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Instantons, Monopoles and the Flux Quantization in the Faddeev-Niemi Decomposition

Toyohiro Tsurumaru[1] and Izumi Tsutsui[2]

Institute of Particle and Nuclear Studies
High Energy Accelerator Research Organization (KEK)
Tsukuba, Ibaraki 305-0801, Japan

Akira Fujii[3]

Department of Physics, Graduate School of Science
Osaka University
Toyonaka, Osaka 560-0043, Japan

Abstract. We study how instantons arise in the low energy effective theory of the $SU(2)$ Yang-Mills theory in the context of the non-linear sigma model recently proposed by Faddeev and Niemi. We find a simple relation between the instanton number ν and the charge m of the monopole that appears in the effective theory. It is given by $\nu = m\Phi/(2\pi)$, where Φ is the quantized flux associated with a $U(1)$ gauge field passing through the loop formed by the singularity of the monopole.

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[1] E-mail: toyohiro.tsurumaru@kek.jp
[2] E-mail: izumi.tsutsui@kek.jp
[3] E-mail: fujii@het.phys.sci.osaka-u.ac.jp

1. Introduction

Yang-Mills (YM) theory is known to provide a basis for describing fundamental interactions, most notably, the strong interaction. In the high energy regime the theory admits perturbative studies thanks to the asymptotic freedom, whereas in the low energy regime it defies a similar systematic analysis due to the strong coupling behavior. Recently, Faddeev and Niemi made a suggestion [1] that the $SU(2)$ YM theory $\mathcal{L}_{\rm YM} = \frac{1}{2g^2}{\rm tr}F_{\mu\nu}F^{\mu\nu}$ in a certain phase of the low energy regime may be described by a modified version of the $O(3)$ nonlinear sigma model (NLSM) in four dimensions often referred to as the Skyrme-Faddeev model,

$$\mathcal{L}_{\rm SF} = \frac{1}{2\lambda^2}(\partial_\mu \mathbf{n})^2 + \frac{1}{4e^2}(\epsilon^{abc}n^a\partial_\mu n^b\partial_\nu n^c)^2. \tag{1.1}$$

This model permits stable soliton solutions [2, 3], which have been conjectured to be a candidate for glueballs [1].

The key ingredient for establishing the connection between the $SU(2)$ YM theory and the Skyrme-Faddeev model is the so-called Faddeev-Niemi (FN) decomposition[4]

$$\mathbf{A} = C\,\mathbf{n} + (1+\sigma)\,{\rm d}\mathbf{n}\times\mathbf{n} + \rho\,{\rm d}\mathbf{n}. \tag{1.2}$$

This is a new version of the Abelian projection which decomposes the YM field $\mathbf{A}$ into a vector field C, a complex scalar $\phi = \rho + i\sigma$ and an iso-vector field $\mathbf{n}$ of unit length $\sum_{a=1}^{3} n^a n^a = 1$. The vector field C is a gauge field associated with the $U(1)$ gauge transformations with respect to $\mathbf{n}$, that is, it transforms as $C \to C + {\rm d}\theta$ under the $U(1)$ subgroup $U(x) = \exp[\theta\, n^a T^a]$ of $SU(2)$ transformations for $\mathbf{A}$. Thus, at on-shell, C has two degrees of freedom when gauge-fixed, and together with $\mathbf{n}$, ρ and σ it comprises the six degrees of freedom of the on-shell $SU(2)$ YM field. The original motivation for the Abelian projection proposed by 't Hooft and Polyakov [4] was to provide a qualitative explanation for the color confinement by asserting that in the low energy regime the YM field exhibits the dual Meissner effect in a condensate of magnetic monopoles. With regard to this, it is very important to study how instantons — a hallmark of the non-perturbative aspect of the YM theory — can affect the magnetic monopole configurations in the Skyrme-Faddeev model obtained under the FN decomposition (1.2). (The implication of the FN decomposition for confinement and chiral symmetry breaking has been discussed in Ref.[5].)

In this note we point out that, when instantons are present in the $SU(2)$ YM theory, monopoles must necessarily appear in the low energy regime, *i.e.*, in the Skyrme-Faddeev

[4] Notation: We denote $su(2)$-valued fields such as $A = A^aT^a = A^a_\mu {\rm d}x^\mu T^a$ often by iso-vectors as $\mathbf{A} = (A^1, A^2, A^3)$. Here d is the space-time exterior derivative, and $T^a := \tau^a/(2i)$, where τ^a are the Pauli matrices, is the $su(2)$ basis we use.

