KEK Preprint 2001-12

Möbius Structure of the Spectral Space of Schrödinger Operators with Point Interaction

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Abstract. The Schrödinger operator with point interaction in one dimension has a $U(2)$ family of self-adjoint extensions. We study the spectrum of the operator and show that (i) the spectrum is uniquely determined by the eigenvalues of the matrix $U \in U(2)$ that characterizes the extension, and that (ii) the space of distinct spectra is given by the orbifold $T^2/\mathbb{Z}_2$ which is a Möbius strip with boundary. We employ a parametrization of $U(2)$ that admits a direct physical interpretation and furnishes a coherent framework to realize the spectral duality and anholonomy recently found. This allows us to find that (iii) physically distinct point interactions form a three-parameter quotient space of the $U(2)$ family.

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1. Introduction

Quantum mechanical motion of a particle subject to a point interaction on a line $\mathbb{R}$ is described by the free Schrödinger (the Laplacian) operator,

$$H = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} \,, \tag{1}$$

with one point perturbation. This is implemented by deleting a point, say $x = 0$ on the line, and thereby considering the family Ω of self-adjoint operators H defined on proper domains in the Hilbert space $\mathcal{H} = L^2(\mathbb{R} \setminus \{0\})$. The theory of self-adjoint extensions then dictates that the family Ω is given by the group $U(2)$, which covers all allowable distinct point interactions [13]. Studies show that the spectrum of the operator H consists of the essential spectrum $[0, \infty)$ together with a discrete spectrum having at most two levels of bound states [2] (see also [4,5] and references therein). Symmetries such as parity or time-reversal are used to classify the family $\Omega \simeq U(2)$ in terms of their invariant subfamilies [3].

Recently, we have examined the spectral properties of this simple system and found a number of interesting features which are usually ascribed to more complex systems. These features include duality in the spectra under strong vs weak coupling exchange [8,14], anholonomy both in the phase of states (the Berry phase) and in levels under a cycle in Ω [6,10], and the double degeneracy which leads to supersymmetry [7]. Meanwhile, a similar study has been made on a circle S^1 with point interaction [11], where it is shown that the spectrum of H does not depend on the entire $U(2)$ parameters as one naïvely expects.

The aim of the present paper is to furnish a comprehensive picture of the spectral structure of the entire family of the Schrödinger operators H on a line $\mathbb{R}$ as well as on an interval $[l, -l]$ (under some innocuous boundary conditions) with the point $x = 0$ removed. Our main results are given in three theorems. Theorem 1 states that the spectrum is uniquely determined by the eigenvalues of the $U(2)$ matrix which characterizes the point interaction, and Theorem 2 shows that, for the case of the interval, the space Σ consisting of all distinct spectra is given by a Möbius strip with boundary, while for the case of the line Σ is a subspace of it. The key observation to reach these statements is that the set of $su(2)$ parity transformations on the operator H which preserve the spectrum [14,7] can be generalized in order to narrow the dependence from $U(2)$ down to its subspace. We also provide a generalization in symmetry transformations in order to associate a pertinent invariant subfamily to any point interaction in Ω. In our treatment emerges a natural parametrization of Ω which admits a direct physical interpretation and furnishes a framework to describe the above mentioned features in a coherent manner. As part of the physical interpretation given as Theorem 3, we find a one-parameter gauge equivalence within Ω and conclude that physically distinct point interactions form a three-parameter quotient space of Ω.

2. Spectral structure

Let us first recall the description of the $U(2)$ family of self-adjoint operators H [14] (see also [1]). The domain of such a self-adjoint operator H is a subspace of $\mathcal{H}$ specified by a boundary condition at the missing point $x = 0$ on the line. Let φ be a state in the domain, and consider the two-component boundary vectors

$$\Phi := \begin{pmatrix} \varphi(0_+) \\ \varphi(0_-) \end{pmatrix}, \qquad \Phi' := \begin{pmatrix} \varphi'(0_+) \\ -\varphi'(0_-) \end{pmatrix}, \tag{2}$$

where 0_+ and 0_- denote the limits at $x = 0$ from the right and the left, respectively.[1] In terms of a matrix $U \in U(2)$ the boundary condition is then given as

$$(U - I)\Phi + iL_0(U + I)\Phi' = 0 , \tag{3}$$

with some constant $L_0 \neq 0$ of length dimension, where I denotes the unit matrix in $U(2)$. We note that the self-adjointness of H is equivalent to the requirement of (global) probability conservation, and that the constant L_0 adds no extra freedom to that given by U [7]. To indicate the $U(2)$-dependence of the operator H, we use the notation H_U.

We now begin our discussion of the spectral structure of the family Ω of the operators H_U by providing the following

Definition 1. A unitary transformation $\mathcal{X} : \mathcal{H} \to \mathcal{H}$ is called *a generalized symmetry* of the family Ω if, for any $U \in U(2)$,

$$\mathcal{X}^{-1} H_U \mathcal{X} = H_{U_\mathcal{X}} , \tag{4}$$

for some $U_\mathcal{X} \in U(2)$.

We note that condition (4) embodies two requirements: first, the domain of H_U is mapped into the domain of $H_{U_\mathcal{X}}$, and secondly, $\mathcal{X}^{-1} H_U \mathcal{X}$ acts on this new domain as the differential operator (1). Note also that the two operators H_U and $H_{U_\mathcal{X}}$ share the same spectrum.

