q)$  for the adjoint  $Q^\dagger$  of  $Q$ . Since the  $\mu$ -term in the supercharge (2.30) is regular, it drops out from the above condition (4.1) leaving only the first derivative term there. We may thus consider the simpler supercharge  $Q = -i\lambda \frac{d}{dx} \otimes \Gamma$  in finding possible domains for  $\mathcal{D}(Q)$  below, based on the theory of self-adjoint extension [1].

To start, let us consider an operator  $Q_0$  given in the same differential form as  $Q$  but defined on the domain

$$\mathcal{D}(Q_0) = \{ \Psi \mid \Psi(x) \in \text{AC}[(0, l]) \otimes \mathbb{C}^4, (Q_0\Psi)(x) \in \mathcal{H}, \Psi(0) = \Psi(l) = 0 \}, \quad (4.2)$$

where  $\text{AC}[(0, l])$  is the space of absolutely continuous functions on  $(0, l]$ . Clearly, its adjoint operator  $Q_0^\dagger$  has also the same operator form as  $Q$  and has the domain  $\mathcal{D}(Q_0^\dagger) = \{ \eta \mid \eta(x) \in \mathcal{H}, (Q_0^\dagger\eta)(x) \in \mathcal{H} \}$ . Now we consider eigenvectors  $v_m$  (with eigenvalues  $a_m$ ) of  $\Gamma$ , *i.e.*,  $\Gamma v_m = a_m v_m$  for  $m = 1 \dots, 4$ , and thereby decompose any state  $\Psi(x)$  as

$$\Psi(x) = \sum_{m=1}^4 g_m(x) v_m, \quad (4.3)$$

where  $g_m(x)$  are coefficient functions. To implement the programme of self-adjoint extension for  $Q_0$ , one needs to find the solutions for  $Q_0^\dagger \Psi_{\pm i} = -i\lambda \frac{d}{dx} \otimes \Gamma \Psi_{\pm i}(x) = \pm i \Psi_{\pm i}(x)$ , which, under the decomposition (4.3), becomes

$$-i\lambda \sum_{m=1}^4 a_m \frac{d}{dx} g_m(x) v_m = \pm i \sum_{m=1}^4 g_m(x) v_m. \quad (4.4)$$

Since the four eigenvectors are independent, the equation (4.4) must hold for each  $m$ , and consequently we obtain  $g_m(x) = e^{\mp x/(\lambda a_m)}$  or

$$\Psi_{\pm i}^{(m)}(x) = e^{\mp x/(\lambda a_m)} v_m, \quad m = 1, \dots, 4. \quad (4.5)$$

In view of (2.34), we find  $\det(\Gamma - aI) = (a + 1)^2(a - 1)^2$  and hence the eigenvalues of  $\Gamma$  are  $\pm 1$  (both doubly degenerate). Accordingly, the deficiency indices are found to be  $(4, 4)$ , implying that the supercharge  $Q$  admits a  $U(4)$  parameter family of self-adjoint domains for systems of two intervals.

The appearance of the  $U(4)$  family may be understood directly from the condition (4.1) which reads

$$\Psi^\dagger(l)\Gamma\Psi(l) - \Psi^\dagger(0)\Gamma\Psi(0) = 0. \quad (4.6)$$

Exploiting the freedom of conjugation by  $\Sigma$ ,  $S$  and  $V$  (and also  $\mathcal{F}_i^\pm$ , if necessary), we may take  $A = I_2$  in  $\Gamma$  with no loss of generality. Then we see by an argument similar to reach (2.4) that (4.6) is ensured if the state satisfies the boundary condition at the ends of the intervals,

$$(U - I)\Psi_- + c_0(U + I)\Psi_+ = 0, \quad U \in U(4), \quad (4.7)$$

where we have introduced  $\Psi_- = (\Psi_1(0), -\Psi_1(l))^T$ ,  $\Psi_+ = (\Psi_2(0), \Psi_2(l))^T$  and a dimensionless real constant  $c_0 \neq 0$ . The choice of the boundary condition is indeed specified by the matrix  $U$  belonging to  $U(4)$ .

