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Symmetry, Duality and Anholonomy of Point Interactions in One Dimension

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Abstract. We analyze the spectral structure of the one dimensional quantum mechanical system with point interaction, which is known to be parametrized by the group $U(2)$. Based on the classification of the interactions in terms of symmetries, we show, on a general ground, how the fermion-boson duality and the spectral anholonomy recently discovered can arise. A vital role is played by a hidden $su(2)$ formed by a certain set of discrete transformations, which becomes a symmetry if the point interaction belongs to a distinguished $U(1)$ subfamily in which all states are doubly degenerate. Within the $U(1)$, there is a particular interaction which admits the interpretation of the system as a supersymmetric Witten model.

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1. Introduction

The point interaction, *i.e.*, the interaction of zero range, is perhaps the simplest among all interactions in physics, and yet it is the most generic in the sense that any local potential can be approximated by the point interaction in the long wavelength limit. The formulation of the point interaction in quantum theory requires some sort of treatment of the singularity that appears in the short range limit of the potential. The problem of short range singularities has been familiar in quantum field theory where the ultraviolet divergence is tamed by the procedure of renormalization.

It is not widely recognized, however, that the same delicate problem already exists in the treatment of a quantum mechanical particle under point (or contact) interaction. In spatial dimension one, a stable zero range limit exists for the family of finite range interactions with a constant volume integral of the strength, which is none other than the Dirac δ-function interaction. However, its straightforward extension to spacial dimensions two and three fails because of the divergence in the Green's function. To put the point interaction on a mathematically sound basis, and thereby uncover its entire physical content in arbitrary dimensions, one has to resort to a more rigorous approach based on functional analysis. With the theory of self-adjoint extensions [1], one finds that the quantum mechanical point interaction can be defined in dimension one, two and three, and it is nontrivial in each of these cases. In particular, in one dimension there arises a $U(2)$ family of interactions [2] (see also [3]), where besides the Dirac δ-function interaction which induces discontinuity in the derivative of the wave function, there is another type of point interaction, called 'ε-function interaction', which induces discontinuity in the wave function itself. From these two typical zero-range forces, one can construct any of the interactions in the $U(2)$ family which in general has discontinuity both in the wave function and its derivative. Although these generic point interactions predict several unusual properties which defy our intuition on pointlike objects [3,4,5], the fact that these interactions are fully expressible as a limit of local potentials [6,7,8] suggests that their experimental realization is feasible, especially in view of the rapid advance of nano-scale technology of recent years.

It has been known that, in dimension two and three where the family of point interactions is given by $U(1)$, the realization of the interactions can be understood through coupling constant renormalization analogous to that encountered in quantum field theory,

and that in two dimensions this offers a prototype of quantum anomaly by the breakdown of scale symmetry [9]. In dimension one, on the other hand, the larger family $U(2)$ of allowed interactions admits a number of more intriguing features. These include the 'fermion-boson duality', which is the phenomenon that two systems with distinct point interactions related by coupling inversion shares an identical spectrum with symmetric and antisymmetric states interchanged [10]. The other notable feature is the spectral anholonomy [11], which is the appearance of a double spiral structure of the energy levels when the subfamily of parity invariant point interactions is considered. However, so far these phenomena have been shown to arise for a specific set of interactions, and one may wonder if this is a generic aspect of the point interaction rather than something accidental.

The aim of the present paper is to provide a coherent framework to understand all of these features on a general basis extending our previous report [12] substantially, and thereby show that these are in fact a generic aspect of the point interaction in one dimension. Since these features are intimately related to symmetries of the system, we shall first study symmetries and associated invariant subfamilies of the point interactions. The symmetries with respect to parity, time reversal and Weyl scaling transformations have been discussed earlier in [13] using a local description of the family. Here we give a fuller account of them by adopting a description valid globally over the $U(2)$ family, and point out that behind the aforementioned nontrivial features underlies an $su(2)$ structure consisting of generators of two novel discrete transformations along with parity. The $su(2)$ will then be shown to become a symmetry if the point interactions belong to a distinct $U(1)$ subfamily in the $U(2)$. The $U(1)$ subfamily forms a singular circle of spectral degeneracy in the parameter space, which is crucial in realizing the duality and the anholonomy. We will also touch upon briefly the possibility of supersymmetry, which is suggested from the degeneracy in the $U(1)$ subfamily.

The plan of the paper is as follows. After this Introduction, we present in Sect.2 a concise account of how the $U(2)$ family of point interactions appears in one dimension. Our argument is elementary [14] and based on the unitarity (probability conservation) of the system, which is, in essence, equivalent to the requirement that the Hamiltonian be self-adjoint. The gap between our derivation and the standard, more involved one which employs the theory of self-adjoint extensions will be filled by Appendix A supplied at the end of the paper. In Sect.3, we discuss symmetries of the point interactions in detail. The

crucial ingredient here is the existence of the $su(2)$ generators which give rise to spectrum-preserving transformations in the $U(2)$ family. We first examine the spectral characteristics such as the spectral flows of the invariant subfamily under the parity transformation. We then move on to the generic invariant subfamily associated with the $su(2)$ generators. This will be important in Sect.4 when we discuss the duality, which can be seen in any of the subfamilies in the $su(2)$. The parity invariant subfamily is again considered in analyzing the spectral anholonomy in the double spiral form, while the Weyl scaling invariant subfamily is found to be the parameter subspace where the Berry phase anholonomy can be found. Furthermore, supersymmetry is observed for a particular point interaction belonging to the $U(1)$ subfamily where the states are all classified according to the 'spin' of the $su(2)$ symmetry. For illustration of these phenomena we sometimes put the particle in a box to obtain an entirely discrete spectrum. The necessary material for the spectrum as well as other basic physical properties of the system occurring under the point interaction is provided in Appendix B. Finally, Sect.5 is devoted to our Summary and Discussions.

2. Systems with Point Interaction

We begin by reviewing how to describe and characterize one-dimensional point interactions in quantum mechanics. The requirement of probability conservation is shown to determine the allowed class of point interactions to be the group $U(2)$. The local description of point interactions often found in the literature is then related to our global description given by the $U(2)$. The set of point interactions missing in the local description is shown to form a subgroup $U(1) \times U(1) \subset U(2)$.

2.1. Point interaction in quantum mechanics

Given a quantum system with point interaction in one dimension, the first question we encounter is how to describe and characterize the interaction in an inclusive and consistent manner. In other words, we want to know what kind of point interactions are allowed quantum mechanically on a line, and how to specify them mathematically.

To answer this, we first need to clarify what we mean by 'point interaction'. As discussed in the Introduction, we regard it to be an idealized long wavelength or infrared limit of localized interactions in one dimension, and hence it is a singular interaction with zero range occurring at one point, say $x = 0$, on a line $\mathbb{R}$. A system with such an interaction can be described by a free system on the line with the singular point removed, namely, on $\mathbb{R} \setminus \{0\}$. Once defined this way, our main concern will be the Hamiltonian operator,

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} \ , \tag{2.1}$$

defined on a proper domain $D(H)$ in the Hilbert space,

$$\mathcal{H} = L^2(\mathbb{R} \setminus \{0\}) \ . \tag{2.2}$$

The Hamiltonian H is meant to be an observable of our quantum system, and if this is the sole requirement, then our question boils down to the problem of seeking for the allowed class of domains $D(H)$ on which the Hamiltonian operator (2.1) becomes self-adjoint. In physical terms, being the generator for time evolution, a self-adjoint Hamiltonian implies probability conservation in the entire system. Probability conservation is guaranteed if the probability current,

$$j(x) = -\frac{i\hbar}{2m}\left((\varphi^*)'\varphi - \varphi^*\varphi'\right)(x) \ , \tag{2.3}$$

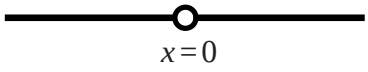

Figure 1. Point interaction in one dimension. The interaction that occurs only at $x = 0$ may be realized simply by removing the point from the real line $\mathbb{R}$. The unitarity of the system demands that the probability current $j(x)$ be continuous at the missing point.

is continuous around the singular point, namely,

$$j(0_-) = j(0_+) \ , \tag{2.4}$$

where 0_+ and 0_- denote the limits to zero from above and from below, respectively.

The requirement (2.4) implies that any state in the domain $D(H)$ must obey a certain set of boundary conditions at $x = 0_\pm$. To exhibit those conditions explicitly, we follow the approach of Ref.[14] and define the two-component vectors,

$$\Phi := \begin{pmatrix} \varphi(0_+) \\ \varphi(0_-) \end{pmatrix}, \qquad \Phi' := \begin{pmatrix} \varphi'(0_+) \\ -\varphi'(0_-) \end{pmatrix}, \tag{2.5}$$

to a given state φ. (Note the minus sign in the second component of Φ'.) In terms of these vectors, the requirement (2.4) becomes $\Phi'^\dagger\Phi = \Phi^\dagger\Phi'$, which is equivalent to

$$|\Phi - iL_0\Phi'| = |\Phi + iL_0\Phi'| \ , \tag{2.6}$$

with L_0 being an arbitrary nonzero constant with the dimension of length. This means that, with a two-by-two unitary matrix $U \in U(2)$, we have

$$(U - I)\Phi + iL_0(U + I)\Phi' = 0 \ . \tag{2.7}$$

It can be shown (see the Appendix) that this matrix U is the same for all states φ in the domain $D(H)$. A standard parametrization for $U \in U(2)$ is given by

$$U = e^{i\xi}\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} = e^{i\xi}\begin{pmatrix} \alpha_R + i\alpha_I & \beta_R + i\beta_I \\ -\beta_R + i\beta_I & \alpha_R - i\alpha_I \end{pmatrix} \ , \tag{2.8}$$

where $\xi \in [0, \pi)$ and α, β are complex parameters satisfying

$$|\alpha|^2 + |\beta|^2 = \alpha_R^2 + \alpha_I^2 + \beta_R^2 + \beta_I^2 = 1 \ . \tag{2.9}$$

Since there is a one-to-one correspondence between a physically distinct point interaction and a self-adjoint Hamiltonian, the foregoing argument shows that the entire family Ω of point interactions admitted in quantum mechanics is exhausted by the group $U(2)$, and that different point interactions are characterized by different boundary conditions (2.7) for the wave functions. In short, in quantum mechanics a point interaction in one dimension is specified by its *characteristic matrix* $U \in U(2)$. (Alternatively, the condition (2.7) can be derived using the theory of self-adjoint extensions [2].)

In passing we point out that the parameter L_0 shows the presence of a scale factor in the system and, consequently, signals the quantum mechanical breakdown of scaling symmetry that is present on the classical level. Scale invariance remains valid quantum mechanically only for a certain subclass of point interactions (for detail, see Sect.3). The parameter L_0 adds no freedom to the boundary condition other than the $U(2)$, because its apparent freedom can be shown to be absorbed by adjusting the $U(2)$ parameters in U (see the Appendix).

2.2. Global vs. local description

In the literature [6,13,15,16,17], a different parametrization for the $U(2)$ point interactions has often been employed. It is the description in terms of the connection condition at the singular point,

$$\begin{pmatrix} \varphi(0_+) \\ \varphi'(0_+) \end{pmatrix} = \Lambda \begin{pmatrix} \varphi(0_-) \\ \varphi'(0_-) \end{pmatrix} \ . \tag{2.10}$$

The (infinitesimal) transfer matrix Λ used in (2.10) takes the form

$$\Lambda = e^{i\chi} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ , \qquad \chi \in [0, \pi), \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1 \ , \tag{2.11}$$

which shows that the family of point interactions covered by (2.10) forms the group $U(1) \times SL(2, \mathbb{R})$ rather than $U(2)$. Actually, it is straightforward to confirm that, for $\beta \neq 0$, our description (2.7) with (2.8) can be put into the connection form (2.10) with

$$\Lambda = \frac{i}{\beta_R - i\beta_I} \begin{pmatrix} \sin\xi - \alpha_I & -L_0(\cos\xi + \alpha_R) \\ L_0^{-1}(\cos\xi - \alpha_R) & \sin\xi + \alpha_I \end{pmatrix} \ . \tag{2.12}$$

The transfer matrix provides a direct description of the physical effect that arises at the singular point, and for this reason the description (2.10) is convenient to use for characterizing the point interaction based on the scattering (reflection and transmission) picture of incident waves (see Appendix B). However, it should be stressed that the transfer matrix provides a *local* description of the entire family $\Omega \simeq U(2)$ of point interactions, since it does not contain the case $\beta = 0$ as seen from the above correspondence. To find out what kind of point interactions are missing from (2.10), we restrict ourselves to the case $\beta = 0$ where we can put $\alpha_R + i\alpha_I = e^{i\rho}$ with $\rho \in [0, 2\pi)$ to obtain $U = e^{i\xi}\, e^{i\rho\sigma_3}$ in terms of the Pauli matrix σ_3. We will also need the projection matrices with respect to σ_i,

$$P_i^{\pm} := \frac{1 \pm \sigma_i}{2} \ , \qquad \text{for} \quad i = 1,\, 2,\, 3, \tag{2.13}$$

each of which satisfies

$$(P_i^{\pm})^2 = P_i^{\pm} \ , \qquad P_i^{\pm} P_i^{\mp} = 0 \ , \qquad P_i^{+} + P_i^{-} = 1 \ . \tag{2.14}$$

A convenient parametrization available for the present case is then provided by using $\phi_{\pm} = \xi \pm \rho \ (\mathrm{mod}\ 2\pi)$ to express

$$U = U(\phi_+, \phi_-) = e^{i(\phi_+ P_3^+ + \phi_- P_3^-)} \ . \tag{2.15}$$

Note that the ranges $\phi_{\pm} \in [0, 2\pi)$ cover the entire U belonging to this case, for which we have the chiral decomposition,

$$U(\phi_+, \phi_-) = U_+(\phi_+)\, U_-(\phi_-) = U_-(\phi_-)\, U_+(\phi_+) \ , \tag{2.16}$$

with

$$U_{\pm}(\phi_{\pm}) := e^{i\phi_{\pm} P_3^{\pm}} = 1 + (e^{i\phi_{\pm}} - 1) P_3^{\pm} \ . \tag{2.17}$$

This shows that the set of these U forms a subfamily given by the subgroup,

$$\Omega_R \simeq U(1) \times U(1) \subset \Omega \simeq U(2) \ . \tag{2.18}$$

We shall find in the next section that Ω_R appears also as the subfamily of point interactions invariant under a certain discrete transformation. These point interactions are those of a perfect wall placed at $x = 0$ through which no probability flow is allowed, separating the left $\mathbb{R}^-$ and the right half line $\mathbb{R}^+$ completely.