[1] φ and its derivative, φ', are required to be absolutely continuous on $\mathbb{R} \setminus \{0\}$ (see [1]).

The following lemmas will be useful in proving Theorem 1.

Lemma 1. *The operators $\mathcal{P}_j$ $(j = 1, 2, 3)$ defined as*

$$
\begin{aligned}
(\mathcal{P}_1\varphi)(x) &:= \varphi(-x) \ , \\
(\mathcal{P}_2\varphi)(x) &:= i[\Theta(-x) - \Theta(x)]\varphi(-x) \ , \\
(\mathcal{P}_3\varphi)(x) &:= [\Theta(x) - \Theta(-x)]\varphi(x) \ ,
\end{aligned}
\tag{5}
$$

(where Θ denotes the Heaviside step function) are generalized symmetries. Further, they are parity-type operators (i.e., $\mathcal{P}_j^2 = \mathrm{id}_{\mathcal{H}}$, $\mathcal{P}_j \neq \pm\mathrm{id}_{\mathcal{H}}$) and satisfy the $su(2)$ commutation relations $[\mathcal{P}_j, \mathcal{P}_k] = 2i\sum_{l=1}^{3}\epsilon_{jkl}\mathcal{P}_l$ and the anticommutation relations $\{\mathcal{P}_j, \mathcal{P}_k\} = 2\delta_{jk}\mathrm{id}_{\mathcal{H}}$.

Proof. It is straightforward to check that these operators are unitary and parity-type, fulfilling the stated commutation and anticommutation relations. To show that they are generalized symmetries, let us observe that, under a $\mathcal{P}_j$, the boundary vectors (2) change as $\Phi \mapsto \sigma_j\Phi$ and $\Phi' \mapsto \sigma_j\Phi'$, where σ_js denote the Pauli matrices. In the boundary condition (3), this change can be absorbed by the change in the matrix U as

$$
U \longmapsto U_{\mathcal{P}_j} := \sigma_j\, U\, \sigma_j \ . \tag{6}
$$

This implies that a $\mathcal{P}_j$ maps the domain of an H_U to the domain of $H_{U_{\mathcal{P}_j}}$ with $U_{\mathcal{P}_j}$ given in (6) (clearly, $\mathcal{P}_j$s preserve the smoothness properties mentioned in Footnote 1, too). It is also easy to see that $\mathcal{P}_j H_U \mathcal{P}_j$ remains the differential operator (1) on this new domain, since, under any of the transformations (5), φ acquires merely an overall complex phase factor that is constant on both $\mathbb{R}_+$ and $\mathbb{R}_-$. *Q.E.D.*

The three transformations defined above are not the only parity-type generalized symmetries. Indeed, operators given by the linear combinations of the three,

$$
\mathcal{P} := \sum_{j=1}^{3} c_j\, \mathcal{P}_j \qquad \text{with} \quad c_j \in \mathbb{R}\,, \quad \sum_{j=1}^{3} c_j^2 = 1 \ , \tag{7}
$$

are all generalized symmetries and fulfill the parity property $\mathcal{P}^2 = \mathrm{id}_{\mathcal{H}}$, where now the induced transformation on U reads

$$
U \longmapsto U_{\mathcal{P}} = \sigma\, U\, \sigma \ , \qquad \sigma := \sum_{j=1}^{3} c_j\, \sigma_j \ . \tag{8}
$$

We therefore arrive at

Lemma 2. *For any $su(2)$ element σ normalized as $\sigma^2 = I$, and for any $U \in U(2)$, H_U and $H_{\sigma U \sigma}$ share an identical spectrum.*

Using Lemma 2, we now show

Theorem 1. *The spectrum of the Schrödinger operator H_U is uniquely determined by the eigenvalues of the matrix U.*

Proof. Let $e^{i\theta_+}$ and $e^{i\theta_-}$ with $\theta_\pm \in [0, 2\pi)$ be the two eigenvalues of the unitary matrix U. These eigenvalues arise in the matrix,

$$D = \begin{pmatrix} e^{i\theta_+} & 0 \\ 0 & e^{i\theta_-} \end{pmatrix} , \tag{9}$$

which appears when one diagonalizes

$$U = V^{-1} D V \ , \tag{10}$$

with an appropriate $V \in SU(2)$. To proceed, let us set

$$D = e^{i\xi}\, e^{i\rho\sigma_3} \ , \qquad \xi = \frac{\theta_+ + \theta_-}{2}, \qquad \rho = \frac{\theta_+ - \theta_-}{2} \ , \tag{11}$$

to rewrite (10) as

$$U = e^{i\xi}\, e^{i\rho\, V^{-1}\sigma_3 V} \ . \tag{12}$$

Note that $V^{-1}\sigma_3 V$ in the exponent is just an element of $su(2)$ obtained by the rotation of σ_3 with respect to an axis determined by V. Note also that, since $\sigma = \frac{1}{i} e^{\frac{\pi}{2} i\sigma} = \sigma^{-1}$, the product $\sigma\sigma_3\sigma$ is an element of $su(2)$ obtained by the rotation of σ_3 with respect to σ by the angle π. This implies that, to a given V, one can always find some σ such that $V^{-1}\sigma_3 V = \sigma\sigma_3\sigma$ holds. With such σ we now have

$$U = e^{i\xi}\, e^{i\rho\, \sigma\sigma_3\sigma} = \sigma\, D\, \sigma \ . \tag{13}$$

