Duality and Anholonomy in Quantum Mechanics of 1D Contact Interactions

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We study systems with parity invariant contact interactions in one dimension. The model analyzed is the simplest nontrivial one — a quantum wire with a point defect — and yet is shown to exhibit exotic phenomena, such as strong vs weak coupling duality and spiral anholonomy in the spectral flow. The structure underlying these phenomena is $SU(2)$, which arises as accidental symmetry for a particular class of interactions.

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Throughout its century-long history, quantum mechanics has been mostly the science of explanation for naturally existing microscopic objects. With the advent of nano-scale engineering, however, we are witnessing the emergence of quantum technology, in which a system of desired specification is designed and manufactured [1]. It is commonly assumed that the operations of nano-scale devices can be understood, in essence, in terms of the quantum mechanics of elementary textbooks. This by no means diminishes the significance of the remarkable technological achievements, but one might still ask the question why they do not admit the richness of exotic phenomena, such as quantum anomaly, supersymmetry and duality, which grace the frontier of high energy physics. One might wonder whether this is a limitation inherent to the low energy physics, where the description by simple quantum mechanics is sufficient, without any real need for invoking quantum field theory or string theory.

In this Letter, we wish to purge such pessimism by investigating one of the simplest quantum systems — a particle in one dimension subject to a contact interaction, an interaction that acts only at a single spatial point. This is an idealized model of a quantum wire with a single defect. It can be also regarded as a solvable limit of a generic quantum system in which the potential has an effective range far shorter than the wavelength of the particle. Contrary to the widely held view, the quantum contact interaction in one dimension is far from trivial [2]. In addition to the well-known Dirac's δ-function potential which induces the discontinuity in the spatial derivative of wave functions, there exists another contact potential, called an 'ε-function potential', which brings about a discontinuity in wave functions [3,4]. By combining δ and ε-function potentials, one obtains a family of generalized one-dimensional contact interactions described by a set of four parameters. Several curious features of the generalized contact interaction have already been pointed out [5–8] over the past few years. We have seen, for instance, evidence of *duality* in the sense of an isospectral mapping between eigenstates of δ and ε-function potentials with reversed coupling strengths [8]. We also know that there arises *spiral anholonomy* displaying the double spiral structure of the energy surface [7]. (Contact interactions in two dimensions have been shown to possess scale anomaly [9].) However, these phenomena seem to be unrelated, and their physical and mathematical origins have not previously been uncovered.

This is precisely the subject of our discussion in this Letter. Specifically, we examine the structure of parity invariant contact interactions, and thereby uncover the general structure of the duality and the spiral anholonomy found earlier. In the course of our analysis, we shall also find accidental $SU(2)$ symmetry as well as supersymmetry that appears for a special class of contact interactions.

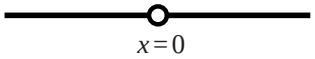

FIG. 1. A particle moves freely on a line except at the singular point, $x = 0$.

We begin by considering a quantum particle moving freely on a line $\mathbf{R}$ except at a single point, $x = 0$, where a singular contact interaction is at work. This system is described by removing the singular point from the line (Fig.1) and using the formal Hamiltonian,

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\ , \tag{1}$$

defined on $\mathbf{R}\backslash\{0\}$. We look for the most general condition for allowable quantum mechanical motion. To this end, we follow the approach of Ref. [10] and define the two-component vectors,

$$\Phi = \begin{pmatrix} \varphi(0_+) \\ \varphi(0_-) \end{pmatrix}, \qquad \Phi' = \begin{pmatrix} \varphi'(0_+) \\ -\varphi'(0_-) \end{pmatrix}. \tag{2}$$

from the values and derivatives of a wave function $\varphi(x)$ at the left $x = 0_-$ and the right $x = 0_+$ of the missing point. Due to the gap at $x = 0$, the wave function can acquire discontinuity there. The allowed discontinuity is dictated by quantum mechanical probability conservation, which requires the probability current