3. Symmetries

In this section we study how symmetry transformations such as parity, time-reversal and Weyl scaling are implemented for point interactions. Our analysis is in a sense analogous to Ref.[13], but we shall soon see the advantage of the global nature of our parametrization. Subfamilies invariant under these transformations are defined, which are then used to classify the point interactions in one dimension. Behind these subfamilies is an $su(2)$ structure, which plays a central role in establishing the duality to be discussed later. The structure of the invariant subfamilies obtained here will also be analyzed.

3.1. Transformations and invariant subfamilies

3.1.1. Parity

To begin with, we consider the *parity* transformation which is defined for any $\varphi \in \mathcal{H}$ by

$$\mathcal{P}: \quad \varphi(x) \longrightarrow (\mathcal{P}\varphi)(x) := \varphi(-x) \ . \tag{3.1}$$

Under this, the boundary values of a wave function and its derivatives at $x = 0_\pm$ are interchanged. Accordingly, the boundary vectors (2.5) transform as

$$\Phi \xrightarrow{\mathcal{P}} \sigma_1 \Phi \ , \qquad \Phi' \xrightarrow{\mathcal{P}} \sigma_1 \Phi' \ . \tag{3.2}$$

As can be seen from (2.7), this induces a map on the $U(2)$ family Ω such that

$$U \xrightarrow{\mathcal{P}} \sigma_1 \, U \, \sigma_1 \ , \tag{3.3}$$

that is, a point interaction (or a domain $D(H)$) specified by U maps to another one specified by $\sigma_1 U \sigma_1$. From this it follows that parity invariant point interactions (or parity invariant domains $D(H)$) are characterized by those U which fulfill

$$\sigma_1 \, U \, \sigma_1 = U \ . \tag{3.4}$$

The general solution of this is given by $\alpha_I = 0$ and $\beta_R = 0$ in (2.8), and if we put $\alpha_R = \cos\phi$ and $\beta_I = \sin\phi$, we have

$$U = e^{i\xi} \begin{pmatrix} \cos\phi & i\sin\phi \\ i\sin\phi & \cos\phi \end{pmatrix} \ , \tag{3.5}$$

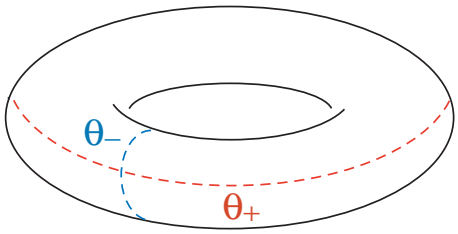

Figure 2. The torus $S^1 \times S^1$ representing the subfamily Ω_P in Ω.

which can be cast into a form analogous to (2.15) as

$$U = U(\theta_+, \theta_-) = e^{i(\theta_+ P_1^+ + \theta_- P_1^-)} \ , \tag{3.6}$$

using angle parameters $\theta_\pm \in [0, 2\pi)$ defined by

$$\theta_\pm := \xi \pm \phi, \tag{3.7}$$

and the projection matrices, $P_1^\pm$ in (2.13). Thus, by an analogous reasoning, we observe that the *parity invariant* subfamily Ω_P is a direct product of the two $U(1)$ groups,

$$\Omega_P \simeq U(1) \times U(1) \subset \Omega \simeq U(2) \ . \tag{3.8}$$

The parameter space of the subgroup $U(1) \times U(1)$ for Ω_P is a torus $S^1 \times S^1$ (see Fig.2) which is, of course, different from the one corresponding to the subgroup (2.18) for Ω_R.

3.1.2. Time-reversal

We next consider the *time-reversal* transformation defined by

$$\mathcal{T} : \quad \varphi(x) \longrightarrow (\mathcal{T}\varphi)(x) := \varphi^*(x) \ . \tag{3.9}$$

This implies that on vectors we have

$$\Phi \xrightarrow{\mathcal{T}} \Phi^* \ , \qquad \Phi' \xrightarrow{\mathcal{T}} \Phi'^* \ . \tag{3.10}$$

From the boundary condition (2.7) we find that the time-reversal transformation induces a map on Ω by

$$U \xrightarrow{\mathcal{T}} U^{\mathsf{T}} \ , \tag{3.11}$$

where T denotes transposition. Thus, a given point interaction is time-reversal invariant if its characteristic matrix U obeys

$$U^{\mathsf{T}} = U \ . \tag{3.12}$$

This condition is fulfilled if $\beta_R = 0$ in (2.8), that is,

$$U = e^{i\xi} \begin{pmatrix} \alpha_R + i\alpha_I & i\beta_I \\ i\beta_I & \alpha_R - i\alpha_I \end{pmatrix} \ . \tag{3.13}$$

Thus we see that the time-reversal invariant subspace $\Omega_T \subset \Omega$ is isomorphic to the coset space,

$$\Omega_T \simeq \frac{U(2)}{U(1)} \simeq S^1 \times S^2 \ , \tag{3.14}$$

where the $U(1)$ is the subgroup generated by σ_2 in the $SU(2) \subset U(2)$.

3.1.3. $\mathcal{PT}$-transformation

The combined $\mathcal{PT}$-transformation can also be considered as

$$\mathcal{PT}: \quad \varphi(x) \longrightarrow (\mathcal{PT}\varphi)(x) := (\mathcal{P}(\mathcal{T}\varphi))(x) = \varphi^*(-x) \ , \tag{3.15}$$

under which the vectors transform as

$$\Phi \xrightarrow{\mathcal{PT}} \sigma_1 \Phi^* \ , \qquad \Phi' \xrightarrow{\mathcal{PT}} \sigma_1 \Phi'^* \ . \tag{3.16}$$

The map on Ω induced by the $\mathcal{PT}$-transformation is then

$$U \xrightarrow{\mathcal{PT}} \sigma_1 \, U^{\mathsf{T}} \, \sigma_1 \ , \tag{3.17}$$

The invariant subspace Ω_{PT} under the $\mathcal{PT}$-transformation is furnished by the set of U obeying the condition,

$$\sigma_1 \, U^{\mathsf{T}} \, \sigma_1 = U \ . \tag{3.18}$$

This condition holds for U if $\alpha_I = 0$ in (2.8), that is,

$$U = e^{i\xi} \begin{pmatrix} \alpha_R & \beta_R + i\beta_I \\ -\beta_R + i\beta_I & \alpha_R \end{pmatrix} \ , \tag{3.19}$$

and hence, again, the invariant subspace $\Omega_{PT} \subset \Omega$ is isomorphic to the coset,

$$\Omega_{PT} \simeq \frac{U(2)}{U(1)} \simeq S^1 \times S^2 \ , \tag{3.20}$$

where now the $U(1)$ is the subgroup generated by σ_3.

3.1.4. Weyl scaling

Unlike the previous three discrete transformations, the *Weyl scaling* transformation is a continuous transformation with parameter $\lambda > 0$ and is given by

$$\mathcal{W}_\lambda : \quad \varphi(x) \longrightarrow (\mathcal{W}_\lambda \varphi)(x) := \lambda^{\frac{1}{2}} \varphi(\lambda x) \ . \tag{3.21}$$

This implies

$$\varphi(0_\pm) \xrightarrow{\mathcal{W}_\lambda} \lambda^{\frac{1}{2}} \varphi(0_\pm) \ , \qquad \varphi'(0_\pm) \xrightarrow{\mathcal{W}_\lambda} \lambda^{\frac{3}{2}} \varphi'(0_\pm) \ , \tag{3.22}$$

and hence

$$\Phi \xrightarrow{\mathcal{W}_\lambda} \lambda^{\frac{1}{2}} \Phi \ , \qquad \Phi' \xrightarrow{\mathcal{W}_\lambda} \lambda^{\frac{3}{2}} \Phi' \ . \tag{3.23}$$

The boundary condition (2.7) proves to be unchanged under the Weyl scaling transformation (3.21) if each of the two terms in (2.7) vanishes separately,

$$(U - I)\Phi = 0 \ , \qquad (U + I)\Phi' = 0 \ . \tag{3.24}$$

Since Φ and Φ' cannot vanish simultaneously, we observe that $U = \pm I$ or else the two eigenvalues of U are $+1$ and -1, namely,

$$\det(U - I) = \det(U + I) = 0 \ . \tag{3.25}$$

This latter is achieved if $\xi = \pi/2$ and $\alpha_R = 0$ in (2.8) where we have

$$U = i \begin{pmatrix} i\alpha_I & \beta_R + i\beta_I \\ -\beta_R + i\beta_I & -i\alpha_I \end{pmatrix} \ . \tag{3.26}$$

We find that the subset $\Omega_W \subset \Omega$ formed by these matrices U is isomorphic to the sphere,

$$\Omega_W \simeq \frac{U(2)}{U(1) \times U(1)} \simeq S^2 \ , \tag{3.27}$$

with the second $U(1)$ being the subgroup generated by σ_3. Together with the isolated points $\{U = \pm I\}$, this continuous subset Ω_W provides the scale invariant subfamily within Ω. We can see from (3.24) that, for scale invariant point interactions, the scale parameter L_0 drops out from the boundary conditions as expected.

3.2. Spectrum-preserving $su(2)$

We have seen that the parity transformation induces a map on Ω which is the conjugation of U by σ_1. We now provide transformations which induce similar maps on Ω by means of the conjugation of U by other Pauli matrices σ_i for $i = 2$ and 3.

The first is the *half-reflection* transformation $\mathcal{R}$ defined by

$$\mathcal{R}: \quad \varphi(x) \longrightarrow (\mathcal{R}\varphi)(x) := [\Theta(x) - \Theta(-x)]\varphi(x) \ , \tag{3.28}$$

where $\Theta(x)$ is the Heaviside step function. On boundary vectors, this leads to

$$\Phi \xrightarrow{\mathcal{R}} \sigma_3\Phi \ , \qquad \Phi' \xrightarrow{\mathcal{R}} \sigma_3\Phi' \ . \tag{3.29}$$

We then see in (2.7) that this induces a map on Ω by

$$U \xrightarrow{\mathcal{R}} \sigma_3\, U\, \sigma_3 \ . \tag{3.30}$$

Hence, the subfamily Ω_R of point interactions invariant under the half-reflection transformation is characterized by those U obeying

$$\sigma_3\, U\, \sigma_3 = U \ . \tag{3.31}$$

The general solution of this is

$$U = e^{i\xi} \begin{pmatrix} e^{i\rho} & 0 \\ 0 & e^{-i\rho} \end{pmatrix} \ , \tag{3.32}$$

with $\xi \in [0, \pi)$ and $\rho \in [0, 2\pi)$, which is the same as the one given in (2.15). This implies that the subfamily Ω_R is in fact the one given in (2.18) which is the set of interactions missing in the local description.

The remaining transformation corresponding to σ_2 is furnished by

$$\mathcal{Q}: \quad \varphi(x) \longrightarrow (\mathcal{Q}\varphi)(x) := i[\Theta(-x) - \Theta(x)]\varphi(-x) \ . \tag{3.33}$$

This is just the combination $\mathcal{Q} = -i\mathcal{R}\mathcal{P}$, and on vectors it implements the transformation,

$$\Phi \xrightarrow{\mathcal{Q}} \sigma_2\Phi \ , \qquad \Phi' \xrightarrow{\mathcal{Q}} \sigma_2\Phi' \ . \tag{3.34}$$

It follows from (2.7) that the induced map on Ω is given by

$$U \xrightarrow{\mathcal{Q}} \sigma_2\, U\, \sigma_2 \ , \tag{3.35}$$

as required. The subfamily Ω_Q of point interactions invariant under this transformation is characterized by

$$\sigma_2\, U\, \sigma_2 = U \ . \tag{3.36}$$

As is clear now, this admits the following solution,

$$U = U(\omega_+, \omega_-) = e^{i(\omega_+ P_2^+ + \omega_- P_2^-)} \ , \tag{3.37}$$

with $\omega_\pm \in [0, 2\pi)$ and the projection matrices $P_2^\pm$ in (2.13). The invariant subfamily Ω_Q is therefore given by

$$\Omega_Q \simeq U(1) \times U(1) \subset \Omega \simeq U(2) \ . \tag{3.38}$$

To sum up, we have altogether three different tori $S^1 \times S^1$ for Ω_P, Ω_Q and Ω_R in $\Omega \simeq S^1 \times S^3$. These tori are formed by distinct subgroups $U(1) \times U(1)$ in $U(2)$, where the first $U(1)$ accounts for the overall phase factor while the second $U(1)$ is the subgroup generated by σ_i for $i = 1,\, 2,\, 3$.