Lemma 2 then ensures that the spectrum of H_U coincides with the spectrum of H_D. *Q.E.D.*

From this theorem we obtain

Corollary 1. *A point interaction characterized by U possesses the* isospectral subfamily

$$\Omega(D) := \{\, H_{V^{-1}DV} \,|\, V \in SU(2) \} \ , \tag{14}$$

where D is the diagonal eigenvalue matrix in the decomposition (10) *of U. The isospectral subspace $\Omega(D)$ is homeomorphic to the coadjoint orbit of $SU(2)$ passing through the element*

$e^{i\rho\sigma_3}$, and hence $\Omega(D) \simeq S^2$ except for the case $D = e^{i\theta} \cdot I$ ($\theta \in [0, 2\pi)$) for which $\Omega(D)$ consists of D alone.

We mention that the exceptional cases ($\theta = \theta_+ = \theta_-$) occur at

$$U = e^{i\theta} \cdot I \ , \qquad \theta \in [0, 2\pi) \ , \tag{15}$$

which form what we call the *self-dual subfamily* $\Omega_{\rm SD} \simeq U(1)$ in the entire set of point interactions $\Omega \simeq U(2)$ (see also Proposition 3 and the remark which follows).

Clearly, the two eigenvalues of U appearing in D are interchangeable, and this is realized for $\Omega(D)$ by setting, *e.g.,* $V \mapsto i\sigma_2 V$. Thus, if we write $D = D(\theta_+, \theta_-)$ for the diagonal matrix D in (9), we have

Corollary 2. *The two isospectral subfamilies associated with $D(\theta_+, \theta_-)$ and $D(\theta_-, \theta_+)$ are identical,*

$$\Omega(D(\theta_+, \theta_-)) = \Omega(D(\theta_-, \theta_+)) \ , \tag{16}$$

and hence the spectrum occurring at $D(\theta_+, \theta_-)$ and that occurring at $D(\theta_-, \theta_+)$ are the same.

The spectral feature discussed above is seen in the discrete spectrum, but it is largely obscured because the spectrum consists mostly of the continuous spectrum $[0, \infty)$. However, the structure becomes manifest if one considers, instead of a line, a box (interval) on which the entire spectrum becomes discrete. This can be done by imposing a boundary condition at both ends of the box in such a way that it does not affect the consequences of the operations of $\mathcal{P}$ in (7). Specifically, if we let the interval $[-l, l]$ be the box where the point interaction is placed at $x = 0$, then we seek for boundary conditions at $x = \pm l$ which remain invariant under any of the transformations induced by $\mathcal{P}$. These are given by

Proposition 1. *The boundary conditions at $x = \pm l$ which are left unchanged under any of the transformations induced by $\mathcal{P}$ (and hence provide a domain for H so that the entire discrete spectrum exhibits the spectral structure manifestly) are*

$$\varphi(l) + L\,\varphi'(l) = 0 \ , \qquad \varphi(-l) - L\,\varphi'(-l) = 0 \ , \tag{17}$$

where $L \in (-\infty, \infty) \cup \{\infty\}$ is an arbitrary parameter.

Proof. The operator H remains self-adjoint if the boundary condition at $x = \pm l$ ensures the probability conservation, and this is exactly the demand we used to obtain the boundary condition (3) at $x = 0$. (More precisely, one needs to require further that the probability current vanish at the both ends, but this will be seen to be satisfied at the end.) This suggests that, if we use the boundary vectors similar to (2),

$$\Psi := \begin{pmatrix} \varphi(l) \\ \varphi(-l) \end{pmatrix}, \qquad \Psi' := \begin{pmatrix} \varphi'(l) \\ -\varphi'(-l) \end{pmatrix}, \tag{18}$$

the boundary conditions at the ends can be given analogously as

$$(\tilde{U} - I)\Psi + iL_0(\tilde{U} + I)\Psi' = 0 \,, \tag{19}$$

in terms of a matrix $\tilde{U} \in U(2)$ characterizing the two ends. The transformation of the operator $\mathcal{P}$ on the boundary vectors (18) is the same as before, and hence it induces the same action $\tilde{U} \mapsto \tilde{U}_{\mathcal{P}} = \sigma\,\tilde{U}\,\sigma$ on the matrix $\tilde{U}$. Thus, the required boundary condition must satisfy $\sigma\,\tilde{U}\,\sigma = \tilde{U}$, that is, we find $\tilde{U} = e^{i\theta} \cdot I$ for $\theta \in [0, 2\pi)$. Putting $L = L_0 \cot\frac{\theta}{2}$ we obtain the statement. *Q.E.D.*

We remark that both the Dirichlet condition $\varphi(l) = \varphi(-l) = 0$ and the Neumann condition $\varphi'(l) = \varphi'(-l) = 0$ are of the type (17).