$j(x) = -i\hbar((\varphi^*)'\varphi - \varphi^*\varphi')/(2m)$ to be continuous at $x = 0$. In terms of Φ and Φ', the requirement is expressed as

$$\Phi'^{\dagger}\Phi - \Phi^{\dagger}\Phi' = 0 \ , \tag{3}$$

which is equivalent to $|\Phi - iL_0\Phi'| = |\Phi + iL_0\Phi'|$ with L_0 being an arbitrary constant in the unit of length. This means that, with a two-by-two unitary matrix $U \in U(2)$, we have the relation,

$$(U - I)\Phi + iL_0(U + I)\Phi' = 0 \ . \tag{4}$$

This shows that the entire family Ω of contact interactions admitted in quantum mechanics is given by the group $U(2)$. Mathematically, we say that the domain in which the Hamiltonian H becomes self-adjoint is parametrized by $U(2)$ — there is a one-to-one correspondence between a physically distinct contact interaction and a self-adjoint Hamiltonian. A standard parametrization for U is

$$U = e^{i\xi}\begin{pmatrix} \alpha_R + i\alpha_I & \beta_R + i\beta_I \\ -\beta_R + i\beta_I & \alpha_R - i\alpha_I \end{pmatrix} \ , \tag{5}$$

where $\xi \in [0, \pi)$ is an angle and α_R, α_I, β_R and β_I are four real numbers constrained by

$$\alpha_R^2 + \alpha_I^2 + \beta_R^2 + \beta_I^2 = 1. \tag{6}$$

For $\beta_R\beta_I \neq 0$ the condition (4) can be cast into the familiar $U(1) \times SL(2, \mathbf{R})$ transition matrix form that describes the gap effect at the singular point [2]. For $\beta_R\beta_I = 0$, on the other hand, the condition (4) leads to a set of separate boundary conditions on the right and left sides of the singular point [11]. The $U(2)$ form of the boundary condition (4) provides a *global* description of the entire parameter space Ω in contrast to the conventional $U(1) \times SL(2, \mathbf{R})$ description which is local.

There are several physically distinct subfamilies in the parameter space $\Omega \simeq U(2)$. Here we concentrate on one which is perhaps physically most interesting, that is, the subfamily consisting of parity invariant contact interactions. The *parity* transformation is defined by

$$\mathcal{P} : \varphi(x) \to (\mathcal{P}\varphi)(x) := \varphi(-x), \tag{7}$$

under which the vectors in (2) transform as

$$\Phi \xrightarrow{\mathcal{P}} \sigma_1\Phi \ , \qquad \Phi' \xrightarrow{\mathcal{P}} \sigma_1\Phi' \ , \tag{8}$$

where σ_1 is the Pauli matrix. Thus, parity invariant boundary conditions arise if we have

$$\sigma_1 U \sigma_1 = U \ , \tag{9}$$

which is satisfied for $\alpha_I = 0$ and $\beta_R = 0$ in (5). The solution can be most conveniently parametrized as

$$U = U(\theta_+, \theta_-) = e^{i\theta_+ P_+}\, e^{i\theta_- P_-} \ , \tag{10}$$

where the angle parameters $\theta_\pm \in [0, 2\pi)$, and the projection operators $P_\pm$ are defined, respectively, by

$$\theta_\pm := \xi \pm \arctan\,(\beta_I/\alpha_R), \tag{11}$$

$$P_\pm := \frac{1}{2}(1 \pm \sigma_1). \tag{12}$$

The operators $P_\pm$ fulfill the relations $P_\pm^2 = P_\pm$, $P_\pm P_\mp = 0$ and $P_+ + P_- = 1$. Eq. (10) shows that parity invariant contact interactions form a subfamily Ω_P given by the torus of two $U(1)$ groups,

$$\Omega_P \simeq U(1) \times U(1) \subset \Omega \simeq U(2) \ . \tag{13}$$

We note that the decomposition $\Phi = P_+\Phi + P_-\Phi$ of any vector Φ corresponds to the decomposition of the wave function into its parity-symmetric and parity-antisymmetric parts, $\varphi(x) = \varphi_+(x) + \varphi_-(x)$ with $\varphi_\pm(-x) = \pm\varphi_\pm(x)$. The parity projection splits Eq. (4) into two separate conditions for even and odd parity eigenstates,

$$\varphi_+(0_+)\sin\frac{\theta_+}{2} + \varphi'_+(0_+)L_0\cos\frac{\theta_+}{2} = 0 \ , \tag{14}$$

$$\varphi_-(0_+)\sin\frac{\theta_-}{2} + \varphi'_-(0_+)L_0\cos\frac{\theta_-}{2} = 0 \ .$$

A crucial point here is that all physical properties of even (or odd) parity states are now determined solely by the parameter θ_+ (or θ_-).