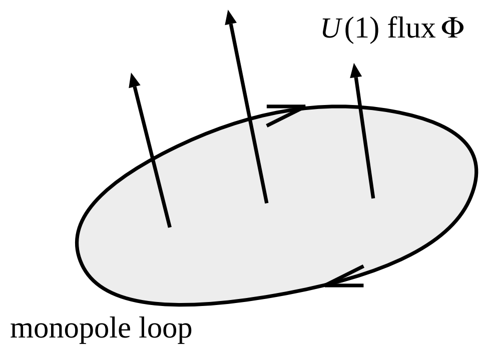

Figure 1. The singularity of $\mathbf{n}$ arises as a closed loop (monopole loop) in the spacetime for non-vanishing instanton sectors. The instanton number ν is given by the monopole charge m times the integer of the quantized flux Φ associated with the $U(1)$ gauge field C passing through the surface encircled by the monopole loop.

model. The relation between instantons and monopoles has been discussed earlier by Christ and Jackiw for physically static field configurations [6], where the instanton number was shown to coincide with the monopole charge for $SU(2)$. In the context of the original Abelian projection, a similar relation has been found in [7] for generic configurations under the Weyl gauge. More recently, it has been shown without reference to gauge fixing that the instanton number is given, modulo the monopole charge, by the (generalized) Hopf invariant associated with an adjoint scalar field [8]. Here we examine the problem in the context of the FN decomposition, and argue that the relation between the instanton number and the monopole charge is again given by a similar formula, involving this time a quantized flux trapped by the monopole. Specifically, if the original YM gauge field that admits the FN decomposition possesses a nonzero instanton number, the corresponding vector field $\mathbf{n}$ must have singularity in closed circles, or 'monopole loops'. We shall find that the flux Φ associated with the $U(1)$ gauge field C trapped by the monopole loop must be quantized topologically, and that the instanton number ν is given by the product of the monopole charge m and the topological integer of the flux Φ (see Fig.1),

$$\nu = m\,\frac{\Phi}{2\pi}. \tag{1.3}$$

The rest of the paper is devoted to show this simple relation.

2. Instanton and singularity

Prior to the discussion on the relation (1.3), we first show that, under the FN decomposition (1.2), the field $\mathbf{n}$ is necessarily singular if the original $\mathbf{A}$ possesses a non-vanishing instanton number. Following the conventional procedure, we assign a configuration of the gauge field on our spacetime diffeomorphic to S^4 by introducing a set of local coordinate patches $V_k \simeq D^4$ for $k = 1, 2$, say. Let A_k be the gauge potential on the patch V_k. On the overlap $V_1 \cap V_2$, the gauge fields are related by

$$A_2 = U^\dagger A_1 U + U^\dagger \mathrm{d}U \ , \tag{2.1}$$

with the transition function $U(x) \in SU(2)$. The instanton number ν of the gauge field is encoded in the transition function in its winding number on the boundary $\partial V_1 \simeq S^3$ (or equivalently on ∂V_2),

$$\nu = -\frac{1}{8\pi^2}\int_{S^4} \mathrm{tr}(F \wedge F) = \frac{1}{24\pi^2}\int_{\partial V_1} \mathrm{tr}(U^\dagger \mathrm{d}U)^3. \tag{2.2}$$

If we now adopt the FN decomposition to each of the gauge fields on the patches, we have

$$\mathbf{A}_k = C_k \mathbf{n}_k + (1+\sigma_k)\, \mathrm{d}\mathbf{n}_k \times \mathbf{n}_k + \rho_k\, \mathrm{d}\mathbf{n}_k \ , \tag{2.3}$$

for $k = 1, 2$. We remark that the FN decomposition implies, in effect, a specific gauge fixing and use of an on-shell condition for $\mathbf{A}$ (except for the part of the $U(1)$ gauge field C), and therefore it is a nontrivial question if the decomposition (2.3) is actually available over the entire patches we have introduced. In what follows we consider the class of configurations for which this is possible, knowing that standard instanton configurations such as Witten's solution (see below) belong to this class. Further, for our effective picture of the Skyrme-Faddeev model which has no local gauge symmetry, we need to regard the field $\mathbf{n}$ *physical*, that is, it represents the low energy dynamical freedom of the YM theory irrespective of the gauge chosen. In other words, we assume that *given a configuration of the $SU(2)$ gauge field $\mathbf{A}$ with gauge equivalent configurations identified, the corresponding field $\mathbf{n}$ in the FN decomposition can be uniquely determined up to a global $O(3)$ transformation.* This is a crucial but necessary demand for the FN decomposition to work in practice, allowing us to discuss the topological aspect of the YM theory in terms of the field $\mathbf{n}$.