A remarkable point to note is that the three discrete transformations, $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$, satisfy the relations,

$$\mathcal{P}\mathcal{Q} = -\mathcal{Q}\mathcal{R} = i\mathcal{R} \ , \qquad \mathcal{Q}\mathcal{R} = -\mathcal{R}\mathcal{Q} = i\mathcal{P} \ , \qquad \mathcal{R}\mathcal{P} = -\mathcal{P}\mathcal{R} = i\mathcal{Q} \ , \tag{3.39}$$

and

$$\mathcal{P}^2 = \mathcal{Q}^2 = \mathcal{R}^2 = 1 \ . \tag{3.40}$$

As a result, the set $\{\mathcal{P}, \mathcal{Q}, \mathcal{R}\}$ forms an $su(2)$ algebra. The vectors Φ and Φ' provide distinctive two-dimensional bases on which the $su(2)$ is represented in terms of the Pauli matrices. In fact, the relations (3.39) and (3.40) are precisely those fulfilled by the Pauli matrices.

Importantly, since the three discrete transformations $\{\mathcal{P}, \mathcal{Q}, \mathcal{R}\}$ of $su(2)$ act as multiplications by constant factors along the positive and the negative half lines and/or the mirror reflection, they are intrinsically *spectrum-preserving*. More explicitly, if φ is an

energy eigenstate $H\,\varphi(x) = E\,\varphi(x)$ of the Hamiltonian H in (2.1), then, *e.g.*, the state $\mathcal{R}\varphi$ is also an eigenstate of the Hamiltonian with the same energy E:

$$H\,(\mathcal{R}\varphi)(x) = E\,(\mathcal{R}\varphi)(x)\ . \tag{3.41}$$

Moreover, if φ is a simultaneous eigenstate of $\mathcal{P}$ as $\mathcal{P}\varphi = \pm\varphi$, then the mapped state has the opposite parity,

$$\mathcal{P}\,(\mathcal{R}\varphi) = -\mathcal{R}\,(\mathcal{P}\varphi) = \mp\mathcal{R}\varphi\ . \tag{3.42}$$

Obviously, the same holds for other generators of $su(2)$ as well. However, this does not necessarily mean that the $su(2)$ is a symmetry, because the transformations alter the domain $D(H)$ of the Hamiltonian, according to (3.3), (3.30) and (3.35). As we shall see shortly, there is only a special subclass of domains characterized by point interactions which remain unaltered under these transformations. When this happens, the $su(2)$ becomes a symmetry of the system, and this $su(2)$ symmetry will be important to realize the duality discussed in the next section.

We have learned that, in the entire family $\Omega = U(2)$ of point interactions allowed in one dimension, there are several subfamilies characterized by their symmetries. Before analyzing the structure of the invariant subfamilies in detail, we mention here that these subfamilies can be used to provide a classification of point interactions. To see this, let $U \in U(2)$ be the characteristic matrix of a generic point interaction. Because of the complementary structure of the two invariant subspaces Ω_W and Ω_R, as seen in (3.27) and (2.18), the matrix U can be decomposed as

$$U = U_W\,U_R\ , \tag{3.43}$$

where U_W is of the form (3.26) while U_R is given by the form (3.32). Namely, a generic point interaction is a 'product' of two point interactions, one is invariant under the Weyl scaling $\mathcal{W}_\lambda$ and the other is invariant under the half-reflection $\mathcal{R}$. More precisely, one can observe that the decomposition (3.43) provides a double covering of the entire family $\Omega \simeq U(2)$. This can be confirmed explicitly by computing the product as

$$U_W\,U_R = e^{i\xi'}\begin{pmatrix} \alpha' & \beta' \\ -\beta'^{*} & \alpha'^{*} \end{pmatrix}\ , \tag{3.44}$$

where $\xi' = \xi + \pi/2 \ (\mathrm{mod}\ \pi)$, $\alpha' = i\alpha_I e^{i\rho}$ and $\beta' = \beta e^{-i\rho}$. Conversely, to a given U as the r.h.s. of (3.44), the parameters in the decomposition are identified as $\rho = \arg\alpha' -$

$\pi/2 \pmod{2\pi}$, $\alpha_I = |\alpha'| \geq 0$ and $\beta = \beta' e^{i\rho}$. Thus we need only non-negative α_I to cover the $U(2)$ once, and negative α_I provide another covering of $U(2)$. From this consideration we learn that, once the choice of the covering is made, the decomposition $U = U_W U_R$ is unique except for the case $\alpha' = 0$ where ρ can be chosen arbitrarily.

Similar decompositions are also available for other choice of invariant subfamilies based on the *local* decomposition of $U(2) \simeq S^1 \times S^3$ by $(S^1 \times S^2) \times S^1$. For instance, one may consider the Abelian subgroup $U(1) \subset \Omega_R$ generated by σ_3, whereby obtain the decomposition $U = U_3\, U_{PT}$ for any $U \in U(2)$ using U_3 and U_{PT} from the $U(1)$ and Ω_{PT}, respectively. These decompositions rest basically on the choice of the spheres S^2 in $U(2)$, which are obtained, *e.g.*, by the condition analogous to (3.25),

$$\det(U - \sigma_i) = \det(U + \sigma_i) = 0 \ . \tag{3.45}$$

Indeed, this leads to the S^2 in Ω_{PT} for $i = 3$ and the one in Ω_T for $i = 2$. We also mention that, albeit being local, the transfer matrix formalism admits a similar classification, a similar decomposition of the family Ω, a decomposition which is physically more sensible in the sense that the sequence of matrices in the decomposition from left to right corresponds to the actual sequence of point interactions placed from left to right at the singularity [7].

3.3. Parity invariant subfamily and spin

We now take a closer look at some of the invariant subfamilies discussed so far. We first analyze the parity invariant subfamily Ω_P given by (3.6). For this, recall that any state φ in the Hilbert space (2.2) can be decomposed into the sum $\varphi = \varphi_+ + \varphi_-$ of a symmetric φ_+ and antisymmetric states φ_-. From $\varphi_+(-x) = \varphi_+(x)$ and $\varphi_-(-x) = -\varphi_-(x)$ we obviously have

$$\varphi_\pm(0_+) = \pm\varphi_\pm(0_-) \ , \qquad \varphi'_\pm(0_+) = \mp\varphi'_\pm(0_-) \ . \tag{3.46}$$

Then, plugging the decomposition into the boundary condition (2.7) and using the relations (3.46), we find that the condition splits into two conditions,

$$\begin{aligned} \sin\frac{\theta_+}{2}\,\varphi_+(0_+) + L_0 \cos\frac{\theta_+}{2}\,\varphi'_+(0_+) = 0 \ , \\ \sin\frac{\theta_-}{2}\,\varphi_-(0_+) + L_0 \cos\frac{\theta_-}{2}\,\varphi'_-(0_+) = 0 \ . \end{aligned} \tag{3.47}$$

Thus we see that the two angles $\theta_\pm$ specify the connection conditions at the gap $x = 0_\pm$ separately for symmetric and antisymmetric states $\varphi_\pm$. An important consequence of this is that the energy spectrum of the symmetric sector is determined solely by the angle θ_+, and in the antisymmetric sector by θ_- alone. The equivalence of the connection condition of the two sectors in (3.47) implies that the spectra are given by the same function of the angles $\theta_\pm$.

At this point it is worth mentioning that, among the one-dimensional parity invariant systems on a line $\mathbb{R}$, there are two typical singular potentials which we are familiar with. One is the delta-function potential, $V(x) = \delta(x;\theta_+) = c(\theta_+)\,\delta(x)$, which gives rise to a gap in the derivative $\varphi'(x)$ of a wave function $\varphi(x)$ at $x = 0$ proportional to the constant $c(\theta_+)$ while keeping the continuity of the value $\varphi(x)$ at $x = 0$. The other, which is less familiar, is the epsilon-function potential, $V(x) = \varepsilon(x;\theta_-)$, which brings about a gap in the value $\varphi(x)$ at $x = 0$ preserving the continuity in the derivative $\varphi'(x)$ there.

We may view our system with point interaction defined on $\mathbb{R} \setminus \{0\}$ as a system on $\mathbb{R}$ with some effective singular potential. Conversely, the system on $\mathbb{R}$ with a delta-function (or epsilon-function) potential may be regarded as a special case of our systems on $\mathbb{R} \setminus \{0\}$. In fact, the boundary conditions implied by the delta-function potential arises precisely at $(\theta_+, \theta_-) = (\theta_+, \pi)$. It is also easy to confirm that the epsilon-function potential is reproduced at $(\theta_+, \theta_-) = (0, \theta_-)$. In particular, at the intersection $(0, \pi)$ we obtain $\varphi'_+(0_+) = \varphi_-(0_+) = 0$, which implies the smooth continuity both in the values and the derivatives, $\varphi(+0) = \varphi(-0)$ and $\varphi'(+0) = \varphi'(-0)$. Hence, the point $(0, \pi)$ yields nothing but the free system.

Now we ask ourselves what happens if we implement other transformations of $su(2)$ in the parity invariant subfamily Ω_P. Consider the half-reflection $\mathcal{R}$ applied to the system of parity invariant point interactions. From (3.30) we observe that

$$U(\theta_+, \theta_-) \xrightarrow{\mathcal{R}} \sigma_3\, U(\theta_+, \theta_-)\, \sigma_3 = U(\theta_-, \theta_+)\ . \tag{3.48}$$

The half-reflection $\mathcal{R}$ therefore induces on Ω_P the interchange of parameters, $(\theta_+, \theta_-) \xrightarrow{\mathcal{R}} (\theta_-, \theta_+)$, which is the map across the diagonal line $\theta+ = \theta_-$ as shown in the diagram in Fig.3. On account of the dual aspect of the spectrum under the map as seen in (3.41) and (3.42) before, we hereafter call the two points (θ_+, θ_-) and (θ_-, θ_+) the *dual* of each other.

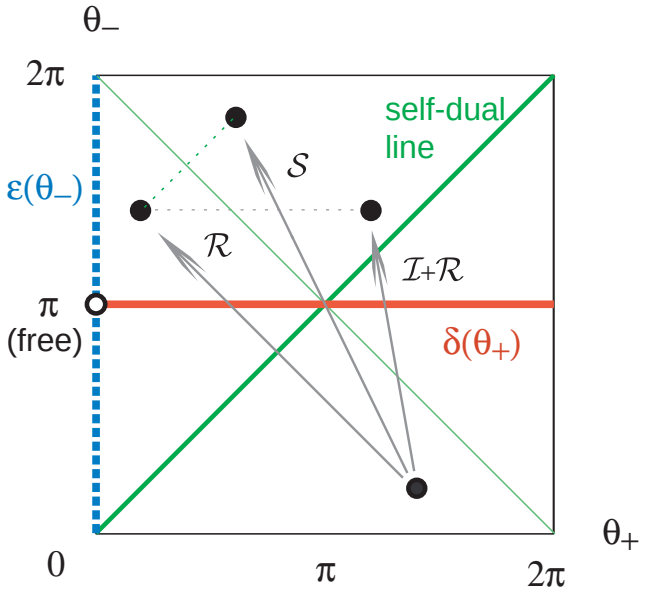

Figure 3. The dissected torus Ω_P. The $\mathcal{R}$ transformation and two other discrete transformations, $\mathcal{I}_+\mathcal{R}$ and $\mathcal{S}$ mentioned in Sect.4, are indicated by the arrows. The horizontal solid line and the vertical dotted line represent the delta-function interactions and the epsilon-function interactions, respectively.

The set of points in Ω_P lying on the diagonal line, *i.e.*, the *self-dual* points, forms the subgroup,

$$\Omega_{SD} \simeq U(1) \subset \Omega_P \simeq U(1) \times U(1) \ , \tag{3.49}$$

consisting of the matrices,

$$U = e^{i\theta} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \qquad \theta \in [0, 2\pi) \ . \tag{3.50}$$

From the foregoing argument we see that points on the diagonal line in Ω_{SD} are left unchanged under the half-reflection $\mathcal{R}$, which means that the domain $D(H)$ of the Hamiltonian defined at a point on Ω_{SD} is invariant under $\mathcal{R}$. Moreover, since $\mathcal{Q}$ is just a product of $\mathcal{P}$ and $\mathcal{R}$, we conclude that the entire $su(2)$ arises as a symmetry algebra on Ω_{SD},

$$[H, \mathcal{P}] = [H, \mathcal{Q}] = [H, \mathcal{R}] = 0 \ . \tag{3.51}$$

It follows that, for point interactions defined at self-dual points, every energy eigenstate falls into an irreducible representation of $su(2)$, *i.e.*, it possesses a *spin* with respect to the

$su(2)$ algebra of the generators for the discrete transformations. More specifically, because $\mathcal{R}$ flips the parity of the state, there always be a double (or even) degeneracy in every level of energy spectrum on Ω_{SD}, and this number of degeneracy is given by the spin of the state.