If we now introduce the space of distinct spectra, $\Sigma := \{\,\mathrm{Spec}\,(H_U)\,|\,U \in U(2)\}$, then from the foregoing argument we find that Σ is a subspace of the torus $T^2 = S^1 \times S^1 = \{(\theta_+, \theta_-)\}$ subject to the identification $(\theta_+, \theta_-) \equiv (\theta_-, \theta_+)$. The quotient space obtained by the identification is the orbifold $T^2/\mathbb{Z}_2$ which is the domain of the triangle shown in Fig.1. The elementary observation in Fig.1 leads to

Theorem 2. *The spectral space Σ of point interactions is a subspace of the orbifold $T^2/\mathbb{Z}_2$ which is homeomorphic to a Möbius strip with boundary. In particular, for the box $[l, -l]$ the spectral space Σ is the entire $T^2/\mathbb{Z}_2$.*

Proof. The first half is already shown (see Fig.1). To show the second half, we observe that for an isospectral subfamily $\Omega(D)$ the spectrum is determined by the boundary condition (3), which splits into

$$\varphi(0_+) + L_+\,\varphi'(0_+) = 0 \,, \qquad \varphi(0_-) - L_-\,\varphi'(0_-) = 0 \,, \tag{20}$$

where we have used

$$L_\pm := L_0 \cot\frac{\theta_\pm}{2} \,. \tag{21}$$

Then for the box $[l, -l]$ the problem boils down to determining the spectrum of the operator in two separate boxes, $[-l, 0_-)$ and $(0_+, l]$, under the combined boundary conditions, (17) and (20). For the interval $(0_+, l]$, for instance, the positive spectrum $E = \hbar^2 k^2/(2m)$ is determined by the condition, $\tan kl = k(L - L_+)/(1 + k^2 L L_+)$, which admits a distinct set of solutions for different L_+ under fixed L. It thus follows that to each pair (L_+, L_-) or (θ_+, θ_-) modulo the exchange $\theta_+ \leftrightarrow \theta_-$ there arises a distinct spectrum. *Q.E.D.*

We have seen that the product form (10) for the matrix U furnishes a useful parametrization for the point interaction in one dimension, where the spectral property resides solely in the diagonal part D. The adjoint part V, on the other hand, may be used to provide a parity transformation pertinent to the point interaction as follows.

Proposition 2. *To a point interaction specified by U there is a parity operator $\mathcal{P}$ of the form* (7) *whose action leaves U invariant. The operator $\mathcal{P}$ is unique (up to the sign) except when $U \in \Omega_{\rm SD}$ for which $\mathcal{P}$ is arbitrary.*

Proof. Consider the $su(2)$ element σ in (8) given by

$$\sigma = \sigma(V) := V^{-1} \sigma_3 V \ , \tag{22}$$

where V is the $SU(2)$ matrix appearing in (10) for the the diagonalization of the matrix $U \notin \Omega_{\rm SD}$. Note that in (10) the matrix V is determined only up to the left action $e^{i\chi\sigma_3} V$, but this ambiguity does not affect in specifying σ in (22). We now expand $\sigma(V)$ in the $su(2)$ basis as $\sigma(V) = \sum_{j=1}^{3} c_j(V)\, \sigma_j$ and define the corresponding parity operator,

$$\mathcal{P}(V) := \sum_{j=1}^{3} c_j(V)\, \mathcal{P}_j \ . \tag{23}$$

We then see at once that, under the transformation induced by $\mathcal{P}(V)$, the matrix U is left invariant, $\sigma(V)\, U\, \sigma(V) \ = U$. The parity $-\mathcal{P}(V)$ corresponding to $-\sigma(V)$ also leaves U invariant. For $U \in \Omega_{\rm SD}$, it is obvious that any σ, and hence any $\mathcal{P}$ in (7) leaves U invariant. *Q.E.D.*

The content of Proposition 2 may equally be stated as

Proposition 2'. *The Schrödinger operator H_U commutes with a parity operator $\mathcal{P}$ given by* (7), $[\, H_U, \mathcal{P}\,] = 0$, *where for $U \notin \Omega_{\rm SD}$ the operator $\mathcal{P}$ is uniquely determined as $\mathcal{P} = \mathcal{P}(V)$ (up to sign) in* (23), *while for $U \in \Omega_{\rm SD}$ it is arbitrary.*

We note that, for an H_U and the parity operator $\mathcal{P}$ commuting with it, the Hilbert space $\mathcal{H}$ can be decomposed into two orthogonal closed linear subspaces, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_+$ and $\mathcal{H}_-$ are the eigenspaces of $\mathcal{P}$ corresponding to the eigenvalues 1 and -1, respectively.