Upon this assumption we can perform, with no loss of generality, a global $O(3)$ transformation so that we have $\mathbf{n}_1 = \mathbf{n}_2$ on the overlap $V_1 \cap V_2$. Thus the transition function U defined on the overlap must be of the form,

$$U(x) = \exp\left[\theta(x) n(x)\right] , \qquad n(x) := \sum_a n^a(x) T^a \ , \tag{2.4}$$

with some function $\theta(x)$ on $V_1 \cap V_2$. Then it is easily seen that if $\mathbf{n}$ is regular everywhere on ∂V_1, the winding number (2.2) of $U(x)$ must be zero. Indeed, if we expand $U(x)$ in (2.4) in the form,

$$U(x) = \alpha(x) + \beta(x)\, n(x) \ , \tag{2.5}$$

with

$$\alpha(x) = \cos\frac{\theta}{2} = \frac{1}{2}\mathrm{tr}\,U\ , \qquad \beta(x) = \sin\frac{\theta}{2} = -2\,\mathrm{tr}\,(U, \mathbf{n})\ , \tag{2.6}$$

then $\alpha(x)$ and $\beta(x)$ are also regular for regular $n(x)$, because $U(x)$ is given regularly over the overlap. Since (α, β) may be regarded as a map from $\partial V_1 \simeq S^3$ to S^1, it can be deformed continuously to the constant map $(\alpha, \beta) = (1, 0)$ on account of $\pi_3(S^1) = 0$. For regular $\mathbf{n}$, this means that $U(x)$ can be continuously deformed to the constant map $U(x) = 1$, for which the winding number vanishes. It follows therefore that $\mathbf{n}$ must necessarily be singular if the gauge field $\mathbf{A}$ has a non-vanishing instanton number. Note that the regularity of $U(x)$ demands that the parameter θ in (2.4) be such that the function β in (2.5) converges to zero sufficiently fast when $\mathbf{n}$ approaches its singularity. Thus, for singular $\mathbf{n}$, the functions (α, β) are still given regularly, even though the deformation of (α, β) mentioned above cannot be performed due to the constraint $\beta = 0$ at the singularity.

The singularity of $\mathbf{n}$ can be classified by the topological property $\pi_2(S^2) = \mathbb{Z}$ by regarding $\mathbf{n}$ as a map from an arbitrary two-dimensional sphere S^2 in the spacetime to the target S^2. The topological integer can be provided by

$$m = \frac{1}{2\pi}\int_{S^2} \mathrm{tr}(n\,\mathrm{d}n \wedge \mathrm{d}n)\ , \tag{2.7}$$

which we call the 'monopole charge'. In the four-dimensional spacetime, we expect that the singularity of $\mathbf{n}$ forms a one-dimensional closed loop [4], *i.e.*, a monopole loop as shown in Fig.1. From the foregoing argument we find that, if $\nu \neq 0$ then $m \neq 0$, or equivalently, if $m = 0$ then $\nu = 0$.

At this point, it is instructive for us to look at explicit solutions of the YM instantons that admit the FN decomposition. For this we take Witten's ansatz [9] which is an axially symmetric self-dual solution to the $SU(2)$ YM equation with arbitrary instanton numbers. With the coordinate $(\vec{\mathrm{x}}, t)$ on a local patch, the ansatz assumes for the solution the most general form which is invariant under $SO(3)$ rotations,

$$A_0^a = \frac{A_0 x^a}{r}\ , \qquad A_j^a = \frac{\varphi_2 + 1}{r^2}\epsilon_{jak}x_k + \frac{\varphi_1}{r^2}[\delta_{ja}r^2 - x_j x_a] + A_1\frac{x_j x_a}{r^2}\ , \tag{2.8}$$

where φ, A_0, A_1 are functions of the radius $r = |\vec{\mathrm{x}}|$ and t. Note that this is already in the form of the FN decomposition (1.2), as seen by setting $\varphi_1 = \rho$, $\varphi_2 = \sigma$, $C_i = (x_i/r)A_1$, $C_0 = A_0$ and $n^a(\vec{\mathrm{x}}) = x^a/r$ [10], where now the monopole loop is located at the origin of the space. If we let $z = r + it$ and define ψ by $A_\mu = \epsilon_{\mu\nu}\partial_\nu\psi$ ($\mu, \nu = 0, 1$ where 0 and 1 refers to t and r, respectively), self-dual solutions with instanton number $\nu = k - 1$ are obtained by $\psi = -\ln[(1 - g^* g)/(2r)]$ with $g(z) = \prod_{i=1}^{k}(a_i - z)/(a_i^* + z)$