3.4. Generic invariant subfamily of $su(2)$

We now argue that the structure of the invariant subfamily discussed for Ω_P is generic, that is, analogous results can also hold for other invariant subfamilies $su(2)$. To see this, let $\mathcal{X}_i$, $i = 1, 2, 3$, be the generators $\{\mathcal{P}, \mathcal{Q}, \mathcal{R}\}$ of the $su(2)$. Since $\mathcal{X}_i^2 = 1$ as seen in (3.40), one can decompose the Hilbert space $\mathcal{H}$ into the eigenspaces $\mathcal{H}_\pm$ with eigenvalues ± 1,

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \ , \tag{3.52}$$

that is, any state $\varphi \in \mathcal{H}$ can be put

$$\varphi = \varphi_+ + \varphi_- \ , \qquad \varphi_\pm := \left(\frac{1 \pm \mathcal{X}_i}{2}\right) \varphi \ . \tag{3.53}$$

The corresponding decomposition of the boundary vectors is

$$\Phi = \Phi_+ + \Phi_- \ , \qquad \Phi' = \Phi'_+ + \Phi'_- \ , \tag{3.54}$$

where

$$\Phi_\pm := P_i^\pm \Phi \ , \qquad \Phi'_\pm := P_i^\pm \Phi' \ . \tag{3.55}$$

Since the $\mathcal{X}_i$ transformation yields $\Phi \xrightarrow{\mathcal{X}_i} \sigma_i \Phi$ and $\Phi' \xrightarrow{\mathcal{X}_i} \sigma_i \Phi'$, and since

$$\sigma_i \Phi_\pm = \pm \Phi_\pm \ , \qquad \sigma_i \Phi'_\pm = \pm \Phi'_\pm \ , \tag{3.56}$$

we find that $\Phi_\pm$ and $\Phi'_\pm$ are indeed the boundary vectors for the eigenstates $\varphi_\pm$. Note that the matrix form available in the subfamily invariant under the $\mathcal{X}_i$ transformation reads

$$U = U(\vartheta_+, \vartheta_-) = e^{i(\vartheta_+ P_i^+ + \vartheta_- P_i^-)} \ , \tag{3.57}$$

where $\vartheta_\pm \in [0, 2\pi)$ are two angle parameters. Then, upon using the identity,

$$U(\vartheta_+, \vartheta_-) \pm 1 = (e^{i\vartheta_+} \pm 1) P_i^+ + (e^{i\vartheta_-} \pm 1) P_i^- \ , \tag{3.58}$$

one can split the boundary condition (2.7) into two conditions imposed on the eigenstates,

$$\begin{aligned} \sin\frac{\vartheta_+}{2}\,\Phi_+ + L_0\cos\frac{\vartheta_+}{2}\,\Phi'_+ = 0\ , \\ \sin\frac{\vartheta_-}{2}\,\Phi_- + L_0\cos\frac{\vartheta_-}{2}\,\Phi'_- = 0\ . \end{aligned} \tag{3.59}$$

By construction, each of the projected boundary vectors has only one independent component. For example, for $\mathcal{X}_1 = \mathcal{P}$ from (3.55) we have

$$\Phi_+ = (\varphi_+(0_+), \varphi_+(0_+))^{\mathsf{T}}\ , \qquad \Phi_- = (\varphi_-(0_+), \varphi_-(0_+))^{\mathsf{T}}\ , \tag{3.60}$$

(and similarly for the derivatives), and hence the conditions (3.59) reduce to (3.47) which we have obtained earlier. On the other hand, for $\mathcal{X}_3 = \mathcal{R}$ we have

$$\Phi_+ = (\varphi_+(0_+), 0)^{\mathsf{T}}\ , \qquad \Phi_- = (0, \varphi_-(0_-))^{\mathsf{T}}\ , \tag{3.61}$$

where now the eigenstates $\varphi_\pm$ are those which have supports only on the left half line $\mathbb{R}^-$ and the right half line $\mathbb{R}^+$, respectively. As a result, the boundary conditions (3.59) read

$$\begin{aligned} \sin\frac{\phi_+}{2}\,\varphi_+(0_+) + L_0\cos\frac{\phi_+}{2}\,\varphi'_+(0_+) = 0\ , \\ \sin\frac{\phi_-}{2}\,\varphi_-(0_-) + L_0\cos\frac{\phi_-}{2}\,\varphi'_-(0_-) = 0\ . \end{aligned} \tag{3.62}$$

An important point to note is that these conditions (3.62) are nothing but the ones which arise when we require the probability conservations $j(0_+) = j(0_-) = 0$ separately on the two half lines, $\mathbb{R}^+$ and $\mathbb{R}^-$. This implies that the subfamily Ω_R of point interactions invariant under the half-reflection $\mathcal{R}$ describes (non-unique) perfect walls at $x = 0$ through which no probability flow can penetrate, separating the left $\mathbb{R}^-$ and the right half lines $\mathbb{R}^+$ completely. For this reason, the subfamily Ω_R may also be called the *separated subfamily*. Finally, for the subfamily invariant under $\mathcal{X}_2 = \mathcal{Q}$ we have

$$\Phi_+ = (\varphi_+(0_+), i\varphi_+(0_+))^{\mathsf{T}}\ , \qquad \Phi_- = (\varphi_-(0_+), i\varphi_-(0_+))^{\mathsf{T}}\ , \tag{3.63}$$

where $\varphi_\pm$ are symmetric and antisymmetric (with phase-twisted) eigenstates of the $\mathcal{Q}$ transformation. Thus, in Ω_Q the boundary conditions (3.59) become

$$\begin{aligned} \sin\frac{\omega_+}{2}\,\varphi_+(0_+) + L_0\cos\frac{\omega_+}{2}\,\varphi'_+(0_+) = 0\ , \\ \sin\frac{\omega_-}{2}\,\varphi_-(0_+) + L_0\cos\frac{\omega_-}{2}\,\varphi'_-(0_+) = 0\ , \end{aligned} \tag{3.64}$$

which are formally identical to the ones (3.47) in Ω_P.

Returning to the generic invariant subfamily of the $su(2)$ transformation $\mathcal{X}_i$, we can also see how other spectrum-preserving transformations in the $su(2)$ act on the subfamily. For this we just observe that the transformation $\mathcal{X}_j$ with $j \neq i$ induces the map on the characteristic matrix (3.57) as

$$U(\vartheta_+, \vartheta_-) \xrightarrow{\mathcal{X}_j} \sigma_j \, U(\vartheta_+, \vartheta_-) \, \sigma_j = U(\vartheta_-, \vartheta_+) \ . \tag{3.65}$$

The dual map $(\vartheta_+, \vartheta_-) \xrightarrow{\mathcal{X}_j} (\vartheta_-, \vartheta_+)$ therefore arises generically in the $su(2)$ invariant subfamily. Thus, at self-dual points the $su(2)$ becomes a symmetry of the system. The set of these self-dual points is always given by Ω_{SD} in (3.49) irrespective of the invariant subfamily one chooses in the $su(2)$.

4. Duality and Anholonomy

Due to the spectrum-preserving $su(2)$ found in the previous section, the system exhibits certain remarkable properties of the energy spectrum. In this section we discuss two notable aspects of them, that is, the strong vs. weak duality and the spectral anholonomy. The former is the duality of two distinct systems having reciprocal coupling constants (one is strong while the other is weak). The latter refers to the global structure in the spectral flow which resembles those of the geometric (Berry) phase. The possibility of interpreting the system as a supersymmetric model will also be mentioned at the end. For definiteness, we confine ourselves to the parity invariant subspace Ω_P.

4.1. Strong vs. weak duality

Prior to the discussion of the strong vs. weak coupling duality, we need to furnish a definition of the coupling constants for point interactions in the first place. In general, a coupling constant is defined to signify the strength of the interaction in action, and it is usually chosen so that it vanishes when the interaction ceases to work.

In the parity invariant subspace Ω_P, we have seen in (3.47) that the two angles θ_+ and θ_- are parameters which characterize independently the two sectors, namely, the sector of symmetric states φ_+ and the one of antisymmetric states φ_-. Further, the point $(\theta_+, \theta_-) = (0, \pi)$ is shown to correspond to the free system. From these, we recognize that for each of the two sectors the coupling constants for parity invariant point interactions may be defined by two functions $g_+(\theta_+)$ and $g_-(\theta_-)$ fulfilling $g_+(0) = 0$ and $g_-(\pi) = 0$. In what follows, we shall adopt the simple reciprocal choice for the two coupling constants,

$$g_+(\theta_+) := \tan\frac{\theta_+}{2} \ , \qquad g_-(\theta_-) := \cot\frac{\theta_-}{2} \ . \tag{4.1}$$

For illustration, we note that under (4.1) the delta-function interactions $V(x) = c(\theta_+)\,\delta(x)$ that appear along the line $(\theta_+, 0)$ have $c(\theta_+) = -\frac{\hbar^2}{mL_0} g_+(\theta_+)$, and we shall regard $g_+(\theta_+)$ as the coupling constant for the delta-function interactions. Analogously, the strength of the epsilon-function interactions $V(x) = \varepsilon(x; \theta_-)$ can be specified by $g_-(\theta_-)$. The further advantage of the definition (4.1) is that it allows us to elucidate the duality discussed below in a simple manner thanks to the identities,

$$g_+(\theta) = \frac{1}{g_-(\theta)} \ , \qquad g_+(\theta \pm \pi) = -g_-(\theta) \ . \tag{4.2}$$

However, any other choice for the two functions would equally serve our purposes, as long as they are single-valued and monotonous functions of θ on Ω_P.

We now recall that the half-reflection $\mathcal{R}$ induces the interchange of the parameters (3.48) in Ω_P. Thus, for coupling constants given in (4.1), we observe

$$(g_+(\theta_+), g_-(\theta_-)) \xrightarrow{\mathcal{R}} (g_+(\theta_-), g_-(\theta_+)) = (1/g_-(\theta_-), 1/g_+(\theta_+)) \ . \tag{4.3}$$

The transformation $\mathcal{R}$ is therefore seen to implement the combination of exchange and inversion for the coupling constants in the two sectors. This implies that the duality of energy spectrum between two systems connected by $\mathcal{R}$ can also be regarded as the duality between two systems with strong and weak coupling constants, namely, it is a *strong vs. weak duality* (see Fig.4). A typical duality appears at $\theta_+ = \theta_- \pm \pi$, for which (4.3) becomes

$$(g_+(\theta_+), g_-(\theta_-)) \xrightarrow{\mathcal{R}} (-1/g_+(\theta_+), -1/g_-(\theta_-)) \ . \tag{4.4}$$

It is also possible to realize systems with partial duality which is the duality between the symmetric sector in one system and the antisymmetric sector in the other, but not necessarily reversely. For this we need the translations $\mathcal{I}_\pm$ by a half-cycle in the two angles $\theta_\pm$,

$$\begin{aligned} U(\theta_+, \theta_-) &\xrightarrow{\mathcal{I}_+} e^{i\pi P_1^+} U(\theta_+, \theta_-) = U(\theta_+ \pm \pi, \theta_-) \ , \\ U(\theta_+, \theta_-) &\xrightarrow{\mathcal{I}_-} U(\theta_+, \theta_-)\, e^{i\pi P_1^-} = U(\theta_+, \theta_- \pm \pi) \ , \end{aligned} \tag{4.5}$$

where the sign of the shift $\pm\pi$ depends on where $\theta_\pm$ lie in the allowed region $[0, 2\pi)$. Under these, the coupling constants become

$$\begin{aligned} (g_+(\theta_+), g_-(\theta_-)) &\xrightarrow{\mathcal{I}_+} (-1/g_+(\theta_+), g_-(\theta_-)) \ , \\ (g_+(\theta_+), g_-(\theta_-)) &\xrightarrow{\mathcal{I}_-} (g_+(\theta_+), -1/g_-(\theta_-)) \ , \end{aligned} \tag{4.6}$$

and hence $\mathcal{I}_\pm$ implement *coupling inversions* followed by the sign change in each of the sectors.

By combining $\mathcal{R}$ and $\mathcal{I}_\pm$ as $\mathcal{I}_\pm\mathcal{R}$ $(= \mathcal{R}\mathcal{I}_\mp)$ we obtain

$$\begin{aligned} (g_+(\theta_+), g_-(\theta_-)) &\xrightarrow{\mathcal{I}_+\mathcal{R}} (-g_-(\theta_-), 1/g_+(\theta_+)) \ , \\ (g_+(\theta_+), g_-(\theta_-)) &\xrightarrow{\mathcal{I}_-\mathcal{R}} (1/g_-(\theta_-), -g_+(\theta_+)) \ . \end{aligned} \tag{4.7}$$

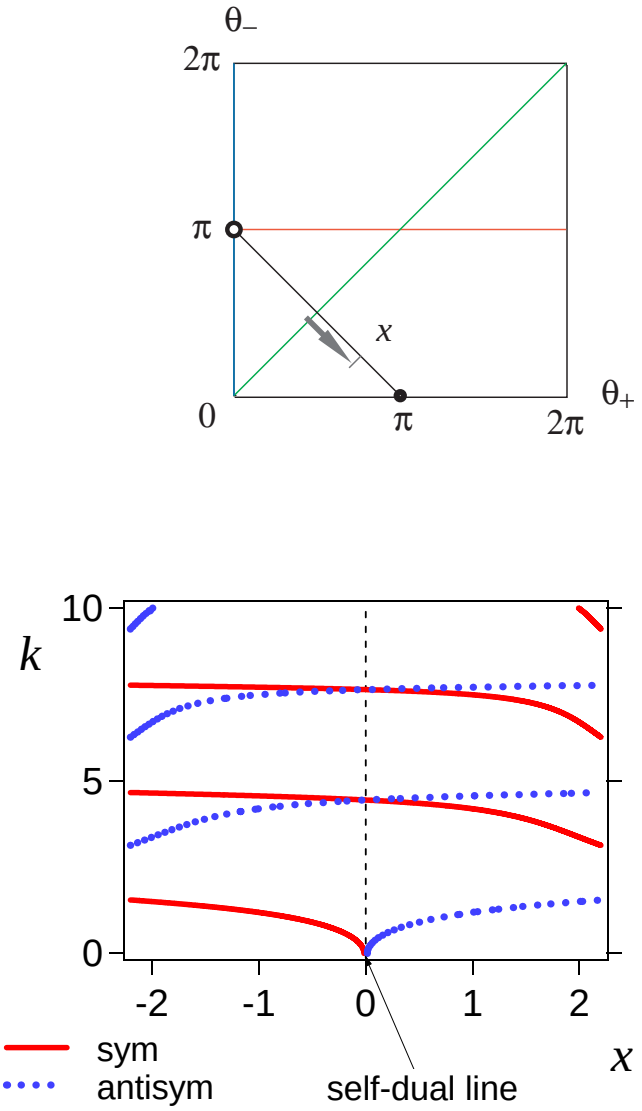

Figure 4. Duality under $\mathcal{R}$ — the eigenvalues k are exactly the same (with the symmetric and antisymmetric states interchanged) at two points of equal distance x from the self-dual line. The eigenvalues are evaluated for the anti-diagonal line connecting the free point $(0,\pi)$ and $(\pi,0)$ in the dissected torus under the Dirichlet boundary conditions, $\varphi_\pm(l) = \varphi_\pm(-l) = 0$ for some l (see Sect.4.2.1).