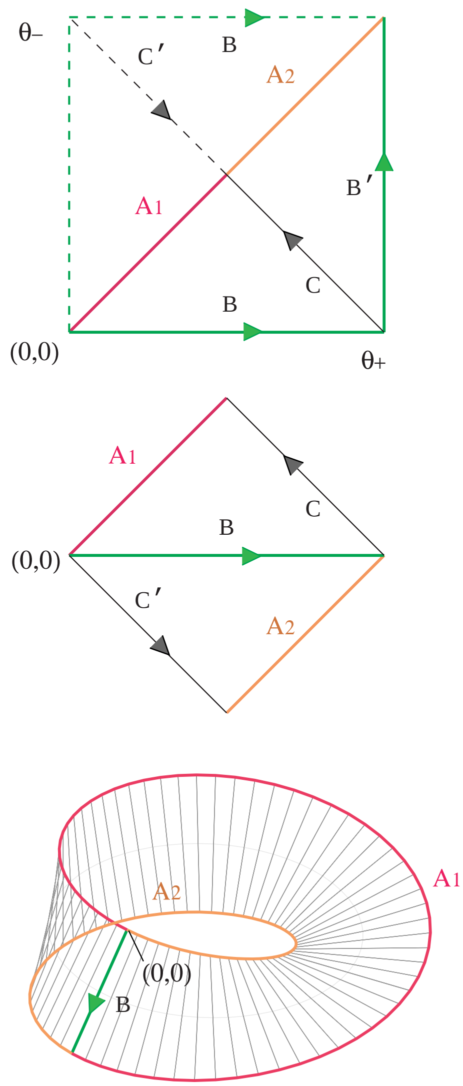

Figure 1. In the top figure, the spectral space Σ is the triangle surrounded by edges $A_1 + A_2$, B and B'. We divide this triangle into two subtriangles B–C–A_1 and B'–C–A_2. Since the latter subtriangle is spectrally identical to its dual image B–C'–A_2, Σ can be represented by the square A_1–C'–A_2–C in the middle figure. When the two spectrally identical edges C and C' are stitched together with the right orientation, we obtain the Möbius strip with boundary A_1–A_2 representing the self-dual subfamily $\Omega_{\rm SD}$ (the bottom figure).

The nondegenerate eigenfunctions of H_U belong to either $\mathcal{H}_+$ or $\mathcal{H}_-$. For doubly degenerate eigenvalues, the eigenfunctions can be chosen such that one belongs to $\mathcal{H}_+$ and the other to $\mathcal{H}_-$. Note that, since the eigenvalue equation is a second order differential equation on both half lines, the eigenvalues of H_U are at most doubly degenerate. Namely, these degenerate solutions contain two free constants each, and the boundary condition (3) reduces this four-parameter freedom to a two-parameter one. These statements are valid for the non-normalizable eigenfunctions (scattering states) of H_U, too, in the rigged Hilbert space sense (note that the definition (5), and correspondingly the definition of $\mathcal{P}(V)$, can be extended to any $\mathbb{R}\setminus\{0\} \to \mathbb{C}$ function in a natural way, which involves the natural extension of $\mathcal{H}_+$ and $\mathcal{H}_-$).

A distinguished family of generalized symmetries which interchange the subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ exist, that is,

Proposition 3. *For an H_U and the associated parity operator $\mathcal{P}$ commuting with it, there exists a $U(1)$ family of generalized symmetries $\mathcal{D}$ such that each $\mathcal{D}$ maps $\mathcal{H}_+$ to $\mathcal{H}_-$ and vice versa, and satisfies $(U_\mathcal{D})_\mathcal{D} = U$.*

Proof. Consider the generalized symmetries $\mathcal{D}$ corresponding to the $U(1)$ family of $su(2)$ elements,

$$\sigma_\mathcal{D} := V^{-1}\,\tilde{\sigma}(\phi)\,V \ , \tag{24}$$

where we have defined $\tilde{\sigma}(\phi) = \cos\phi\,\sigma_1 + \sin\phi\,\sigma_2$ for $\phi \in [0, 2\pi)$, and introduced $\mathcal{D} := \sum_{j=1}^{3} c'_j \mathcal{P}_j$ using the expansion $\sigma_\mathcal{D} = \sum_{j=1}^{3} c'_j \sigma_j$ of $\sigma_\mathcal{D}$. On $U = V^{-1}\,D(\theta_+, \theta_-)\,V$ these $\mathcal{D}$ induce

$$\begin{aligned} U \longmapsto U_\mathcal{D} &= \sigma_\mathcal{D}\,U\,\sigma_\mathcal{D} \\ &= V^{-1}\,\tilde{\sigma}(\phi)\,D(\theta_+, \theta_-)\,\tilde{\sigma}(\phi)\,V = V^{-1}\,D(\theta_-, \theta_+)\,V \ , \end{aligned} \tag{25}$$

and hence implement the interchange $\theta_+ \leftrightarrow \theta_-$. From this $(U_\mathcal{D})_\mathcal{D} = U$ is clear. To prove that a $\mathcal{D}$ maps any eigenfunction of $\mathcal{P}$ to another one with opposite eigenvalue, we show that $\{\mathcal{P}, \mathcal{D}\} = 0$. Indeed, from $\{\mathcal{P}_j, \mathcal{P}_k\} = \mathrm{Tr}\,(\sigma_j \sigma_k)\,\mathrm{id}_\mathcal{H}$ and (23) it follows that

$$\{\mathcal{P}, \mathcal{D}\} = \mathrm{Tr}\,\left(\sum_{j=1}^{3} c_j \sigma_j\ \sum_{k=1}^{3} c'_k \sigma_k\right)\mathrm{id}_\mathcal{H} = \mathrm{Tr}\,(\sigma\sigma_\mathcal{D})\,\mathrm{id}_\mathcal{H} = \mathrm{Tr}\,(\sigma_3\,\tilde{\sigma}(\phi))\,\mathrm{id}_\mathcal{H} = 0\,. \tag{26}$$

Q.E.D.