For systems of two lines, on the other hand, the condition for  $Q$  to be self-adjoint arises only from the contribution at  $x = 0$  in the foregoing argument. Thus the deficiency indices of the operator  $Q$  become  $(2, 2)$  (since half of the eigenfunctions (4.5) are no longer square integrable), and hence one obtains a  $U(2)$  family of the self-adjoint domains for the supercharge  $Q$ . These are characterized by the boundary condition (4.7) restricted to the half subspace associated with  $x = 0$ , where the group for the matrix  $U$  reduces to  $U(2)$ .

In seeking a SUSY system in the preceding sections, we have tacitly assumed that the supercharge is defined, at least, on eigenstates of some, self-adjoint Hamiltonian. In regard to this, it is assuring and also interesting to observe that, in fact, the domain of any self-adjoint Hamiltonian is contained in some domain of a self-adjoint supercharge. To see this, for brevity we consider only the conditions associated with the endpoint  $x = 0$  for a general characteristic matrix  $U \in U(4)$ , since an analogous argument applied to the other endpoint  $x = l$ .

Now, given a  $U$ , we choose some orthonormal set of eigenvectors  $f_i$ ,  $i = 1, \dots, 4$ , of  $U$ , that is,  $Uf_i = e^{i\theta_i}f_i$  with the vectorial inner-product  $\langle f_i, f_j \rangle = \delta_{ij}$ . In terms of these, we consider the decomposition of the boundary vectors  $\Psi(0) = \sum_{i=1}^4 \langle f_i, \Psi(0) \rangle f_i$  and  $\Psi'(0) = \sum_{i=1}^4 \langle f_i, \Psi'(0) \rangle f_i$ . The boundary condition (2.4) then becomes

$$\sum_{i=1}^4 [(e^{i\theta_i} - 1)\langle f_i, \Psi(0) \rangle + iL_0(e^{i\theta_i} + 1)\langle f_i, \Psi'(0) \rangle] f_i = 0. \quad (4.8)$$

Since the eigenvectors are independent each other, the condition (4.8) must hold for each eigenvector, separately. But since a SUSY system has  $-1$  for two eigenvalues, which we choose  $e^{i\theta_3}$  and  $e^{i\theta_4}$ , we obtain

$$\langle f_i, \Psi(0) \rangle = 0, \quad \text{for } i = 3, 4. \quad (4.9)$$

To get a condition for other eigenvectors,  $f_1$  and  $f_2$ , we replace  $D$  with the general  $U$  in (2.20) (note that the diagonalization exploited in section 2 may not be available in the interval systems) and multiply (2.20) by  $f_i$  from the right, and similarly (2.20) by  $f_j^\dagger U^\dagger$  from the left to find

$$f_j^\dagger \Gamma f_i = 0, \quad \text{for } i, j = 1, 2. \quad (4.10)$$

We then see that the second term in (4.6), decomposed similarly in terms of the eigenvectors, becomes

$$\Psi^\dagger(0) \Gamma \Psi(0) = \sum_{i,j=1}^4 \langle f_i, \Psi(0) \rangle \langle \Psi(0), f_j \rangle f_j^\dagger \Gamma f_i = 0. \quad (4.11)$$

on account of the identities (4.9) and (4.10). By a similar argument, the first term in (4.6) can also be seen to vanish.

It is important to recognize that (4.10) provides a relation between the characteristic matrix  $U$  and the corresponding supercharge  $Q$ , whereas (4.9) furnishes the self-adjoint boundary condition for the supersymmetric Hamiltonian  $H$ . (For intervals we need to add extra conditions at  $x = l$  analogously.) Thus, what we have seen here is that the combination of the two conditions are sufficient to ensure that the supercharge  $Q$  be self-adjoint, namely, the domain of such  $H$  is contained in the domain of a self-adjoint  $Q$ .