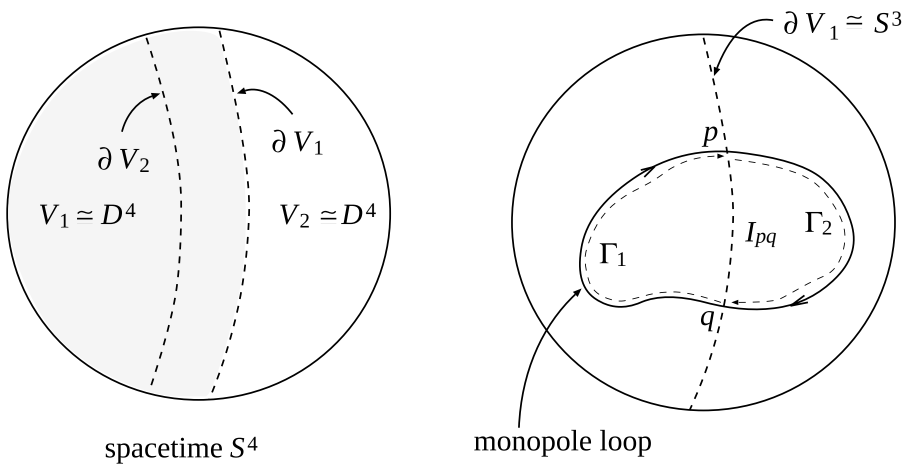

Figure 2. The spacetime S^4 is covered with two local patches V_1, $V_2 \simeq D^4$. The monopole loop lies across the overlap $V_1 \cap V_2$, and hence can be cut into two pieces Γ_1 and Γ_2 at the two intersections p and q with the boundary ∂V_1.

using arbitrary complex constants a_i satisfying $\mathrm{Re}\, a_i > 0$. Clearly, the instanton number ν is not determined (and hence can be put even to zero) under the field $n^a(\vec{\mathrm{x}}, t) = x^a/r$ which is singular at the origin and has the monopole charge $m = 1$ (see (2.7)).

The above example shows that the instanton number ν is not determined by the monopole charge alone, and that to account for the instanton number the configuration of the $U(1)$ gauge field C must also be considered. Indeed, for the ansatz (2.8) the instanton number (2.2) reduces to

$$\nu = \frac{1}{2\pi} \int_{D^2} \mathrm{d}^2 x \left[\partial_\mu (\epsilon_{ij} \epsilon_{\mu\nu} \varphi_i D_\nu \varphi_j) + \frac{1}{2} \epsilon_{\mu\nu} F_{\mu\nu} \right] , \tag{2.9}$$

where D^2 is a two-dimensional disc whose boundary is the monopole loop. We find that the second term yields the flux of the gauge field C passing through the monopole loop, and this will be seen the only contribution to the instanton number ν (*i.e.*, the first term given by the surface integral vanishes) on a general basis shortly.

3. Instanton number vs monopole charge

The formula (2.9) suggests that the flux associated with C penetrating through the surface encircled by the monopole loop contributes to the instanton number ν. We now

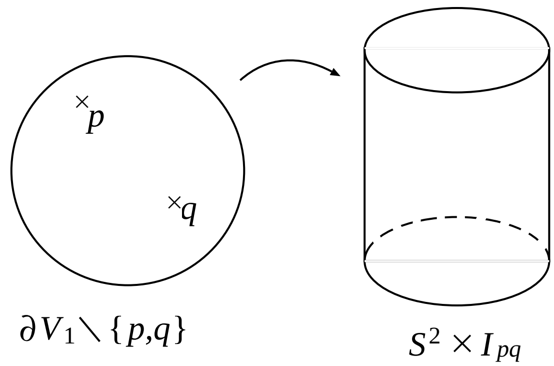

Figure 3. The removal of the two points p and q from ∂V_1 yields the cylinder $\partial V_1 \setminus \{p,q\} \simeq S^2 \times I_{pq}$.

show that ν is in fact proportional to the flux in just the way (1.3) we mentioned in the Introduction. We do this in two steps, first showing that the instanton number is proportional to the monopole charge, and then arguing that the proportional factor is given by the flux which is quantized.