We take a closer look at the parity invariant torus $\Omega_P \simeq S^1 \times S^1$. Since the system splits into even and odd sectors, it is possible to define the *coupling strength* of the interactions independently in each sector. Note, however, that there is an inherent asymmetry between the two sectors. Namely, in the even sector, the system becomes free (*i.e.*, both φ and φ' are continuous at $x = 0$) when one has $\varphi'_+(0_+) = 0$, whereas in the odd sector the free system is achieved with $\varphi_-(0_+) = 0$. These occur at $\theta_+ = 0$ in the even sector and $\theta_- = \pi$ in the odd sector. This observation leads us to the following definition of the strength g_+ (g_-) for the even (odd) sector,

$$g_+ := \tan(\theta_+/2) \ , \tag{15}$$

$$g_- := \cot(\theta_-/2) \ .$$

We learn from (15) that there exist contact interactions which act only on even (or odd) parity sector leaving the other sector free, and if we are to express the interactions in terms of potentials, these are nothing but δ and ε-functions. More explicitly, in the θ_+–θ_- plane (Fig. 2), the δ-function interactions show up on the horizontal line $\theta_- = \pi$, and the ε-function interactions arise on the vertical line $\theta_+ = 0$. Their joint $(\theta_+, \theta_-) = (0, \pi)$ is the point where both of the sectors become free, *i.e.*, it represents the free system.

We now define the *half-reflection* transformation $\mathcal{R}$ by

$$\mathcal{R}:\ \varphi(x) \to (\mathcal{R}\varphi)(x) := [\Theta(x) - \Theta(-x)]\varphi(x)\ , \qquad (16)$$

where $\Theta(x)$ is the Heaviside step function. This causes the parity interchange of states φ_+ and φ_-, which is implemented on vectors Φ and Φ' as

$$\Phi \xrightarrow{\mathcal{R}} \sigma_3\Phi\ , \qquad \Phi' \xrightarrow{\mathcal{R}} \sigma_3\Phi'\ . \qquad (17)$$

Like the parity $\mathcal{P}$ transformation, $\mathcal{R}$ induces a map on the parameter space $\Omega \simeq U(2)$. In particular, in the parity invariant subfamily Ω_P we find

$$U(\theta_+,\theta_-) \xrightarrow{\mathcal{R}} \sigma_3\, U(\theta_+,\theta_-)\,\sigma_3 = U(\theta_-,\theta_+)\ , \qquad (18)$$

that is, the half-reflection transformation (16) induces the interchange $\theta_+ \leftrightarrow \theta_-$ yielding the mirror transformation with respect to the diagonal axis $\theta_+ = \theta_-$ in the θ_+–θ_- plane. It follows that the invariant subfamily Ω_{SD} under the half-reflection transformation is given by the set of points on the diagonal line $\theta_+ = \theta_-$, which we call *self-dual*, which forms a $U(1)$ subgroup of Ω_P, namely,

$$\Omega_{SD} \simeq U(1) \subset \Omega_P \simeq U(1)\times U(1)\ . \qquad (19)$$

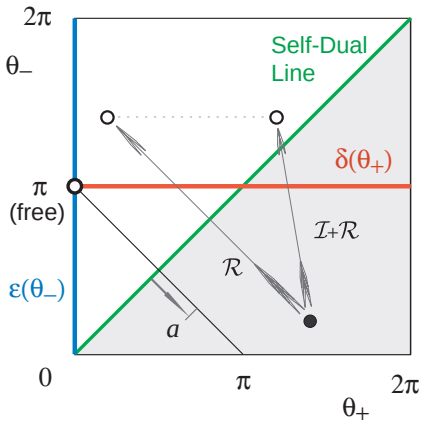

FIG. 2. (color) The dissected torus Ω_P. Various transformations are indicated by the arrows. The horizontal red line and the vertical blue line represent the δ-function interactions and the ε-function interactions, respectively.