## 5. Conclusion and discussions

In the present paper we have studied the possibility of SUSY in systems consisting of a pair of lines/intervals each of which possesses a point singularity. These two point singularities are in general different and can be specified by the matrix  $U \in U(2) \times U(2)$  given in (2.7). The line systems are thus specified by such  $U$  and found to possess an  $N = 1$  SUSY if the matrix  $U$ , when properly diagonalized into  $\bar{D}$ , takes the form (2.22). The SUSY is enhanced to  $N = 4$  if  $B = 0$ , *i.e.*, if the two angle parameters  $\theta_1$  and  $\theta_2$  in  $\bar{D}$  and the constant  $\mu$  in the supercharge  $Q$  satisfy (2.39). The SUSY is broken except for a restricted class of point singularities allowing for supercharges with  $B = 0$ . To specify the interval systems, besides the matrix  $U$  we further need an extra matrix  $D_l \in [U(1)]^4$  that characterizes the walls at the two ends. Exploiting the freedoms in the two matrices, we have found various types of SUSY systems which have either (good or broken)  $N = 2$  or  $N = 4$  SUSY. These newly found SUSY systems include the known SUSY system with the Dirac  $\delta(x)$ -potential as a special case of type (B1) for line systems and also provide a similar example for interval systems as type (c3). The spectra and the SUSY properties for all of these SUSY systems are summarized in the table in Appendix B.

One of the important points in our analysis is that our supercharge  $Q$  in (2.8) has the  $\mu$ -term in addition to the conventional derivative term. This  $\mu$ -term allows us to acquire the variety of the SUSY systems having two independent scale parameters  $L(\theta_1)$  and  $L(\theta_2)$ , at the expense of the constant shift of energy by  $\mu^2$  in the Hamiltonian for realizing the standard SUSY algebra (2.10). Without the  $\mu$ -term, we obtain only a restricted class of SUSY systems given by a combination of Dirichlet and Neumann boundary conditions [7]. For interval systems, we have essentially exhausted the SUSY systems for the  $B = 0$  case, but a large number of novel SUSY systems for the general  $B \neq 0$  case will exist under the  $\mu$ -term. The introduction of the  $\mu$ -term may in a sense be regarded as a generalization of the SUSY potential in the Witten model. In fact, our supercharge  $Q$  in (2.30) has a structure analogous to (the doubly graded form of) the one used in the Witten model. Moreover, for  $B = 0$  the supercharge  $Q$  reduces essentially to the supercharge of the Witten model with a constant SUSY potential [7].

Another point to be noted is the notion of SUSY itself. Namely, to seek SUSY systems we adopted the criterion [6] that eigenstates of the Hamiltonian  $H$  satisfy the same boundary condition even after the SUSY transformation by  $Q$  is performed. This is a necessary condition for the complete SUSY invariance of the boundary condition (*i.e.*, valid for any state, not just for energy eigenstates) but may be shown to be sufficient, too. We note that the issues such as the complete SUSY invariance or the self-adjointness of

the supercharge  $Q$  discussed in section 4 have not been fully addressed in generic SUSY quantum mechanics, despite that they become important if we wish to put SUSY on a firm basis in systems with boundaries or singularities. The relation between the self-adjoint domains of  $Q$  and  $H$  pointed out in this paper may provide a first step toward the full investigation on these issues.

Related to the above two points, we mention that independent supercharges fulfilling the criterion may not, in general, admit a basis set realizing the orthogonal SUSY algebra (2.38). In our analysis, we have defined the number  $N$  of SUSY by the number of supercharges fulfilling the orthogonal SUSY algebra, rather than by the total number of the independent supercharges. One may instead accept the full SUSY algebra — though it may be fairly involved — formed by the entire set of the supercharges as an extended version of SUSY, adopting the total number of the charges for the number  $N$  of SUSY.

To extend our work, perhaps the most straightforward is to put point singularities on more complicated one dimensional systems, such as a circle or networks with loops and vertices. We may also add a potential  $V(x)$  to our systems without changing our argument, as long as  $V(x)$  is regular at the singularities. In fact, the possibility of SUSY of a circle system with two point singularities has been studied recently in Ref.[7], where the introduction of regular potentials in the framework of the Witten model has also been discussed. We wish to stress, however, that the regularity of the potential is not essential, that is, even if  $V(x)$  is singular (like the Coulomb potential), we can treat it by generalizing slightly our procedure of assigning the boundary/connection conditions at the singularities [19]. We believe that various novel SUSY systems will be obtained under singular potentials if one employs the generalized approach developed in this paper.