To this end, recall that for nonzero ν the field $\mathbf{n}$ is singular on the boundary ∂V_1. This implies that the monopole loop (which is the line of singularity of $\mathbf{n}$) must intersect with ∂V_1 twice or more generally even times. Suppose, for simplicity, that there are two intersections on ∂V_1, which we denote as p and q. (When there are more than two intersections we may choose a different set of coordinate patches so that the boundary ∂V_1 intersects with the monopole loop only twice.) From (2.6) we find that in the overlap $\partial V_1 \cap \partial V_2$ the function θ can be defined regularly. At the singular points p and q, we have $\beta = \sin(\theta/2) = 0$ to ensure the regularity of U and hence

$$\theta = 2\pi \times \text{integer} \ , \tag{3.1}$$

at both p and q.

Consider then the cylinder $\partial V_1 \setminus \{p,q\} \simeq S^2 \times I_{pq}$ (without edges) obtained by removing the two points p and q from $\partial V_1 \simeq S^3$, where I_{pq} is an interval from p to q on the cylinder (see Fig.3). Since $U(x)$ is regular, the removal of the two points from the domain of integration for the instanton number (2.2) does not alter the outcome. Thus, instead of the domain $\partial V_1 \simeq S^3$ we may use the cylinder to evaluate the instanton number (2.2) as

$$\nu = \frac{1}{24\pi^2} \int_{S^2 \times I_{pq}} \mathrm{tr}\,(U^\dagger \mathrm{d}U)^3 = \frac{1}{4\pi^2} \int_{S^2 \times I_{pq}} (\,1 - \cos\theta\,)\,\mathrm{d}\theta\ \mathrm{tr}\,(n\mathrm{d}n \wedge \mathrm{d}n) \ , \tag{3.2}$$

where we have used

$$U^\dagger \mathrm{d}U = (n\,\mathrm{d}\theta + \sin\theta\,\mathrm{d}n) - [n, \mathrm{d}n](1 - \cos\theta) \ . \tag{3.3}$$

Let us choose the coordinates of the cylinder $S^2 \times I_{pq}$ such that θ is constant on S^2 at each point of I_{pq}. This choice of coordinates allows us to evaluate the integral (3.2) by two separate integrals over θ and $\mathbf{n}$ as

$$\nu = \frac{m}{2\pi} \int_{I_{pq}} (\, 1 - \cos\theta \,)\, \mathrm{d}\theta = \frac{m}{2\pi} \int_{I_{pq}} \mathrm{d}\theta \ , \tag{3.4}$$

where m is the monopole charge (2.7), and we have used (3.1) for the second equality. The condition (3.1) also ensures that the r.h.s. of (3.4) gives an integer as required.

4. Flux quantization

It remains to show that the integer factor multiplying m in (3.4) is the flux Φ associated with the $U(1)$ gauge field C passing through the monopole loop. Let Γ_k, $k = 1, 2$, be the contours obtained by cutting the monopole loop in half at p and q (see Fig.2). Then the flux Φ_k penetrating the surface encircled by Γ_k and I_{pq} is given by $\Phi_k = (\int_{\Gamma_k} + \int_{I_{pq}}) C_k$. Noting that C_1 and C_2 are related by the relation $C_2 = C_1 + \mathrm{d}\theta$ on I_{pq}, we find that the total flux is given by

$$\Phi = \Phi_1 + \Phi_2 = \int_{\Gamma_1} C_1 + \int_{\Gamma_2} C_2 + \int_{I_{pq}} \mathrm{d}\theta \ . \tag{4.1}$$

We shall argue that the component of the $U(1)$ field C_k along the monopole loop actually vanishes at the monopole loop, and hence the contributions from the two contours Γ_1 and Γ_2 in (4.1) disappear.