If φ is an energy eigenstate, $H\varphi = E\varphi$, then the mapped state $\mathcal{R}\varphi$ is also an eigenstate with the same energy E, but in general, of different connection condition. In mathematical terms, the domain of H specified by U is changed under an isospectral transformation $\mathcal{R}$, and H does not commute with $\mathcal{R}$ in general. The physical consequence of the half-reflection transformation $\mathcal{R}$ is now evident: the spectra of even states and odd states are interchanged leaving the whole spectrum unchanged. The point is that this is a *strong vs weak coupling duality*, because under $\mathcal{R}$ the coupling strengths, g_+ and g_- defined in (15), are interchanged and then inverted,

$$g_+ \xrightarrow{\mathcal{R}} 1/g_-, \qquad g_- \xrightarrow{\mathcal{R}} 1/g_+. \qquad (20)$$

For illustration, take the point $(\theta_+,\theta_-) = (\pi, 0)$, for instance. Under $\mathcal{R}$ this is dual to the free point $(0,\pi)$, and hence the spectrum built on it is that of the free system even though it is not free (since both of the coupling strengths diverge there). The situation should become clearer by inspecting Fig. 3, in which the momentum flow k, which is directly related to the spectral flow by $E = (\hbar^2k^2)/(2m)$, is plotted as a function of the distance from the self-dual line.

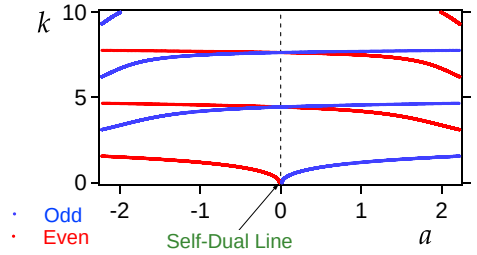

FIG. 3. (color) Momentum flow k along the line in Fig. 2 connecting $(0,\pi)$ and $(\pi,0)$ obtained under the Dirichlet boundary conditions, $\varphi_\pm(L) = \varphi_\pm(-L) = 0$ for some L. Two points with an equal distance from the self-dual line in the opposite direction are dual partners connected by $\mathcal{R}$.

We define yet another transformation by

$$\mathcal{Q}:\ \varphi(x) \to (\mathcal{Q}\varphi)(x) := i[\Theta(-x) - \Theta(x)]\varphi(-x)\ . \qquad (21)$$

This is just the combination $\mathcal{Q} = -i\mathcal{R}\mathcal{P}$ and implements the transformation on vectors,

$$\Phi \xrightarrow{\mathcal{Q}} \sigma_2\Phi\ , \qquad \Phi' \xrightarrow{\mathcal{Q}} \sigma_2\Phi'\ . \qquad (22)$$

One then observes that the transformations, $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$, anti-commute each other, and obey exactly the same relations as those of the Pauli matrices:

$$\mathcal{P}\mathcal{Q} = i\mathcal{R}, \quad \mathcal{Q}\mathcal{R} = i\mathcal{P}, \quad \mathcal{R}\mathcal{P} = i\mathcal{Q}, \qquad (23)$$
$$\mathcal{P}^2 = \mathcal{Q}^2 = \mathcal{R}^2 = 1.$$

The existence of the $su(2)$ algebra formed by $\{\mathcal{P},\mathcal{Q},\mathcal{R}\}$ suggests that, in the preceding analysis (7)-(20), we may exchange the role of $\mathcal{P}$ and $\mathcal{R}$, or replace either of them by $\mathcal{Q}$. For example, in the $\mathcal{R}$-invariant subfamily defined by $\sigma_3 U \sigma_3 = U$, the generator $\mathcal{P}$ (or $\mathcal{Q}$) acts as the operator for isospectral interchange of states.

We now see why the double degeneracy takes place along the self-dual diagonal line $\theta_+ = \theta_-$. On this line the generators do not alter the domain of H, and hence they commute with H. In other words, at self-dual points, all of the three generators $\{\mathcal{P},\mathcal{Q},\mathcal{R}\}$ keep the boundary condition intact, and hence the $SU(2)$ generated by them becomes *symmetry* there. One might also wonder if there arises *supersymmetry* on the line in view of the

fact that the degeneracy occurs with symmetric and antisymmetric states. Here we simply state the fact that, at the middle point (π, π), the system indeed becomes a supersymmetric Witten model [12], while other self-dual points do not admit a similar interpretation because the supercharges fail to be self-adjoint there.