Equivalently, in the angle coordinates on Ω_P we have $(\theta_+,\theta_-) \overset{\mathcal{I}_+\mathcal{R}}{\longrightarrow} (\theta_- \pm \pi, \theta_+)$ and $(\theta_+,\theta_-) \overset{\mathcal{I}_-\mathcal{R}}{\longrightarrow} (\theta_-, \theta_+ \pm \pi)$, under which the energy spectrum in one sector is preserved in the other sector, showing that $\mathcal{I}_\pm\mathcal{R}$ are partial duality transformations. In particular, we find that $\mathcal{I}_+\mathcal{R}$ maps (θ_+,π) to $(0,\theta_+)$ for $\theta_+ \in [0,2\pi)$, which corresponds to sending the

delta-function interaction to the epsilon-function interaction,

$$\delta(x;\theta_+) \stackrel{\mathcal{I}_+\mathcal{R}}{\longrightarrow} \varepsilon(x;\theta_+) \ . \tag{4.8}$$

This is an example of strong vs. weak (semi) duality because the coupling constant is inverted $g_+(\theta_+) \stackrel{\mathcal{I}_+\mathcal{R}}{\longrightarrow} 1/g_+(\theta_+)$ as seen in (4.7). This semi-duality has been noticed earlier in Ref.[10].

There are two other discrete transformations worth mentioning. One of them is given by

$$U(\theta_+,\theta_-) \stackrel{\mathcal{S}}{\longrightarrow} U^{-1}(\theta_+,\theta_-) = U(-\theta_+,-\theta_-) \ , \tag{4.9}$$

which causes the *signature change* in the coupling constants,

$$(g_+(\theta_+), g_-(\theta_-)) \stackrel{\mathcal{S}}{\longrightarrow} (-g_+(\theta_+), -g_-(\theta_-)) \ . \tag{4.10}$$

The other is the combination $\mathcal{C} := \mathcal{S}\mathcal{I}_+\mathcal{I}_-\mathcal{R} = \mathcal{R}\mathcal{I}_+\mathcal{I}_-\mathcal{S}$ which provides the *coupling exchange* of even and odd strengths,

$$(g_+(\theta_+), g_-(\theta_-)) \stackrel{\mathcal{C}}{\longrightarrow} (g_-(\theta_-), g_+(\theta_+)) \ . \tag{4.11}$$

Unlike the half-reflection transformation $\mathcal{R}$, this does not preserve the spectra. We mention that the coupling exchange $\mathcal{C}$ is in fact induced by the interchange of Φ and Φ' in the boundary condition (2.7), and that it is related to the charge conjugation when the system is reformulated relativistically.

We remark at this point that, in view of the fact that all information about a point interaction is encoded in the boundary condition (2.7), any physical quantity $\mathcal{O}$, such as the transmission rate or the energy levels obtained under the point interactions, is a function of the parameters θ_+ and θ_- only through the coupling constants, $\mathcal{O} = \mathcal{O}(g_+(\theta_+), g_-(\theta_-))$. Thus, dualities similar to the one found here can always arise if $\mathcal{O}$ admits a simple relation with the $su(2)$ elements, like the one enjoyed by the Hamiltonian.

4.2. Anholonomy

4.2.1. Geometric (Berry) phase on Ω_W

It is well-known that spectral anholonomy can appear in the presence of spectral singularity, *i.e.*, degenerate points in the spectral parameter space. Thus the scale invariant

sphere $\Omega_W \simeq S^2$ which enjoys the isospectral property may furnish a suitable place to observe the spectral anholonomy if there exists some spectral singularity inside the sphere S^2. Indeed, if one puts the radius of the sphere zero by setting $\alpha_I = \beta_R = \beta_I = 0$ while keeping $\alpha_R = 1$ and $\xi = \pi/2$, one finds that the corresponding point in Ω belongs to Ω_{SD} where there occurs a double degeneracy at each level. The type of spectral anholonomy we find here is the geometric (Berry) phase that arises when one completes a cycle along a loop on the sphere.

To discuss it, for definiteness we put the particle in a box given, say, by the interval $-l \leq x \leq l$, and place the Dirichlet condition $\varphi_\pm(l) = \varphi_\pm(-l) = 0$. The energy eigenfunction is then given by (see Appendix B)

$$\varphi_k(x) = A_k e^{ikx} + B_k e^{-ikx} \ , \tag{4.12}$$

with appropriate coefficients A_k, B_k and the momentum k satisfying (2.7) which reads

$$kL_0 \cot kl = \frac{\sin\xi + \sqrt{1-\alpha_R^2}}{\cos\xi + \alpha_R} \ . \tag{4.13}$$

On the the scale invariant sphere $\Omega_W \simeq S^2$ given by $\alpha_I^2 + \beta_R^2 + \beta_I^2 = 1$ this is simplified into $\cos kl = 0$ and hence the allowed momenta are

$$k(\alpha_I, \beta_R, \beta_I) = \left(n - \frac{1}{2}\right)\pi \ , \qquad n = 1, 2, \ldots \tag{4.14}$$

which are independent of the parameters. Using the polar coordinates,

$$\alpha_I = \cos\theta \ , \qquad \beta_R = \sin\theta\cos\phi \ , \qquad \beta_I = \sin\theta\sin\phi \ , \tag{4.15}$$

the eigenfunction reads

$$\varphi_k(x;\theta,\phi) = \cos\frac{\theta}{2}\,\xi_+(x) + \sin\frac{\theta}{2}e^{i(\phi+\frac{\pi}{2})}\,\xi_-(x) \ , \tag{4.16}$$

where

$$\xi_\pm(x) := \sqrt{\frac{1}{l}}\sin k(x \mp l)\Theta(\pm x) \ . \tag{4.17}$$

Let us now take an arbitrary loop $\mathcal{C}$ on the sphere and evaluate the phase $e^{i\gamma(\mathcal{C})}$ that the eigenstate acquires when it completes a cycle along the loop. According to Berry (see, *e.g.*, [19] and references therein) it is given by

$$\gamma(\mathcal{C}) = \oint_{\mathcal{C}} A \ , \tag{4.18}$$

where A is the Berry connection 1-form,

$$A := \langle\varphi|\, i\frac{\partial}{\partial\theta}\,\varphi\rangle\; d\theta + \langle\varphi|\, i\frac{\partial}{\partial\phi}\,\varphi\rangle\; d\phi = -\sin^2\frac{\theta}{2}\, d\phi\ . \tag{4.19}$$

The curvature 2-form F is then found to be

$$F = dA = -\frac{1}{2}\sin\theta\, d\theta d\phi\ , \tag{4.20}$$

which is precisely the potential for the Dirac monopole with strength $g = -1$. The geometric phase factor $\gamma(\mathcal{C})$ is therefore the magnetic flux which is given by the magnetic field $B_r = -1/2$ times the solid angle subtended by the loop. (A Berry phase similar to this has been mentioned in [20].)

4.2.2. Anholonomy on Ω_P

Another type of spectral anholonomy can be observed in the parity invariant torus $\Omega_P \simeq S^1 \times S^1$ where the degenerate, self-dual subfamily Ω_{SD} is given by a circle that winds the torus diagonally in the two independent cycles. Again, we consider the periodic Dirichlet boundary condition (See Appendix B) and obtain the spectrum determined by

$$kL_0 \cot kl = \tan\frac{\theta_+}{2}\ , \qquad kL_0\cot kl = \tan\frac{\theta_-}{2}\ , \tag{4.21}$$

in the symmetric and the antisymmetric sector, respectively. Since each of the momenta is a monotonously decreasing function of the angle parameter as $k = k(\theta_+)$ and $k = k(\theta_-)$, it is evident that each energy level acquires a spiral anholonomy as one completes a cycle along any of the two associated cycles of the torus $\Omega_P \simeq S^1 \times S^1$. The double spiral arises when the cycle is completed simultaneously in the two parameters θ_+ and θ_- crossing the self-dual circle Ω_{SD}, on account of the fact that even and odd eigenstates arise alternately in the spectrum (see Fig.5). This is the origin of the double spiral anholonomy pointed out earlier in [11].

4.3. Supersymmetry

As observed in Sect.3.3, point interactions at self-dual points in Ω_{SD} of the form (3.50) give rise to a double (or even) degeneracy at each level in the spectrum. Since these degenerate levels are paired into two states with opposite parity $\mathcal{P}$, or more generally with $\pm$ eigenstates of any of the $su(2)$ generators, we may ask if the systems with those point

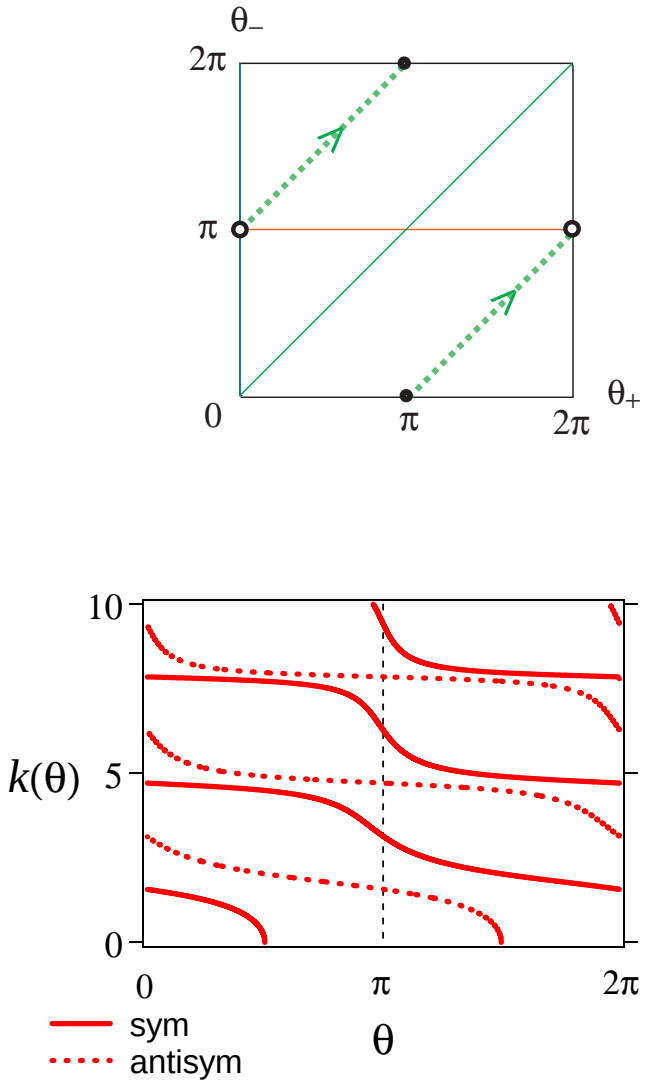

Figure 5. Spiral anholonomy — each level of k gets shifted after the cycle θ $(=\theta_+)$ $\to \theta + 2\pi$ along the diagonal line passing through the free point $(0, \pi)$ is completed, even though the spectrum is left unchanged as a whole.

interactions exhibit some kind of supersymmetry (SUSY). This issue will be examined here.

For this we recall briefly the basics of SUSY quantum mechanics (see, *e.g.*, [21]). A SUSY quantum system is defined by the set $\{\widehat{H}, \widehat{Q}_1, \ldots, \widehat{Q}_N; \mathcal{H}\}$ where $\widehat{H}$ is the Hamiltonian operator, $\widehat{Q}_i$ for $i = 1, \ldots, N$ are self-adjoint operators called supercharges and $\mathcal{H}$ is the Hilbert space. The operators are subject to the relations,

$$\{\widehat{Q}_i,\, \widehat{Q}_j\} = \widehat{H}\, \delta_{ij} \ . \tag{4.22}$$

Note that from (4.22) it follows that $\widehat{H} = 2\widehat{Q}_i^2$ and $[\widehat{H},\, \widehat{Q}_i] = 0$ for any i.

The scheme of SUSY quantum mechanics which is most familiar for us is the $N = 2$ Witten model [22]. In the standard formulation on a line $\mathbb{R}$, the model presupposes the Hilbert space,

$$\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2 \ , \tag{4.23}$$

which is graded by $\mathbb{C}^2$. The supercharges are then provided by

$$\begin{aligned} \widehat{Q}_1 &= \frac{1}{\sqrt{2}} \left(-\frac{i\hbar}{\sqrt{2m}} \frac{d}{dx} \otimes \sigma_1 + \Lambda(x) \otimes \sigma_2 \right) \ , \\ \widehat{Q}_2 &= \frac{1}{\sqrt{2}} \left(-\frac{i\hbar}{\sqrt{2m}} \frac{d}{dx} \otimes \sigma_2 - \Lambda(x) \otimes \sigma_1 \right) \ , \end{aligned} \tag{4.24}$$

with a real-valued, continuously differentiable function $\Lambda(x)$. The Hamiltonian is then found to be

$$\widehat{H} = \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \Lambda^2(x) \right) \otimes 1 + \Lambda'(x) \otimes \sigma_3 \ . \tag{4.25}$$

Note that our basis in the graded space is chosen so that $\widehat{H}$ be diagonal with respect to σ_3 which is the element associated with the half-reflection $\mathcal{R}$ in the $su(2)$, but any other choice for the basis works as well.