For this, we choose a local coordinate patch in V_1 and parameterize it by $(\vec{\mathrm{x}}, t)$, where the origin $\vec{\mathrm{x}} = \vec{0}$ is taken to be at the monopole loop, and t is the local coordinate along the monopole loop. With $\hat{\mathrm{x}} = \vec{\mathrm{x}}/|\vec{\mathrm{x}}|$ and $r = |\vec{\mathrm{x}}|$, we consider the limit,

$$\mathbf{n}_0(\hat{\mathrm{x}}, t) := \lim_{r \to 0} \mathbf{n}(r\hat{\mathrm{x}}, t) \ . \tag{4.2}$$

Here we suppose that $\mathbf{n}(x)$ has a certain limit dependent on the direction $\hat{\mathrm{x}}$. To specify the direction, let us use the polar coordinates (r, ϑ, φ) and consider the unit vectors $\vec{\mathrm{e}}_r$, $\vec{\mathrm{e}}_\vartheta$, $\vec{\mathrm{e}}_\varphi$ associated with them. (Note that $\mathbf{n}$ has a direction-independent limit if and only if it is regular, for which we have $m = 0$ and hence the relation (1.3) holds trivially.) For $\mathbf{n}$ with $m \neq 0$, the radius of the sphere S^2 on which the monopole charge (2.7) is evaluated can be taken as small as we wish without changing the value m. From this we recognize that the derivative,

$$\vec{\bigtriangledown}\mathbf{n} = \vec{\mathrm{e}}_r \partial_r \mathbf{n} + \frac{\vec{\mathrm{e}}_\vartheta}{r} \partial_\vartheta \mathbf{n} + \frac{\vec{\mathrm{e}}_\varphi}{r} \partial_\varphi \mathbf{n} \ , \tag{4.3}$$

diverges as $1/r$ on average, or more precisely, on an area of finite volume on the S^2. Under the existence of the limit (4.2), we obtain

$$\lim_{r\to 0}\frac{|\partial_t \mathbf{n}|}{\left|\vec{\nabla}\mathbf{n}\right|}=\lim_{r\to 0}\frac{|\partial_t \mathbf{n}_0|}{\left|\vec{\nabla}\mathbf{n}\right|}=0\ , \tag{4.4}$$

where we have used the notation $\left|\vec{\nabla}\mathbf{n}\right| = \sqrt{\sum_{i,a}(\partial_i n^a)^2}$. On the other hand, from the inequality,

$$|(1+\sigma)\partial_t\mathbf{n}\times\mathbf{n}+\rho\partial_t\mathbf{n}| = \left|(1+\sigma)\vec{\nabla}\mathbf{n}\times\mathbf{n}+\rho\vec{\nabla}\mathbf{n}\right|\frac{|\partial_t \mathbf{n}|}{\left|\vec{\nabla}\mathbf{n}\right|}\le|\vec{\mathbf{A}}|\frac{|\partial_t \mathbf{n}|}{\left|\vec{\nabla}\mathbf{n}\right|}\ , \tag{4.5}$$

and the fact that $|\vec{\mathbf{A}}|$ is finite, we observe that the l.h.s. of (4.5) converges to zero. Hence it follows that

$$\mathbf{A}_t \to C_t\mathbf{n} \qquad \text{for} \quad r\to 0\ . \tag{4.6}$$

However, since $\mathbf{A}_t$ is smooth while $\mathbf{n}$ is singular at the origin $r = 0$, we must have $\lim_{r\to 0}\mathbf{A}_t = 0$ and

$$C_t \to 0 \qquad \text{for} \quad r\to 0\ . \tag{4.7}$$

Thus we see that the component C_t along the monopole loop vanishes identically at the monopole loop. This in turn implies that the flux Φ in (4.1) is quantized as $\Phi = 2\pi\times$integer, where the integer is given by the difference in the integers (3.1) of the angle θ at the singular points p and q. (The quantization condition takes a more familiar form $\Phi = 2\pi\hbar/g\times$integer if the flux is evaluated for C/g with a properly rescaled g.) In conclusion, we have shown that the relation (1.3) holds for the class of those $\mathbf{n}$ for which the limit (4.2) exists. We note that Witten's ansatz for instantons has such a limit, and we expect that any physically interesting configurations will also have it, because violent fluctuations will be smeared out in the low energy regime. For completeness, however, in the Appendix we shall provide an outline of the argument for more generic configurations for $\mathbf{n}$.