Hence, in the light of these properties, $\mathcal{D}$ may be called *duality transformation*. The duality found in [14,7] is a special case of $\mathcal{D}$.

The role of the point interaction and the parity operator in Proposition 2 can be reversed to obtain

Proposition 4. *To a parity operator $\mathcal{P}$ given in* (7) *there is a subfamily of point interactions which are left invariant under $\mathcal{P}$. For any $\mathcal{P}$ the subfamily $\Omega_{\mathcal{P}}$ is homeomorphic to a torus T^2.*

Proof. The subfamily $\Omega_{\mathcal{P}}$ is given by

$$\Omega_{\mathcal{P}} := \{\, U \in U(2) \mid \sigma\, U\, \sigma = U \,\} \ , \tag{27}$$

where σ is determined from $\mathcal{P}$ by (8). The matrices U belonging to $\Omega_{\mathcal{P}}$ are then found to be of the form,

$$U = e^{i\xi}\, e^{i\rho\sigma} \ , \qquad \xi \in [0, \pi) \ , \quad \rho \in [0, 2\pi) \ , \tag{28}$$

which is homeomorphic to a torus T^2 for any $\mathcal{P}$. *Q.E.D.*[2]

For instance, if we choose $\mathcal{P} = \mathcal{P}_1$, the subfamily $\Omega_{\mathcal{P}_1}$ is just the set of parity invariant (left-right symmetric) point interactions in the usual sense of the word. If, on the other hand, we choose $\mathcal{P} = \mathcal{P}_3$, then the resultant subfamily $\Omega_{\mathcal{P}_3}$ becomes the so-called separated subfamily where no probability flow through the gap $x = 0$ is allowed. One may also choose for $\mathcal{P}$ the one $\mathcal{P}(V)$ that corresponds to a specific U. The invariant subfamily $\Omega_{\mathcal{P}(V)}$ then contains U by construction, and becomes a subfamily pertinent to the point interaction characterized by U. One then finds from Propositions 3 and 4 that $\Omega_{\mathcal{P}(V)} \simeq T^2$ except when $U \in \Omega_{\rm SD}$ for which $\Omega_{\mathcal{P}(V)}$ coincides with the entire family $\Omega \simeq U(2)$.

The self-dual subfamily $\Omega_{\rm SD}$ has also the following distinguished characteristics:

Proposition 5. *For any point interaction belonging to $\Omega_{\rm SD}$ (i.e., $U \in \Omega_{\rm SD}$), all eigenvalues of H_U (including the generalized ones) are doubly degenerate.*

Proof. For any $U \in \Omega_{\rm SD}$, we have from Proposition 2' that $[\, H_U, \mathcal{P}_j \,] = 0$ for $j = 1$, 2, 3. This implies that, on any eigenspace of H_U, a representation of $su(2)$ formed by $\{\mathcal{P}_j\}_{j=1,2,3}$ is given. Since an eigenfunction of, say, $\mathcal{P}_1$ cannot be an eigenfunction of $\mathcal{P}_2$, the eigenspaces of H_U must be doubly degenerate. This argument is valid for the generalized eigenvalues (scattering state energies) and the corresponding eigenspaces as well. *Q.E.D.*

The double degeneracy implies that the system with point interaction belonging to $\Omega_{\rm SD}$ may be regarded as supersymmetric. As shown in [7], this is in fact the case for $U = -I$,

[2] To derive the above results one may consider, instead of the parities $\mathcal{P}$ in (7), more general transformations $\mathcal{F}_W$ given by $(\mathcal{F}_W\psi)(x) := W_{11}\Theta(x)\psi(x) + W_{12}\Theta(x)\psi(-x) + W_{21}\Theta(-x)\psi(-x) + W_{22}\Theta(-x)\psi(x)$ with the matrix W of the coefficients W_{ij} belonging to $U(2)$. These generalized symmetries $\mathcal{F}_W$ realize the arbitrary boundary conjugations $U_{\mathcal{X}} = WUW^{-1}$ and obey several useful properties, such as $\mathcal{F}_{W_1W_2} = \mathcal{F}_{W_1}\mathcal{F}_{W_2}$ and $\mathcal{F}_{\lambda_1 W_1 + \lambda_2 W_2} = \lambda_1 \mathcal{F}_{W_1} + \lambda_2 \mathcal{F}_{W_2}$ for $\lambda_1, \lambda_2 \in \mathbb{C}$.

where the energy of the two bound states vanishes yielding an $N = 2$ Witten model with a 'good SUSY' [12]. Generically, however, the ground state energy of the system is nonvanishing and the system is not supersymmetric even though it admits a formally supersymmetric reformulation for any U of Ω_{SD}. The obstacle for being supersymmetric is the fact that the presumed supercharges are not self-adjoint unless $U = -I$.