In order to fit our system with the Witten model, we need to introduce the graded structure $\mathbb{C}^2$ to our Hilbert space. In fact, such a structure has already been equipped with our system under the use of the two-dimensional boundary vectors in (2.7). Explicitly, we consider our Hilbert space (2.2) to be the sum (3.52) of two eigenspaces $\mathcal{H}_\pm$ of $\mathcal{R}$, which are just $\mathcal{H}_\pm = L^2(\mathbb{R}^\pm)$. Since $L^2(\mathbb{R}^+) \simeq L^2(\mathbb{R}^-)$, we have

$$\mathcal{H} = L^2(\mathbb{R}^+) \otimes \mathbb{C}^2 \ , \tag{4.26}$$

where the state decomposition $\varphi = \varphi_+ + \varphi_-$ for $\varphi_\pm \in L^2(\mathbb{R}^\pm)$ in (3.53) can now be implemented by means of the projection operators $P_3^\pm$ for the two-dimensional vector state,

$$\Phi(x) := (\varphi_+(x), \varphi_-(x))^{\mathsf{T}} \ . \tag{4.27}$$

At the boundary, the vector state $\Phi(x)$ reduces to the boundary vector Φ as required, and this is consistent with the decomposition (3.54) and (3.61) discussed earlier. Accordingly,

the projected Hamiltonian $H_\pm$ acting on the states $\varphi_\pm$ can be obtained from (4.25) as $\widehat{H} = H_+ P_3^+ + H_- P_3^-$ with

$$H_\pm = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_\pm(x) \ , \tag{4.28}$$

where

$$V_\pm(x) = \Lambda^2(x) \pm \frac{\hbar}{\sqrt{2m}}\Lambda'(x) \ . \tag{4.29}$$

Thus our question becomes if there exists a potential $\Lambda(x)$ defined on the half-line $\mathbb{R}^+$ that can reproduce our Hamiltonian H in (2.1). Apparently, this is trivially answered since if we wish to have $H_\pm = H$ formally on each of the half-lines $\mathbb{R}^\pm$, we need $\Lambda(x) = 0$ identically. However, there still remains a nontrivial aspect because in our case the characteristics of the interaction is encoded in the boundary condition rather than in the formal differential operator. The real question, therefore, is whether the boundary condition (2.7) for $U \in \Omega_{SD}$ admits supercharges $\widehat{Q}_1$ and $\widehat{Q}_2$ in (4.24) which are self-adjoint.

This can be answered by observing that, for $\Lambda(x) = 0$, the supercharges are basically the momentum operator $p := -i\hbar\frac{d}{dx}$. The self-adjointness of the supercharges can thus be examined by the self-adjointness of p. Evidently, p becomes self-adjoint in both of $\mathcal{H}_\pm$ if and only if the wave functions vanish at the point of interaction, $\varphi_+(0_+) = \varphi_-(0_-) = 0$. Since for $U \in \Omega_{SD}$ the boundary condition is given by (3.62) with $\phi_+ = \phi_- = \theta \in [0, 2\pi)$, we see that this is achieved at $(\theta_+, \theta_-) = (\pi, \pi)$. From our previous argument, we notice that this corresponds to the delta-function interaction with infinite strength $g_+(\theta_+) \to \pm\infty$.

To sum up, we have learned that, among the self-dual points (θ, θ) for $\theta \in [0, 2\pi)$ where the $su(2)$ symmetry arises, there exists an exceptional point (π, π) which enjoys the standard SUSY as a Witten model. For other points $\theta \neq \pi$, however, the question of being a SUSY model — possibly under generalized SUSY charges other than (4.24) — remains to be explored, although the double degeneracy indicates that that is the case in general.

5. Summary and Discussions

In this paper, we have investigated the physical properties of the quantum mechanical point interaction in one dimension. The allowed types of interactions in one dimension are classified by the parameter group $U(2)$, in which we focused on a number of subfamilies characterized by symmetries. In particular, we have presented a detailed analysis of the parity invariant subfamily $U(1) \times U(1)$ which is the set of left-right symmetric point interactions. The coupling constants of the point interaction, which describe separately the strengths of the interaction for symmetric and antisymmetric states, have been defined by means of the parameters of the subfamily. It has been found that the strong vs. weak duality observed in the parity invariant subfamily is a direct consequence of the existence of spectrum-preserving discrete maps in the subfamily. More generally, we have shown that the generators of these discrete maps form an $su(2)$ algebra, and that any of the $su(2)$ generators defines an invariant subfamily in which a similar duality can be observed. We have mentioned a distinguished $U(1)$ subfamily, consisting of interactions invariant under all of the $su(2)$ transformations, which furnishes a singular circle in the parameter space $U(2)$ where every energy level of states is doubly degenerate. Because of this singularity in the spectral space, one can expect some kind of quantum anholonomy to arise when one completes a cycle in the space. Indeed, for the parity invariant subfamily we have found a double spiral structure of the energy levels, whereas for the Weyl scaling invariant subfamily we have observed an induced magnetic monopole leading to a geometric (Berry) phase. Further, we have pointed out that at one point in the $U(1)$ subfamily the system can be regarded as a Witten model, where the double degeneracy can be accounted for in terms of the supersymmetry.

These features, the strong vs. weak duality, quantum anholonomy and supersymmetry, are usually associated to more involved systems of quantum field theories or string theories. However, what we have found in this paper is that they may arise as generic, not accidental, features of a vastly simpler setting of one dimensional quantum mechanical system with a single point defect. In other words, the low-energy single-particle quantum mechanics already possesses nontrivial characteristics which have not been widely recognized so far. This implies that, if we can fabricate one dimensional devices with a point defect whose type of interactions is under our control, it is possible to observe these exotic physical phenomena at the laboratory level without invoking the large facility for high energy

experiments. This is not entirely inconceivable under the present day progress in nano-scale technology, and for applications we may ponder on such things as the quantum interferometer using quantum wires with point defects, or the 'quantum pump' which is the device that can pump the energy through a cycle in the parameter space exploiting the anholonomy of the spectral structure. In fact, attempts are currently being made at constructing the quantum filters mentioned in Appendix B, which seems to have immediate relevance to the field of quantum information processing or quantum computing [23].

We end this paper by stating our hope that our analysis presented here serve as a basis for realizing the potential use as well as the physics of the one dimensional system with point interaction, which is seemingly innocent but actually full of intriguing features especially when it is controllable, and that effort be made for making the potential a reality in the near future.

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Appendix A

In Sect.2, we presented an elementary argument which shows that the point interaction in one dimension is characterized by the group $U(2)$. The essence of our argument was that the probability conservation (2.4) — which is equivalent to the self-adjointness of the Hamiltonian (2.1) — requires the condition (2.6), that is, the norm of the two vectors,

$$\Phi^{(\pm)} := \Phi \pm iL_0\,\Phi' \ , \tag{A.1}$$

constructed from the boundary vectors in (2.5) with some fixed $L_0 \neq 0$ be equal,

$$|\Phi^{(+)}| = |\Phi^{(-)}| \ , \tag{A.2}$$

and, therefore, these vectors must be related by an unitary matrix $U \in U(2)$. Upon a rigorous ground, however, there are two points which have remained to be shown. The first point is that the matrix U is actually independent of the choice of the state φ used for the vectors in (2.5). The second is that the parameter L_0 adds no extra freedom to the variety of point interactions other than the $U(2)$ group. For completeness, in this Appendix we wish to prove these two points, and thereby fill the remaining gap between our simple derivation and the technically more involved one based on the theory of self-adjoint extensions.

A.1. State-independence of the matrix U

Before we start, let us observe that, if a state φ which belongs to a self-adjoint domain $D(H)$ of the Hamiltonian operator H has $\Phi^{(+)} = \Phi^{(-)} = 0$, the condition (A.2) is trivially fulfilled and no question of the independence arises. Thus, in what follows we consider only states for which $\Phi^{(+)}$ and/or $\Phi^{(-)}$ is nonvanishing. Note that such states must exist in $D(H)$ because otherwise the domain $D(H)$ would not be closed as required by the self-adjointness of H.

First, we show that there exists a pair of two states $\varphi_1, \varphi_2 \in D(H)$ for which the corresponding $\Phi_1^{(+)}$ and $\Phi_2^{(+)}$ are linearly independent, and/or $\Phi_1^{(-)}$ and $\Phi_2^{(-)}$ are linearly independent. Here the linear independence is understood by viewing the vectors in the two dimensional vector space $\mathbb{C}^2$ which is equipped with the inner product $\langle \Phi_1, \Phi_2 \rangle := \Phi_1^\dagger \Phi_2$. Indeed, if such a pair does not exist, then all $\varphi \in D(H)$ are such that the corresponding

two $\Phi^{(+)}$s are each other's multiple and, similarly, the $\Phi^{(-)}$s are each other's multiple. Then it is easy to find a smooth function φ_{new} with $|\Phi^{(+)}_{new}| = |\Phi^{(-)}_{new}| > 0$ such that $\Phi^{(+)}_{new}$ is orthogonal to these $\Phi^{(+)}$s and $\Phi^{(-)}_{new}$ is orthogonal to these $\Phi^{(-)}$s. It follows that, for any $\varphi \in D(H)$ and $\alpha, \beta \in \mathbb{C}$, the linear combination $\alpha\varphi + \beta\varphi_{new}$ will satisfy

$$|\alpha\Phi^{(+)} + \beta\Phi^{(+)}_{new}| = |\alpha\Phi^{(-)} + \beta\Phi^{(-)}_{new}| \;, \tag{A.3}$$

because

$$\begin{aligned} |\alpha\Phi^{(+)} + \beta\Phi^{(+)}_{new}|^2 &= |\alpha|^2|\Phi^{(+)}|^2 + |\beta|^2|\Phi^{(+)}_{new}|^2 \\ &= |\alpha|^2|\Phi^{(-)}|^2 + |\beta|^2|\Phi^{(-)}_{new}|^2 = |\alpha\Phi^{(-)} + \beta\Phi^{(-)}_{new}|^2 \;. \end{aligned} \tag{A.4}$$

This implies that the domain $D(H)$ can be enlarged by the linear combinations $\alpha\varphi+\beta\varphi_{new}$ to the bigger domain $D(H)_{new} = \overline{D(H) \oplus \{\beta\varphi_{new} \,|\, \beta \in \mathbb{C}\}}$ on which the Hamiltonian operator is still self-adjoint. However, this is impossible since the domain of a self-adjoint operator cannot be extended any further (maximal symmetric), see [1].

Having shown the existence of a pair $\varphi_1, \varphi_2 \in D(H)$ for which $\Phi^{(+)}_1$ and $\Phi^{(+)}_2$, and/or $\Phi^{(-)}_1$ and $\Phi^{(-)}_2$, are linearly independent, we can now find (via a Gram-Schmidt orthogonalization) a new pair φ_1 and φ_2 such that $\Phi^{(+)}_1 \perp \Phi^{(+)}_2$ and/or $\Phi^{(-)}_1 \perp \Phi^{(-)}_2$ holds. Without loss of generality we can assume that, *e.g.,* $\Phi^{(+)}_1 \perp \Phi^{(+)}_2$. But then we can show that $\Phi^{(-)}_1 \perp \Phi^{(-)}_2$ holds, too. For this, recall that for φ_1, $\varphi_2 \in D(H)$ we have $\varphi_1+e^{i\omega}\varphi_2 \in D(H)$ for an arbitrary $\omega \in [0, 2\pi)$, and hence we have $|\Phi^{(+)}_1+e^{i\omega}\Phi^{(+)}_2| = |\Phi^{(-)}_1+e^{i\omega}\Phi^{(-)}_2|$. Squaring both sides and using $|\Phi^{(\pm)}_i|^2 = |\Phi^{(\mp)}_i|^2$ for $i = 1, 2$, we find $\mathrm{Re}[e^{i\omega}\langle\Phi^{(-)}_1, \Phi^{(-)}_2\rangle] = 0$ for any ω and, consequently, $\langle\Phi^{(-)}_1, \Phi^{(-)}_2\rangle = 0$.