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## Appendix A. Solutions for (3.5) and (3.6)

### A.1. type (a)

The choice of (a) for  $D$  and  $D_l$  implies  $S = S_l = I$ , and hence (3.5) is fulfilled if

$$V_1^\dagger A = A_l. \quad (\text{A.1})$$

On the other hand, from (3.6) we have  $V_1^\dagger K A = K_l A_l$  which, in view of (A.1), becomes  $(V_1^\dagger K V_1 - K_l) A_l = 0$  or

$$V_1^\dagger K V_1 = K_l, \quad (\text{A.2})$$

because  $A_l \in U(2)$  is invertible. With  $K$  and  $K_l$  in (3.10), (A.2) is seen to be

$$(1 + s_2 - s_1^l - s_2^l) I_2 + (1 - s_2) V_1^\dagger \sigma_3 V_1 - (s_1^l - s_2^l) \sigma_3 = 0, \quad (\text{A.3})$$

that is,

$$s_2 - s_1^l - s_2^l + 1 = 0, \quad (1 - s_2) V_1^\dagger \sigma_3 V_1 = (s_1^l - s_2^l) \sigma_3. \quad (\text{A.4})$$

Using the parametrization of  $V_1$  in (3.11), the second equation of (A.4) becomes

$$(1 - s_2) e^{-i\delta_1 \sigma_2} \sigma_3 e^{i\delta_1 \sigma_2} = (s_1^l - s_2^l) \sigma_3, \quad (\text{A.5})$$

which is satisfied under the three cases:

$$\begin{aligned} (1) \quad & s_2 = 1, \quad s_1^l = s_2^l, & (2) \quad & s_2 + s_1^l - s_2^l = 1, \quad \delta_1 = 0, \quad (s_1^l - s_2^l \neq 0), \\ (3) \quad & s_2 - s_1^l + s_2^l = 1, \quad \delta_1 = \pi/2, & & (s_1^l - s_2^l \neq 0). \end{aligned} \quad (\text{A.6})$$

Combining these with (A.1) and the first equation of (A.4), we obtain the solutions (a1) – (a3).

### A.2. type (b)

In this case we have  $S = X$  and  $S_l = I$ . Then (3.5) is solved by

$$V_2^\dagger A = A_l^\dagger. \quad (\text{A.7})$$

Similarly to type (a), with the solution (A.7) the condition (3.6) implies  $A^\dagger K A = -K_l$  from which we find

$$s_2 + s_1^l + s_2^l + 1 = 0, \quad (1 - s_2) A^\dagger \sigma_3 A + (s_1^l - s_2^l) \sigma_3 = 0. \quad (\text{A.8})$$

Using (3.11), the second equation of (A.8) becomes

$$(1 - s_2) e^{-i\beta\sigma_2} \sigma_3 e^{i\beta\sigma_2} + (s_1^l - s_2^l) \sigma_3 = 0, \quad (\text{A.9})$$

which is satisfied under the three cases:

$$\begin{aligned} (1) \quad & s_2 = 1, \quad s_1^l = s_2^l, & (2) \quad & s_2 = s_1^l = -1, \quad s_2^l = 1, \quad \beta = 0 \\ (3) \quad & s_2 = s_2^l = -1, \quad s_1^l = 1, & & \beta = \pi/2. \end{aligned} \quad (\text{A.10})$$

Combining these with (A.7) and the first equation of (A.8), we obtain (b1) – (b3).

### A.3. type (c)

Here we have  $S = Y$  and  $S_l = I$ . Then (3.5) implies

$$\begin{aligned} \sigma_+ A \sigma_2 \sigma_- - \sigma_- \sigma_2 A^\dagger \sigma_+ &= \sigma_+ \sigma_2 A \sigma_- - \sigma_- A^\dagger \sigma_2 \sigma_+ = 0, \\ V_1^\dagger (\sigma_+ A \sigma_- - \sigma_- \sigma_2 A^\dagger \sigma_2 \sigma_+) V_2 &= A_l, \end{aligned} \quad (\text{A.11})$$

where  $\sigma_\pm$  are defined in (2.32), whereas (3.6) gives

$$\begin{aligned} \sigma_+ A \sigma_2 \sigma_- + \sigma_- \sigma_2 A^\dagger \sigma_+ &= \sigma_+ \sigma_2 A \sigma_- + \sigma_- A^\dagger \sigma_2 \sigma_+ = 0, \\ V_1^\dagger (\sigma_+ A \sigma_- + s_2 \sigma_- \sigma_2 A^\dagger \sigma_2 \sigma_+) V_2 &= (s_1^l \sigma_+ + s_2^l \sigma_-) A_l. \end{aligned} \quad (\text{A.12})$$