We have learned that the spectrum of the operator H_U is determined by the two parameters in D in the decomposition $U = V^{-1}DV$, and that, in particular, for the box the space Σ of the spectra is given by a Möbius strip. One can proceed further and assign more general physical meaning to the parameters in the matrix U. To see this we first rewrite the boundary condition (3) using the decomposition as

$$V\Phi + \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix} V\Phi' = 0 \ , \tag{29}$$

with $L_\pm$ given in (21). We further parametrize V by the Euler angles (with the first factor $e^{i\chi\sigma_3}$ which does not affect U being dropped),

$$V = e^{i\frac{\mu}{2}\sigma_2} e^{i\frac{\nu}{2}\sigma_3} \ , \qquad \mu \in [0, \pi], \quad \nu \in [0, 2\pi) \, , \tag{30}$$

and thereby present

$$\left\{ (L_+,\, L_-,\, \mu,\, \nu) \,\middle|\, L_\pm \in (-\infty, \infty) \cup \{\infty\},\, \mu \in [0, \pi],\, \nu \in [0, 2\pi) \right\} \ , \tag{31}$$

as a basic set for the parametrization of the point interactions on a line. On account of the double specification of the eigenvalues of U, the set (31) is in a two-to-one correspondence to U, providing a double covering[3] of the whole family $\Omega \simeq U(2)$ (see Fig.2). We then have

Theorem 3. *The parameters in the set* (31) *possess the following physical properties:*

(i) *The two parameters $L_\pm$ furnish two independent length scales to the point interaction.*

(ii) *The angle ν is physically irrelevant (unobservable).*

(iii) *The angle μ measures the extent of mixture of states between the positive and negative half lines.*

Proof. (i) is evident because in the boundary condition (29) $L_\pm$ are the only parameters with length dimension. To show (ii), we observe from (30) and (29) that the angle ν can be

[3] This redundancy is introduced to avoide unwanted discontinuity which arises when we study the response, such as the level change or the anholonomy phase, of the system under smooth changes over Ω.

absorbed by introducing the new vectors $e^{i\nu\sigma_3/2}\Phi$ and $e^{i\nu\sigma_3/2}\Phi'$ which arise if we replace $\varphi(0_\pm) \mapsto e^{\pm i\nu/2}\varphi(0_\pm)$ and $\varphi'(0_\pm) \mapsto e^{\pm i\nu/2}\varphi'(0_\pm)$. This is implemented by the $U(1)$ phase transformation (gauge transformation) on the state,

$$\varphi(x) \longmapsto e^{\frac{i}{\hbar}\vartheta(x)}\varphi(x) \ , \qquad \vartheta(x) := \frac{\nu}{2}\,[\Theta(x) - \Theta(-x)]\,\hbar \ . \tag{32}$$

Since the phase shift $\vartheta(x)$ is constant over $\mathbb{R}\setminus\{0\}$, and since the phase gap (which occurs at the missing point $x = 0$) cannot be observed on a line,[4] the transformed state is equivalent to the original state in quantum theory, that is, the angle ν is irrelevant physically. Finally, (iii) is also evident in the boundary condition (29) because the factor $e^{i\frac{\mu}{2}\sigma_2}$ mixes the two rows of the boundary vectors by rotation according to the angle μ. *Q.E.D.*

An important point to be noted here is that the existence of the one-parameter gauge equivalence within Ω implies that point interactions which are distinct physically — not just on the spectral basis — form a three-parameter quotient space of Ω.

The properties stated in Theorem 3 can be seen explicitly in the solutions of the Schrödinger equation under the operator H_U being the Hamiltonian. For instance, the bound states allowed under H_U on the line are given by

$$\varphi_\kappa(x) = \begin{cases} A_\kappa^- e^{\kappa x}, & x < 0 \\ B_\kappa^+ e^{-\kappa x}, & x > 0 \end{cases} \tag{33}$$

where κ determines the bound state energy $E_{\mathrm{bound}} = -\hbar^2\kappa^2/(2m)$, and the constants A_κ^- and B_κ^+ are subject to the normalization condition $|A_\kappa^-|^2 + |B_\kappa^+|^2 = 2\kappa$. A nonvanishing solution is then ensured if

$$\kappa = \frac{1}{L_+} \qquad \text{or} \qquad \kappa = \frac{1}{L_-} \ , \tag{34}$$

which shows that there exist two bound states if $L_+ > 0$ and $L_- > 0$, and one if $L_+L_- < 0$, and none if $L_+ < 0$ and $L_- < 0$. The parameters $L_\pm$ thus give (in case they are positive) the scales of the trapped particle. In terms of (30) the coefficients are found to be

$$\begin{pmatrix} B_\kappa^+ \\ A_\kappa^- \end{pmatrix} = \sqrt{\frac{2}{L_+}}\begin{pmatrix} e^{-2i\nu}\cos\frac{\mu}{2} \\ \sin\frac{\mu}{2} \end{pmatrix} \ , \qquad \begin{pmatrix} B_\kappa^+ \\ A_\kappa^- \end{pmatrix} = \sqrt{\frac{2}{L_-}}\begin{pmatrix} -e^{-2i\nu}\sin\frac{\mu}{2} \\ \cos\frac{\mu}{2} \end{pmatrix} \ , \tag{35}$$

for $\kappa = 1/L_+$ and $1/L_-$, respectively. Note that the relative phase factor $e^{-2i\nu}$ attached to the coefficients of the states on the positive half line can be removed by (32). Similarly, the

[4] However, a phase gap may be observed, for instance, on a circle by interference.