Thus we have established that there is a pair φ_1, $\varphi_2 \in D(H)$ such that $\Phi^{(+)}_1 \perp \Phi^{(+)}_2$ and $\Phi^{(-)}_1 \perp \Phi^{(-)}_2$. These vectors can be normalized as $|\Phi^{(\pm)}_1| = |\Phi^{(\pm)}_2| = 1$ by rescaling the states of the pair appropriately. From this we see that there exists a unique unitary matrix $U : \mathbb{C}^2 \to \mathbb{C}^2$ such that

$$U\Phi^{(+)}_1 = \Phi^{(-)}_1, \qquad U\Phi^{(+)}_2 = \Phi^{(-)}_2. \tag{A.5}$$

It remains to show that this U is actually universal for $D(H)$, that is,

$$U\Phi^{(+)} = \Phi^{(-)} \;, \qquad \forall\varphi \in D(H) \;. \tag{A.6}$$

To this end, we note that each of the sets $\{\Phi_1^{(+)}, \Phi_2^{(+)}\}$ and $\{\Phi_1^{(-)}, \Phi_2^{(-)}\}$ forms an orthonormal basis in $\mathbb{C}^2$, and hence the vectors $\Phi^{(\pm)}$ corresponding to φ can be expanded as

$$\Phi^{(+)} = \alpha\Phi_1^{(+)} + \beta\Phi_2^{(+)}, \qquad \Phi^{(-)} = \alpha'\Phi_1^{(-)} + \beta'\Phi_2^{(-)}, \tag{A.7}$$

with some coefficients α, β, α' and $\beta' \in \mathbb{C}$. Since the condition (A.2) must be fulfilled for any vectors given by a linear combination of φ_1, φ_2 and φ, we consider, in particular, the combination $e^{i\omega_1}\varphi_1 + e^{i\omega_2}\varphi_2 + \varphi$ with $\omega_1, \omega_2 \in [0, 2\pi)$. For this state, the condition (A.2) then reads

$$|e^{i\omega_1}\Phi_1^{(+)} + e^{i\omega_2}\Phi_2^{(+)} + \Phi^{(+)}|^2 = |e^{i\omega_1}\Phi_1^{(-)} + e^{i\omega_2}\Phi_2^{(-)} + \Phi^{(-)}|^2 \ . \tag{A.8}$$

Making use of the expansions (A.7) and the orthonormality of the bases, we obtain

$$2\mathrm{Re}[e^{-i\omega_1}(\alpha - \alpha')] + 2\mathrm{Re}[e^{-i\omega_2}(\beta - \beta')] = |\alpha'|^2 - |\alpha|^2 + |\beta'|^2 - |\beta|^2. \tag{A.9}$$

Since the right hand side is independent of the arbitrary parameters ω_1 and ω_2, this equality can hold if and only if $\alpha = \alpha'$ and $\beta = \beta'$. Plugging these into (A.7) one obtains the identity (A.6) as claimed.

A.2. L_0 adds no extra freedom

Next we show that the parameter L_0 appearing in (2.7) does not give any additional freedom in characterizing the boundary conditions other than those given by the $U(2)$ group. For this, we first note that any $U(2)$ matrix U can be diagonalized using some appropriate unitary matrix V as

$$U \to VUV^{-1} = D := \begin{pmatrix} e^{i\mu_+} & 0 \\ 0 & e^{i\mu_-} \end{pmatrix}, \qquad \mu_+, \mu_- \in [0, 2\pi). \tag{A.10}$$

These parameters, μ_+ and μ_- in D, may be regarded as two of the four parameters of U that arise under the decomposition $U = V^{-1}DV$. In fact, as we will see in Appendix B, they are related to ξ and $\rho = \arccos\alpha_R$ as $\mu_\pm = \xi \pm \rho \ (\mathrm{mod}\ 2\pi)$ (cf. (B.16)). We now define new basis vectors,

$$\Psi = \begin{pmatrix} \psi(0_+) \\ \psi(0_-) \end{pmatrix} := V\Phi \ , \qquad \Psi' = \begin{pmatrix} \psi'(0_+) \\ -\psi'(0_-) \end{pmatrix} := V\Phi' \ . \tag{A.11}$$

We note that the components in the new basis (A.11) are a mixture of the components of the original basis (2.5) and hence their arguments are only symbolic, but they are still independent of L_0. In terms of the new basis, the boundary conditions (2.7) become

$$(D - I)\Psi + iL_0\,(D + I)\Psi' = 0\ , \tag{A.12}$$

or in components,

$$\begin{aligned} &\psi(0_+) + L_0 \cot\frac{\mu_+}{2}\,\psi'(0_+) = 0\ , \\ &\psi(0_-) - L_0 \cot\frac{\mu_-}{2}\,\psi'(0_-) = 0\ . \end{aligned} \tag{A.13}$$

It is then obvious that the freedom of changing the value L_0 can be absorbed by the corresponding change in the two parameters μ_+ and μ_- (by $\delta\mu_+ = \sin\mu_+\,\delta L_0\,/L_0$ and $\delta\mu_- = \sin\mu_-\,\delta L_0\,/L_0$ for $L_0 \to L_0 + \delta L_0$). This shows that, for describing distinct boundary conditions, the parameter L_0 does not provide an additional freedom which is independent of the $U(2)$ parameters in U.

It is interesting to observe that, if both μ_+ and μ_- are 0 or π, a change of L_0 does not modify the parameters μ_+ and μ_-, that is, a scale change does not affect the point interaction at all. These occur at the values $(\xi, \alpha_R) = (\frac{\pi}{2}, 0)$, $(0, 1)$, and $(0, -1)$, which correspond exactly to the scale independent systems (the continuous family, and the two isolated points, respectively, cf. Sect. 3.1) as expected.

Appendix B

In this Appendix we shall present some basic results concerning the spectral as well as scattering properties of the systems with point interaction (see, *e.g.*, [2,6]). We first deal with systems defined on a line, and then study systems placed in a box where the entire spectrum becomes discrete. The latter case is used to demonstrate the spectral anholonomy and supersymmetry in Sect.4.

B.1. System with point interaction on a line

To begin with, we consider eigenfunctions with positive energy $E > 0$. With $k = \sqrt{2mE}/\hbar$ the general form of the positive energy eigenfunctions is given by

$$\varphi_k(x) = \begin{cases} A_k^- e^{ikx} + B_k^- e^{-ikx}, & x < 0, \\ A_k^+ e^{ikx} + B_k^+ e^{-ikx}, & x > 0, \end{cases} \tag{B.1}$$

where the coefficients $A_k^\pm$ and $B_k^\pm$ are constants. This expression, along with the boundary condition (2.7) yields two linear equations for the coefficients, resulting in a two-parameter family of solutions φ_k to a given matrix U, *i.e.*, point interaction. The spectrum for $E > 0$ is of course continuous for any U, but the characteristics of the point interaction can still be seen in the scattering processes. For this we set $B_k^+ = 0$ and $A_k^- = 1/\sqrt{2\pi}$ to obtain a plane wave incoming from the left plus the reflected one in $x < 0$ and the transmitted one in $x > 0$ by the scattering at $x = 0$,

$$\varphi_k^{(l)}(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} e^{ikx} + r^{(l)} e^{-ikx}, & x < 0, \\ t^{(l)} e^{ikx}, & x > 0. \end{cases} \tag{B.2}$$

The reflection and the transmission amplitudes turn out to be

$$r^{(l)} = \frac{\alpha\, q + \alpha^* q^{-1} - (\eta + \eta^*)}{\eta\, q + \eta^* q^{-1} - (\alpha + \alpha^*)}\ , \qquad t^{(l)} = \frac{-\beta(q - q^{-1})}{\eta\, q + \eta^* q^{-1} - (\alpha + \alpha^*)}\ , \tag{B.3}$$

where we have used

$$\eta = e^{i\xi}\ , \qquad q = \frac{1 - kL_0}{1 + kL_0}\ . \tag{B.4}$$

Similarly, if we set $A_k^- = 0$ and $B_k^+ = 1/\sqrt{2\pi}$ we obtain a plane wave which is incoming from the right and scattered off at $x = 0$,

$$\varphi_k^{(r)}(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} t^{(r)} e^{-ikx}, & x < 0, \\ e^{-ikx} + r^{(r)} e^{ikx}, & x > 0, \end{cases} \tag{B.5}$$

where

$$r^{(r)} = \frac{\alpha^* q + \alpha\, q^{-1} - (\eta + \eta^*)}{\eta\, q + \eta^* q^{-1} - (\alpha + \alpha^*)} \ , \qquad t^{(r)} = \frac{\beta^* (q - q^{-1})}{\eta\, q + \eta^* q^{-1} - (\alpha + \alpha^*)} \ . \tag{B.6}$$

The amplitudes $r^{(l)}$, $t^{(l)}$, $r^{(r)}$ and $t^{(r)}$ obey the unitarity conditions,

$$|r^{(l)}|^2 + |t^{(l)}|^2 = 1 \ , \qquad |r^{(r)}|^2 + |t^{(r)}|^2 = 1 \ , \tag{B.7}$$

and

$$r^{(l)^*} t^{(r)} + t^{(l)^*} r^{(r)} = 0 \ . \tag{B.8}$$

With the aid of these relations and

$$\int_0^\infty dx\, e^{i(k-k')x} = \pi \delta(k - k') \ , \qquad k,\, k' > 0 \ , \tag{B.9}$$

we find that $\varphi_k^{(l)}$ and $\varphi_k^{(r)}$ are orthonormalized as

$$\int_{-\infty}^\infty dx\, |\varphi_k^{(l)}|^2 = \int_{-\infty}^\infty dx\, |\varphi_k^{(r)}|^2 = \delta(k - k') \ , \qquad \int_{-\infty}^\infty dx\, (\varphi_k^{(l)})^* \varphi_k^{(r)} = 0 \ . \tag{B.10}$$

If we let k take negative values, too, and noting that $k \to -k$ implies $q \to q^{-1}$, we get further relations among the reflection and transmission amplitudes as

$$r_{-k}^{(l)} = r_k^{(l)^*}, \qquad r_{-k}^{(r)} = r_k^{(r)^*}, \qquad t_{-k}^{(l)} = t_k^{(r)^*}, \qquad t_{-k}^{(r)} = t_k^{(l)^*} \ . \tag{B.11}$$

A general positive energy eigenfunction (B.1) is given by the linear combination of these two solutions,

$$\varphi_k(x) = C_k^{(l)}\ \varphi_k^{(l)}(x) + C_k^{(r)}\ \varphi_k^{(r)}(x) \ , \tag{B.12}$$

where the constants $C_k^{(l)}$, $C_k^{(r)}$ are related to $A_k^\pm$ and $B_k^\pm$ as $C_k^{(l)} = \sqrt{2\pi} A_k^-$ and $C_k^{(r)} = \sqrt{2\pi} B_k^+$.

Turning to the eigenfunctions with negative energy $E < 0$, we first note that the general form of a negative energy eigenfunction is

$$\varphi_\kappa(x) = \begin{cases} A_\kappa^- e^{\kappa x} + B_\kappa^- e^{-\kappa x}, & x < 0, \\ A_\kappa^+ e^{\kappa x} + B_\kappa^+ e^{-\kappa x}, & x > 0, \end{cases} \tag{B.13}$$

with $\kappa = \sqrt{2m|E|}/\hbar$. Since normalizability requires $A_\kappa^+ = 0$ and $B_\kappa^- = 0$, the boundary condition (2.7) gives

$$U \begin{pmatrix} B_\kappa^+ \\ A_\kappa^- \end{pmatrix} = \frac{1 + i\kappa L_0}{1 - i\kappa L_0} \begin{pmatrix} B_\kappa^+ \\ A_\kappa^- \end{pmatrix} . \tag{B.14}$$

The unit factor on the r.h.s. can be written as $e^{i\omega_\kappa}$ with $\omega_\kappa = 2\arctan\kappa L_0$. As κ runs in $(0,\infty)$, ω_κ runs in $(0,\pi)$. Thus we find that there can arise maximally two negative energy bound states under point interactions, and that there is a one-to-one correspondence between a bound state and an eigenvalue λ of the matrix U with $\arg\lambda \in (0,\pi)$ (note that $|\lambda| = 1$ since U is unitary).

Under the parametrization (2.8), the eigenvalues λ of U are determined by the equation,

$$\lambda^2 - 2\alpha_R e^{i\xi}\lambda + e^{2i\xi} = 0 \ . \tag{B.15}$$

This has the roots,

$$\lambda_\pm = e^{i\xi}\left(\alpha_R \pm i\sqrt{1-\alpha_R^2}\right) \ , \tag{B.16}$$

or $\lambda_\pm = e^{i(\xi\pm\rho)}$ with $\alpha_R = \cos\rho$ for $\rho \in [0,\pi]$. Since $\xi - \rho \in [-\pi,\pi)$ and $\xi + \rho \in [0,2\pi)$, we learn that there arises a bound state to λ_- for $\xi > \rho$ and analogously to λ_+ for $0 < \xi + \rho < \pi$. A doubly degenerate bound state may arise when $\alpha_R = \pm 1$, namely, $\rho = 0$ and π. The case $\rho = \pi$, however, is not allowed because it implies $\xi + \rho \geq \pi$, whereas the case $\rho = 0$ is allowed for $\xi > 0$. The latter case in fact belongs to the self-dual subfamily Ω_{SD}, where now the characteristic matrix U is given by (3.50) with $\theta \in (0,\pi)$ which is the angle ξ here. To each eigenvalue $\lambda_\pm$ corresponds the value,

$$\kappa = \frac{1}{iL_0}\,\frac{\lambda+1}{\lambda-1} = \frac{1}{L_0}\cot\left(\frac{\xi\pm\rho}{2}\right) \ . \tag{B.17}$$

The existence of a zero energy $E = 0$ eigenfunction may be examined by looking at the limit $\kappa \to 0$ of the negative energy eigenfunctions discussed above. It is then readily confirmed that a zero energy state occurs when there arise roots $\lambda_\pm$ corresponding to $\arg\lambda = 0$, that is, at $\xi = \pm\rho$. In particular, at $\xi = \rho = 0$ there appears a doubly degenerate state. Like those appearing for U with negative energy, this degeneracy is a consequence of the $su(2)$ spin symmetry possessed by the interactions of self-dual points discussed in section 3. These negative and zero energy states which are doubly degenerate give the ground states for the subfamily Ω_{SD}, whose energy is given by

$$E^{SD}_{\text{ground}} = -\frac{\hbar^2}{2mL_0^2}\tan^2\frac{\theta}{2} \ , \tag{B.18}$$

for (3.50) with $\theta \in [0,\pi)$.