The first equations of (A.11) and (A.12) are satisfied by  $\beta = \pi/2$ . From the second equations of (A.11) and (A.12), we have

$$((1 - s_1^l) \sigma_+ + (1 - s_2^l) \sigma_-) V_1^\dagger \sigma_+ A \sigma_- + ((s_2 + s_1^l) \sigma_+ + (s_2 + s_2^l) \sigma_-) V_1^\dagger \sigma_- \sigma_2 A^\dagger \sigma_2 \sigma_+ = 0, \quad (\text{A.13})$$

which implies

$$\begin{aligned} (1 - s_1^l) \sigma_+ V_1^\dagger \sigma_+ A \sigma_- &= (1 - s_2^l) \sigma_- V_1^\dagger \sigma_+ A \sigma_- = 0, \\ (s_2 + s_1^l) \sigma_+ V_1^\dagger \sigma_- \sigma_2 A^\dagger \sigma_2 \sigma_+ &= (s_2 + s_2^l) \sigma_- V_1^\dagger \sigma_- \sigma_2 A^\dagger \sigma_2 \sigma_+ = 0, \end{aligned} \quad (\text{A.14})$$

form which we obtain

$$\begin{aligned} (1 - s_1^l) \sin \beta \cos \delta_1 &= (1 - s_2^l) \sin \beta \sin \delta_1 = 0, \\ (s_2 + s_1^l) \sin \beta \sin \delta_1 &= (s_2 + s_2^l) \sin \beta \cos \delta_1 = 0. \end{aligned} \quad (\text{A.15})$$

By  $\beta = \pi/2$ , eq.(A.15) is seen to be satisfied under the there cases:

$$\begin{aligned} (1) \quad & \delta_1 = 0, \quad s_2 = s_1^l = 1, \quad s_2^l = -1, & (2) \quad & \delta_1 = \pi/2, \quad s_2 = s_2^l = 1, \quad s_1^l = -1, \\ (3) \quad & s_2 = -1, \quad s_1^l = s_2^l = 1. \end{aligned} \quad (\text{A.16})$$



Solving the second equation of (A.11) for the three cases of (A.16), we obtain (c1) – (c3).

#### A.4. type (d)

Here we have  $S = S_l = Y$ , and from (3.5) we find

$$\begin{aligned}
& (\sigma_+ V_1^\dagger \sigma_- + \sigma_- V_2^T \sigma_+) \sigma_2 A^\dagger (\sigma_+ V_1 \sigma_+ + \sigma_- V_2^* \sigma_-) \\
& \quad - (\sigma_+ V_1^\dagger \sigma_+ + \sigma_- V_2^T \sigma_-) A \sigma_2 (\sigma_- V_1 \sigma_+ + \sigma_+ V_2^* \sigma_-) = 0, \\
& (\sigma_+ V_1^T \sigma_+ + \sigma_- V_2^\dagger \sigma_-) A^\dagger \sigma_2 (\sigma_- V_1^* \sigma_+ + \sigma_+ V_2 \sigma_-) \\
& \quad - (\sigma_+ V_1^T \sigma_- + \sigma_- V_2^\dagger \sigma_+) \sigma_2 A (\sigma_+ V_1^* \sigma_+ + \sigma_- V_2 \sigma_-) = 0, \\
& (\sigma_+ V_1^\dagger \sigma_+ + \sigma_- V_2^T \sigma_-) A (\sigma_+ V_1^* \sigma_+ + \sigma_- V_2 \sigma_-) \\
& \quad - (\sigma_+ V_1^\dagger \sigma_- + \sigma_- V_2^T \sigma_+) \sigma_2 A^\dagger \sigma_2 (\sigma_- V_1^* \sigma_+ + \sigma_+ V_2 \sigma_-) = A_l.
\end{aligned} \tag{A.17}$$