Discrete transformations may also be defined directly as a map on the space Ω_P rather than being induced by transformations on states. For example, we consider

$$U(\theta_+, \theta_-) \xrightarrow{\mathcal{I}_+} e^{i\pi P_+} U(\theta_+, \theta_-) = U(\theta_+ \pm \pi, \theta_-) \ , \quad (24)$$
$$U(\theta_+, \theta_-) \xrightarrow{\mathcal{I}_-} U(\theta_+, \theta_-)\, e^{i\pi P_-} = U(\theta_+, \theta_- \pm \pi) \ ,$$

which are the translations in the θ_+–θ_- plane by a half-cycle (the sign $\pm\pi$ depends on where $\theta_\pm$ lies). These are *coupling inversions* (with minus sign) because their effects on coupling strengths are

$$g_+ \xrightarrow{\mathcal{I}_+} -1/g_+, \qquad g_- \xrightarrow{\mathcal{I}_-} -1/g_-. \quad (25)$$

By combining $\mathcal{R}$ and $\mathcal{I}_\pm$ as $\mathcal{I}_+\mathcal{R} = \mathcal{R}\mathcal{I}_-$ or $\mathcal{I}_-\mathcal{R} = \mathcal{R}\mathcal{I}_+$, one obtains the transformation that brings a δ-function interaction to an ε-function interaction, and vise versa. Note that only even spectra of δ and odd spectra of ε are identical because of the presence of $\mathcal{I}_+$ and $\mathcal{I}_-$ which change the spectra of the other sectors. Thus we learn that the even-odd duality found earlier [8] is a special case of a wider class of duality found in the whole $U(2)$ parameter space.

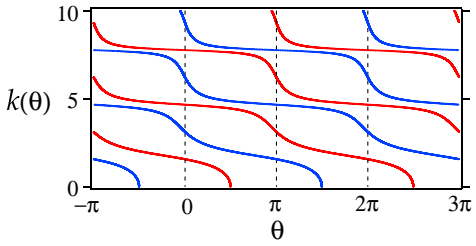

FIG. 4. (color) The origin of the spiral anholonomy — the eigenvalues k as a function of the angle $\theta = \theta_+ = \theta_- + \pi$. (The π shift ensures the curve's passage through the free point $(\theta_+, \theta_-) = (0, \pi)$.) The cycle $\theta \to \theta + 2\pi$ leaves the spectra unchanged as a whole, but each level gets shifted when one follows the spectral flow.

We now examine the origin of the double spiral anholonomy in energy levels found in Ref. [7]. We look at the momentum eigenvalues k as functions of the parameters $\theta_\pm$. Under the Dirichlet boundary condition $\varphi_\pm(L) = \varphi_\pm(-L) = 0$, the eigenvalues arise as solutions of

$$kL_0 \cot kL = \tan(\theta_+/2) \quad \text{(even states)}, \quad (26)$$
$$kL_0 \cot kL = \tan(\theta_-/2) \quad \text{(odd states)}.$$

Since any solution, $k = k(\theta_+)$ or $k = k(\theta_-)$, of (26) is a monotonously decreasing function as depicted in Fig. 4, it is evident that each energy level acquires a spiral anholonomy as one completes a cycle along the spectral curve in the parameter space. The double spiral arises when the cycle is completed simultaneously in the two parameters θ_+ and θ_-, on account of the fact that even and odd eigenstates arise alternately in the spectrum.

Finally, we mention that an immediate extension of our study of quantum contact interactions is in the analysis of junctions of more than two lines [6,13]. In the case of a junction of two lines, or the 'x-junction', the problem can also be regarded as contact interactions on a line for a particle with spin, pointing to the formulation of relativistic contact interactions [14]. In the light of the fact that some of them have a potential application in designing a device for quantum computation [13], their analysis along the line of the present work may be worth pursuing. We end this paper with the restatement of our contention that there is still nontrivial physics left to be discovered in the seemingly innocent settings of low-energy quantum mechanics.

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