Let us specialize to some of the subfamilies encountered before and see how the physical quantities we just obtained look like. We start with the scale invariant subfamily Ω_W, where we have $\xi = \rho = \pi/2$ and hence no bound state can arise. This is in fact expected, since a bound state energy E_κ (or κ) would require a length parameter which is lacking in a scale invariant system. For the scattering states, one finds that the reflection and transmission coefficients are

$$r^{(l)} = \alpha_I, \qquad r^{(r)} = -\alpha_I, \qquad t^{(l)} = i\beta, \qquad t^{(r)} = -i\beta^* \ . \tag{B.19}$$

For the exceptional cases $U = \pm I$, we obtain

$$r^{(l)} = r^{(r)} = \pm 1, \qquad t^{(l)} = t^{(r)} = 0 \ . \tag{B.20}$$

In all cases, we can observe that the coefficients are momentum independent, again as a consequence of the absence of any length parameter.

In the separating (or half-reflection invariant) subfamily Ω_R, we have the boundary conditions given by (3.59) which are just the ones for two 'half-line plus infinite wall' systems. With $L_\pm := L_0 \cot(\vartheta_\pm/2)$, the existence of bound states is ensured for $0 < L_+ < \infty$ and $0 < L_- < \infty$. If, for example, $0 < L_+ < \infty$ then we find a bound state,

$$\varphi^{L_+}_{\text{bound}}(x) = \sqrt{\frac{2}{L_+}}\ \Theta(x)\ e^{-x/L_+}. \tag{B.21}$$

For the scattering states, on the other hand, we find

$$r^{(l)} = -\frac{1 - ikL_-}{1 + ikL_-} \ , \qquad r^{(r)} = -\frac{1 - ikL_+}{1 + ikL_+} \ , \tag{B.22}$$

and

$$t^{(l)} = t^{(r)} = 0 \ . \tag{B.23}$$

These results are also in accordance with the ones found for the 'half-line plus infinite wall' systems [18].

In the parity invariant subfamily Ω_P one may put $\beta = \beta_I = i \sin\rho$ along with $\alpha = \alpha_R = \cos\rho$. One then finds that bound states exist if $0 < \xi \pm \rho \pmod{2\pi} < \pi$, while the scattering states do not have much special feature with respect to the generic case, except that

$$r^{(r)} = r^{(l)} \qquad \text{and} \qquad t^{(r)} = t^{(l)} \ . \tag{B.24}$$

These properties reflect the fact that the '$+ \leftrightarrow -$' symmetry is just the '$r \leftrightarrow l$' symmetry of this subfamily.

The subfamily Ω_Q exhibits properties similar to Ω_P. With $\beta = \beta_R = \sin\rho$, the existence of bound states is again connected to the condition, $0 < \xi \pm \rho \pmod{2\pi} < \pi$. The scattering states, while having again the generic k-dependence, possess the amplitudes satisfying

$$r^{(r)} = r^{(l)} \qquad \text{and} \qquad t^{(r)} = -t^{(l)} \ . \tag{B.25}$$

The transfer matrix formalism mentioned in Sect. 2 provides a method which is more appealing physically than the direct method we just used. We demonstrate its usefulness here to recover some of the the scattering data and the spectral properties first, and then later use it to study the discrete spectra that appear under certain Dirichlet and Neumann boundary conditions. To this end, let us introduce a vector from the wavefunction by

$$\Psi(x) = \begin{pmatrix} \varphi(x) \\ \varphi'(x) \end{pmatrix}, \tag{B.26}$$

and define the *transfer matrix* $M(x,y)$ by

$$\Psi(x) = M(x,y)\,\Psi(y). \tag{B.27}$$

In the absence of interaction, one has the free transfer matrix,

$$M_0(x,y) = \begin{pmatrix} \cos k(x-y) & \frac{1}{k}\sin k(x-y) \\ -k\sin k(x-y) & \cos k(x-y) \end{pmatrix}. \tag{B.28}$$

Consider then the eigenvalue problems for $M_0(x,y)$ and its conjugate $M_0^\dagger(x,y)$,

$$M_0(x,y)\,u_\pm = e^{\pm ik(x-y)}\,u_\pm \ , \qquad M_0^\dagger(x,y)\,v_\pm = e^{\pm ik(x-y)}\,v_\pm \ . \tag{B.29}$$

The eigenvectors $u_\pm$ and $v_\pm$ are given by

$$u_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm ik \end{pmatrix}, \qquad v_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mp 1/ik \end{pmatrix}, \tag{B.30}$$

which satisfy the bi-orthogonal relations,

$$v_\pm^\dagger u_\pm = 1, \qquad v_\mp^\dagger u_\pm = 0. \tag{B.31}$$

The advantage of using the transfer matrix is that the boundary condition (2.10) characteristic to the point interaction — which is a local description of (2.7) — provides precisely the connection condition for the transfer matrix at $x = 0_-$ and 0_+ as $M(0_+, 0_-) = \Lambda$ or

$$\Psi(0_+) = \Lambda\, \Psi(0_-) \ , \tag{B.32}$$

which is nothing but the condition (2.10).

By means of the transfer matrix, the scattering process induced by an incoming plane wave from the right, for instance, can easily be determined as follows. First, the vector (B.26) corresponding to the wavefunction (B.5) is given by

$$\Psi_k^{(r)}(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} t^{(r)} e^{-ikx} u_-, & x < 0, \\ e^{-ikx} u_- + r^{(r)} e^{ikx} u_+, & x > 0, \end{cases} \tag{B.33}$$

At the point singularity, the vector must satisfy (B.32) which reads

$$u_- + r^{(r)} u_+ = t^{(r)} \Lambda u_- \ . \tag{B.34}$$

From this, with the help of bi-orthogonality, we obtain immediately the amplitudes,

$$t^{(r)} = \frac{1}{v_-^\dagger \Lambda u_-} \ , \qquad r^{(r)} = \frac{v_+^\dagger \Lambda u_-}{v_-^\dagger \Lambda u_-} \ , \tag{B.35}$$

where we have

$$\begin{aligned} v_-^\dagger \Lambda u_- &= \frac{\Lambda_{11} + \Lambda_{22}}{2} - i \frac{\Lambda_{12} k - \Lambda_{21}/k}{2} \ , \\ v_+^\dagger \Lambda u_- &= \frac{\Lambda_{11} - \Lambda_{22}}{2} - i \frac{\Lambda_{12} k + \Lambda_{21}/k}{2} \ , \end{aligned} \tag{B.36}$$

in terms of the components of Λ. It can be readily confirmed using (2.12) that these amplitudes in (B.35) agree with the ones in (B.6) obtained under the global description. The negative energy bound states can also be found from the amplitude $t^{(r)}$ (or $r^{(r)}$) by looking at the poles on the imaginary axis in the complex k-plane.

It is instructive to look at the scattering amplitudes for the two limiting cases of delta and epsilon function potentials. For delta function potential, $\Lambda_{21} = 0$, the wave function is symmetric, *i.e.*,

$$\Psi(0_+) = \begin{pmatrix} \varphi_+(0_+) \\ \varphi'_+(0_+) \end{pmatrix}, \qquad \Psi(0_-) = \begin{pmatrix} \varphi_+(0_+) \\ -\varphi'_+(0_+) \end{pmatrix}. \tag{B.37}$$

From this and eqs.(3.47) and (4.1), one has

$$\Lambda = \begin{pmatrix} 1 & 0 \\ -2g_+/L_0 & 1 \end{pmatrix}, \tag{B.38}$$

which results in

$$t^{(r)} = \frac{1}{1 - ig_+/(kL_0)} , \qquad r^{(r)} = \frac{1 + ig_+/(kL_0)}{1 - ig_+/(kL_0)} . \tag{B.39}$$

Thus we have $|t^{(r)}|^2 = 0$ at $k = 0$ and $|t^{(r)}|^2 \to 1$ at $k \to \infty$. In other words, the delta function potential works as a high-pass (or low-cut) filter for the incoming quantum wave. Similarly, for epsilon function potential, $\Lambda_{12} = 0$, whose wave functions are antisymmetric,

$$\Psi(0_+) = \begin{pmatrix} \varphi_-(0_+) \\ \varphi'_-(0_+) \end{pmatrix}, \qquad \Psi(0_-) = \begin{pmatrix} -\varphi_-(0_+) \\ \varphi'_-(0_+) \end{pmatrix}, \tag{B.40}$$

and therefore

$$\Lambda = \begin{pmatrix} 1 & -2g_-L_0 \\ 0 & 1 \end{pmatrix}, \tag{B.41}$$

one obtains

$$t^{(r)} = \frac{1}{1 + ig_-kL_0} , \qquad r^{(r)} = \frac{1 - ig_-kL_0}{1 + ig_-kL_0} , \tag{B.42}$$

thus $|t^{(r)}|^2 = 1$ at $k = 0$ and $|t^{(r)}|^2 \to 0$ at $k \to \infty$. One thus sees, for instance, that the ε-function potential works as a low-pass (or high-cut) filter. In the general case (B.35), one has a quantum filter that passes only certain range of momentum value cutting out both high and low frequency components from the transmitted waves.

B.2. System with point interaction in a box

So far our system has been that of a particle moving freely on a line $\mathbb{R}$ under a point singularity at $x = 0$. However, even if we put the particle in a box given, say, by the interval $-l_- \le x \le l_+$, the effect of the singularity at $x = 0$ remains the same and can be characterized by the same condition (2.7). The extra requirement we need to take into account is the boundary conditions at the edges $x = l_+$ and $x = -l_-$, and for this we shall consider some typical cases below.

We begin by the simple case $l_+ = l_- = l$ with a periodic or antiperiodic boundary condition, $\Psi(l) = \pm\Psi(-l)$, that is,

$$\Psi(0_-) = \pm M_0(2l)\, \Psi(0_+) . \tag{B.43}$$

Combining this with the connection condition (B.32) at the singularity, we obtain the eigenvalue problem,

$$M_0(-2l)\,\Psi(0_-) = \pm\Lambda\,\Psi(0_-)\ . \tag{B.44}$$

Energy eigenstates arise at the roots for k in the eigenvalue equation,

$$\det[\Lambda \mp M_0(-2l)] = 0\ , \tag{B.45}$$

which is

$$2\sin\xi\cos 2kl + \left[\cos\xi\left(kL_0 + \frac{1}{kL_0}\right) + \alpha_R\left(kL_0 - \frac{1}{kL_0}\right)\right]\sin 2kl \mp 2\beta_I = 0. \tag{B.46}$$

The periodic case is equivalent to the system of a circle of length $2l$, for which the spectral classification has been discussed in various subfamilies in [14].

Next we consider two cases of an ideal box where 'infinite' walls are placed at the edges so that no probability current can leak from the box. One of them is given by the Dirichlet boundary condition $\varphi(l_+) = \varphi(-l_-) = 0$ at the edges. In terms of the transfer matrix, we then have

$$M_0(l_+)\,\Lambda\,M_0(l_-)\begin{pmatrix} 0 \\ \varphi'(l_-)\end{pmatrix} = \begin{pmatrix} 0 \\ \varphi'(l_+)\end{pmatrix}, \tag{B.47}$$

from which we obtain the eigenvalue equation,

$$(M_0(l_+)\,\Lambda\,M_0(l_-))_{12} = 0\ . \tag{B.48}$$

Similarly, for the Neumann boundary condition $\varphi'(l_+) = \varphi'(-l_-) = 0$ we find

$$M_0(l_+)\,\Lambda\,M_0(l_-)\begin{pmatrix} \varphi(l_-) \\ 0\end{pmatrix} = \begin{pmatrix} \varphi(l_+) \\ 0\end{pmatrix}\ , \tag{B.49}$$

which leads to

$$(M_0(l_+)\,\Lambda\,M_0(l_-))_{21} = 0\ . \tag{B.50}$$

In each case, the spectrum is determined by the eigenvalue equation. Explicitly, for the case of the Dirichlet boundary condition, the equation leads to

$$\begin{aligned}&(kL_0)^2\cot kl_+\cot kl_-(\cos\xi + \alpha_R)\\ &- kL_0\left[\cot kl_+(\sin\xi - \alpha_I) + \cot kl_-(\sin\xi + \alpha_I)\right] - (\cos\xi - \alpha_R) = 0\ ,\end{aligned} \tag{B.51}$$

whereas for the case of the Neumann boundary condition, it yields

$$\begin{aligned}&(kL_0)^2 \tan kl_+ \tan kl_-(\cos\xi + \alpha_R) \\ &+ kL_0\left[\tan kl_+(\sin\xi + \alpha_I) + \tan kl_-(\sin\xi - \alpha_I)\right] - (\cos\xi - \alpha_R) = 0\ .\end{aligned} \tag{B.52}$$

In particular for $l_+ = l_- = l$, we have

$$kL_0 \cot kl = \frac{\sin\xi \pm \sqrt{1-\alpha_R^2}}{\cos\xi + \alpha_R} \tag{B.53}$$

for the Dirichlet case, and

$$kL_0 \tan kl = \frac{-\sin\xi \pm \sqrt{1-\alpha_R^2}}{\cos\xi + \alpha_R} \tag{B.54}$$

for the Neumann case, respectively. In particular, for $\alpha_R = 0$, $\xi = \pi/2$, *i.e.,* in the scale invariant subfamily Ω_W, one simply has

$$\cos kl = 0\ , \tag{B.55}$$

for the Dirichlet case, and

$$\sin kl = 0\ , \tag{B.56}$$

for the Neumann case, which shows that the scale invariant sphere $\Omega_W \simeq S^2$ in Ω is isospectral.

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