From (3.6) we have

$$\begin{aligned}
& (\sigma_+ V_1^\dagger \sigma_- + \sigma_- V_2^T \sigma_+) \sigma_2 A^\dagger (\sigma_+ V_1 \sigma_+ + s_2 \sigma_- V_2^* \sigma_-) \\
& \quad + (\sigma_+ V_1^\dagger \sigma_+ + s_2 \sigma_- V_2^T \sigma_-) A \sigma_2 (\sigma_- V_1 \sigma_+ + \sigma_+ V_2^* \sigma_-) = 0, \\
& (\sigma_+ V_1^T \sigma_+ + \sigma_- V_2^\dagger \sigma_-) A^\dagger \sigma_2 (\sigma_- V_1^* \sigma_+ + s_2 \sigma_+ V_2 \sigma_-) \\
& \quad + (\sigma_+ V_1^T \sigma_- + s_2 \sigma_- V_2^\dagger \sigma_+) \sigma_2 A (\sigma_+ V_1^* \sigma_+ + \sigma_- V_2 \sigma_-) = 0, \\
& (\sigma_+ V_1^\dagger \sigma_+ + s_2 \sigma_- V_2^T \sigma_-) A (\sigma_+ V_1^* \sigma_+ + \sigma_- V_2 \sigma_-) \\
& \quad + (\sigma_+ V_1^\dagger \sigma_- + \sigma_- V_2^T \sigma_+) \sigma_2 A^\dagger \sigma_2 (\sigma_- V_1^* \sigma_+ + s_2 \sigma_+ V_2 \sigma_-) = (s_1^l \sigma_+ + s_2^l \sigma_-) A_l.
\end{aligned} \tag{A.18}$$

From the first equation of (A.17) we obtain

$$\begin{aligned}
& \sigma_+ V_1^\dagger (\sigma_- \sigma_2 A^\dagger \sigma_+ - \sigma_+ A \sigma_2 \sigma_-) V_1 \sigma_+ = 0, \\
& \sigma_+ V_1^\dagger (\sigma_- \sigma_2 A^\dagger \sigma_- - \sigma_+ A \sigma_2 \sigma_+) V_2^* \sigma_- = 0, \\
& \sigma_- V_2^T (\sigma_+ \sigma_2 A^\dagger \sigma_- - \sigma_- A \sigma_2 \sigma_+) V_2^* \sigma_- = 0,
\end{aligned} \tag{A.19}$$

whereas from the first equation of (A.18) we find

$$\begin{aligned}
& \sigma_+ V_1^\dagger (\sigma_- \sigma_2 A^\dagger \sigma_+ + \sigma_+ A \sigma_2 \sigma_-) V_1 \sigma_+ = 0, \\
& \sigma_+ V_1^\dagger (s_2 \sigma_- \sigma_2 A^\dagger \sigma_- + \sigma_+ A \sigma_2 \sigma_+) V_2^* \sigma_- = 0, \\
& \sigma_- V_2^T (\sigma_+ \sigma_2 A^\dagger \sigma_- + \sigma_- A \sigma_2 \sigma_+) V_2^* \sigma_- = 0.
\end{aligned} \tag{A.20}$$

The first and third equations of (A.19) and (A.20) imply

$$\begin{aligned}
& \sigma_+ V_1^\dagger \sigma_- \sigma_2 A^\dagger \sigma_+ V_1 \sigma_+ = \sigma_+ V_1^\dagger \sigma_+ A \sigma_2 \sigma_- V_1 \sigma_+ = 0, \\
& \sigma_- V_2^T \sigma_+ \sigma_2 A^\dagger \sigma_- V_2^* \sigma_- = \sigma_- V_2^T \sigma_- A \sigma_2 \sigma_+ V_2^* \sigma_- = 0.
\end{aligned} \tag{A.21}$$

(A.21) is satisfied by

$$\sin 2\delta_1 \cos \frac{\beta}{2} = 0, \quad \sin 2\delta_2 \cos \frac{\beta}{2} = 0. \quad (\text{A.22})$$