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Appendix

We here discuss the general case in which $\mathbf{n}$ may not have the limit (4.2). Let us regard A_μ^a as a four times three matrix and consider the quantity rank $\mathbf{A}$ which is the number of linearly independent (column or, equivalently, row) vectors in the matrix A_μ^a. Choose a point on the monopole loop to which we assign the value $t = 0$. Consider then a neighbourhood of the point with four-dimensional radius ϵ such that

$$\mathrm{rank}\mathbf{A}(x) \geq \mathrm{rank}\mathbf{A}(x=0) \qquad \text{for} \quad |x| = \sqrt{x_\mu x^\mu} < \epsilon. \tag{A.1}$$

Since rank $\mathbf{A}(x=0) = 0$ means $\mathbf{A}(x=0) = 0$ and hence $\lim_{r\to 0} C_\mu = 0$, we only need to consider the cases where rank $\mathbf{A} \geq 1$ at $x = 0$.

(i) rank $\mathbf{A}(x=0) = 3$: In this case we have $k^\mu(x)$ satisfying $k^\mu \mathbf{A}_\mu = 0$ given by

$$k^\mu := \frac{1}{3!}\epsilon^{\mu\alpha\beta\gamma}\epsilon_{abc}A_\alpha^a A_\beta^b A_\gamma^c\ , \tag{A.2}$$

which is a non-vanishing smooth function for rank $\mathbf{A} = 3$. As can be easily seen from (1.2), we have $(1+\sigma)^2 + \rho^2 \neq 0$ for $r \neq 0$, and hence $k^\mu \mathbf{A}_\mu = 0$ means that

$$k^\mu \partial_\mu \mathbf{n} = 0\ . \tag{A.3}$$

Thus k^μ is a Killing vector for $\mathbf{n}$, and hence it must point to the t-direction along the monopole loop at $x = 0$. It then follows that $\mathbf{A}_t(x=0) = 0$ and, therefore, $C_t(x=0) = 0$.

(ii) rank $\mathbf{A}(x=0) = 2$ and 1: For $\mathbf{A}_\mu$ with rank $= 2$, we can put $\mathbf{A}_\mu$ and $\partial_\mu \mathbf{n}$ (which also has rank $= 2$ because $\mathbf{n}$ has two independent freedoms) in the form,

$$\mathbf{A}_\mu(x) = \alpha_\mu \mathbf{a} + \beta_\mu \mathbf{b} + O(|x|)\ , \qquad \partial_\mu \mathbf{n}(x) = \xi_\mu(x)\mathbf{e}(x) + \chi_\mu(x)\mathbf{f}(x)\ , \tag{A.4}$$

where α_μ, β_μ, $\mathbf{a}$, $\mathbf{b}$ are constant vectors while other vectors ξ_μ, χ_μ, $\mathbf{e}$, $\mathbf{f}$ are coordinate dependent. Plugging these into the identity,

$$(\mathbf{A}_\mu \times \mathbf{n}) \times \mathbf{n} = -(1+\sigma)\,\partial_\mu \mathbf{n} \times \mathbf{n} + \rho\,\partial_\mu \mathbf{n}\ , \tag{A.5}$$

which holds for $r > 0$, we see that the vectors $\xi_\mu(x)$ and $\chi_\mu(x)$ are given by a linear combination of α_μ and β_μ with coordinate dependent coefficients. We thus find that, for $|x| \to 0$, $\partial_\mu \mathbf{n}$ takes the form,

$$\partial_\mu \mathbf{n} \to \alpha_\mu \mathbf{u}(\mathbf{n}) + \beta_\mu \mathbf{v}(\mathbf{n}), \tag{A.6}$$

where $\mathbf{u}(\mathbf{n})$, $\mathbf{v}(\mathbf{n})$ are vectors orthogonal to $\mathbf{n}$. Let γ_μ be a vector which is orthogonal to both α_μ and β_μ and has no time-component. If we choose the z-direction of space to be parallel to the vector γ_μ, then we get

$$\lim_{|x|\to 0} \frac{|\partial_z \mathbf{n}|}{\left|\vec{\nabla}\mathbf{n}\right|} = 0 \ . \tag{A.7}$$

This is, however, impossible because $\partial_i \mathbf{n}$ must diverge uniformly for all i as $r \to 0$ if $\mathbf{n}$ has a non-vanishing monopole charge $m \neq 0$. Thus, we conclude that $\mathbf{A}_\mu$ with rank $= 2$ cannot arise at the monopole loop. The case $\mathbf{A}_\mu$ with rank $= 1$ can also be denied by a similar argument.

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