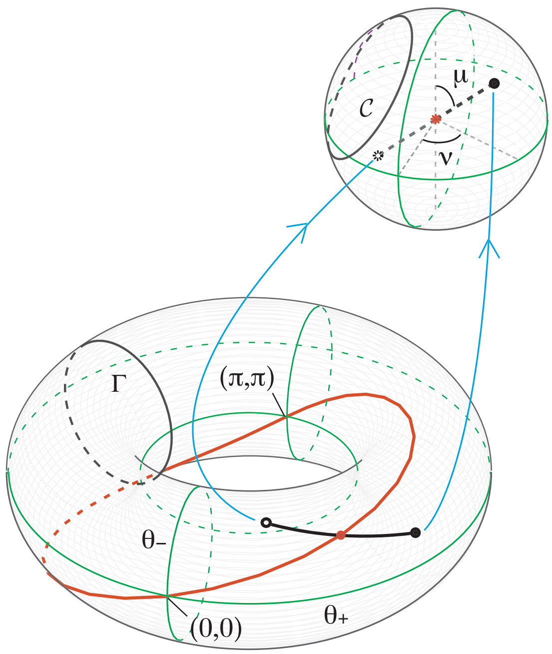

Figure 2. The parameter space $\{(\theta_+, \theta_-, \mu, \nu)\}$ is a product of the spectral torus T^2 specified by the angles (θ_+, θ_-) and the isospectral sphere S^2 specified by the angles (μ, ν) with radius $\rho = (\theta_+ - \theta_-)/2$ (cf. Corollary 1) which collapses to a point for the self-dual case $\theta_+ = \theta_-$. A cyclic path $\mathcal{C}$ on the sphere yields a phase anholonomy (the Berry phase) proportional to the area enclosed by $\mathcal{C}$ due to the degeneracy present at the center of the sphere. A cyclic path Γ on the torus, on the other hand, yields a level anholonomy (level shifts) if Γ is homotopically nontrivial. A generic cycle is a combination of the two, and hence yields an anholonomy in both phase and level. The parametrization shown here provides a double covering of the entire family $\Omega \simeq U(2)$, where the two antipodal points on the spheres equidistant from the self-dual line $\theta_+ = \theta_-$ are identified. This identification determines the spectral space Σ to be given by $T^2/\mathbb{Z}_2$ which is a Möbius strip with boundary.

scattering states for the particle (with velocity $v = \hbar k/m$) incident, say, from the positive side,

$$\varphi_k^{(+)}(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} t_k^{(+)} e^{-ikx}\,, & x < 0 \\ e^{-ikx} + r_k^{(+)} e^{ikx}\,, & x > 0 \end{cases} \tag{36}$$

have the reflection and transmission coefficients

$$\begin{pmatrix} r_k^{(+)} \\ t_k^{(+)} \end{pmatrix} = -\frac{1}{(1+ikL_+)(1+ikL_-)} \begin{pmatrix} 1 + k^2 L_+ L_- - ik(L_+ - L_-)\cos\mu \\ -ik(L_+ - L_-)\sin\mu\, e^{i\nu} \end{pmatrix} . \tag{37}$$

We observe that, in accordance with the interpretation, the factor $e^{i\nu}$ is simply the phase which is acquired by the transmitted wave when the incoming wave passes the point $x = 0$. We can also see that, unlike ν, each of the other three parameters plays an independent and physical role in the eigenstates of H_U.

Finally, let us illustrate the basic structure of the $U(2)$ family by considering a generic point interaction specified by U in the $U(2)$ parameter space which is shown in Fig.2 as a product of a torus representing (θ_+, θ_-) and a sphere with radius ρ (see (13) and Corollary 1) representing (μ, ν). On this torus, two point interactions connected by the duality transformation (25) are represented by two equidistant points from the self-dual loop, $\theta_+ = \theta_-$. The double covering of the parametrization implies that the two spheres attached to these dual points are actually the same, with antipodal points on the two spheres identified. Under a cyclic process on the sphere one can expect a phase anholonomy (the Berry phase) to arise, since the spectrum becomes degenerated at the center $\theta_+ = \theta_-$ which belongs to Ω_{SD} (see Proposition 5). One can also expect a level shift if the cycle is homotopically nontrivial on the torus (see, *e.g.*, [9]). The anholonomy both in phase and level has indeed been observed [7] for cycles passing through $U = \sigma_3$, that is, $(\theta_+, \theta_-) = (\pi, 0)$ and $(\mu, \nu) = (0, 0)$. We note that this point $U = \sigma_3$ is rather special because it has the invariant parity $\mathcal{P}(V = I) = \mathcal{P}_3$ and hence its invariant subfamily is just the separated subfamily $\Omega_{\mathcal{P}_3}$. Further, its isospectral subfamily $\Omega(D = \sigma_3)$ is (the continuous part of) the scale invariant subfamily Ω_{W} [3] in view of the fact [7] that such U satisfies the condition for scale invariance, $\det(U \pm I) = \det(\sigma_3 \pm I) = 0$. We stress, however, that the anholonomy in phase and/or level is a generic phenomenon observed for any cyclic process in the parameter space $\Omega \simeq U(2)$.

Acknowledgement: I.T. is indebted to S. Tanimura for useful comments. T.C. thanks members of the Theory Group of KEK for the hospitality extended to him during his stay. This work has been supported in part by the Grant-in-Aid for Scientific Research (C) (Nos. 10640301, 11640396 and 13640413) by the Japanese Ministry of Education, Science, Sports and Culture.

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