From the second equations of (A.19) and (A.20), we have

$$\begin{aligned} \sin \delta_1 \cos \delta_2 \sin \frac{\beta}{2} = \cos \delta_1 \sin \delta_2 \sin \frac{\beta}{2} = 0 \quad \text{for } s_2 = 1, \\ (\tan \delta_1 e^{i\xi} - \tan \delta_2 e^{-i\xi}) \sin \frac{\beta}{2} = 0 \quad \text{for } s_2 = -1. \end{aligned} \quad (\text{A.23})$$

The second equations of (A.17) and (A.18) hold automatically by (A.22) and (A.23) which are satisfied under the six cases:

$$\begin{aligned} (1) \quad & \delta_1 = \delta_2 = 0, & (2) \quad & \delta_1 = \delta_2 = \pi/2, & (3) \quad & \delta_1 = 0, \quad \delta_2 = \pi/2, \quad \beta = 0, \\ (4) \quad & \delta_1 = \pi/2, \quad \delta_2 = 0, \quad \beta = 0, \\ (5) \quad & \delta_1 = \delta_2 \neq 0 \text{ or } \pi/2, \quad \beta = \pi, \quad \xi = 0 \text{ or } \pi \quad \text{for } s_2 = -1, \\ (6) \quad & \delta_2 = \pi - \delta_1 \neq \pi/2, \quad \beta = \pi, \quad \xi = \pi/2 \text{ or } 3\pi/2 \quad \text{for } s_2 = -1. \end{aligned} \quad (\text{A.24})$$

Solving the third equations of (A.17) and (A.18) for the six cases of (A.24), we obtain (d1) – (d6).

## Appendix B. Spectral and SUSY properties

The following table summarizes the spectral as well as SUSY properties of the various types of SUSY systems obtained for two lines and two intervals.

- Two lines

Type	Number of supercharges	SUSY	Spectrum of regular series	Degeneracy of regular series	Number of isolated eigenstates
(A1)	$N = 1$	$\times$	$k > 0$	4	0 or 1 or 2
(A2)	$N = 1$	$\times$	$k > 0$	4	0 or 2
(B1)	$N = 4$	$\bigcirc$	$k > 0$	4	1 or 2
(B2)	$N = 4$	$\times$	$k > 0$	4	0

- Two intervals

Type	Number of supercharges	SUSY	Spectrum of regular series	Degeneracy of regular series	Number of isolated eigenstates
(a1)	$N = 4$	$\bigcirc$	$k_n$	4	2
(a2)	$N = 4$	$\bigcirc$	$k_n$	4	2
(a3)	$N = 4$	$\bigcirc$	$k_n$	4	2
(b1)	$N = 2$	$\times$	$k_n^+, k_n^-$	2+2	0 or 2
(b2)	$N = 2$	$\times$	$k_n^+, k_n^-$	2+2	0 or 2
(b3)	$N = 4$	$\times$	$k_n^-$	4	0 or 4
(c1)	$N = 2$	$\bigcirc$	$k_n, k_n^-$	2+2	1 or 3
(c2)	$N = 2$	$\bigcirc$	$k_n, k_n^-$	2+2	1 or 3
(c3)	$N = 2$	$\bigcirc$	$k_n, k_n^+$	2+2	1 or 3
(d1)	$N = 2$	$\bigcirc$	$k_n, k_n^{-s_2}$	2+2	1 or 3
(d2)	$N = 2$	$\bigcirc$	$k_n, k_n^-$	2+2	1 or 3
(d3)'	$N = 2$	$\times$	$k_n + \delta_1/l$	2	0
(d4)'	$N = 2$	$\times$	$k_n + \delta_1/l$	2	0
(d5)	$N = 4$	$\bigcirc$	$k_n$	4	2
(d6)	$N = 4$	$\times$	$k_n^-$	4	0 or 4

Note: ‘2+2’ means that two distinct types of doubly degenerate eigenstates exist;  $\bigcirc$  and  $\times$  denote good and broken SUSY, respectively;  $k_n$  and  $k_n^\pm$  are given in (3.14) and (3.22); and (d3)’ and (d4)’ refer to the special  $\theta = 0$  case in type (d3) and (d4), respectively.

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