Dark Energy Problem and Renormalization Group Flow *

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0. Introduction

$$S = \int d^4x \sqrt{-g}\{\frac{1}{16\pi G_N}(R - 2\lambda) + \mathcal{L}_{matter}\} \tag{1}$$

Dark Energy Problem $\approx$ Cosmological Constant Problem

Biggest Discrepancy between Observation and Theory

$$\frac{\lambda_{th}}{\lambda_{obs}} = 10^{124} \quad \text{So HUGE !!} \tag{2}$$

Causes of Difficulty

- Quantum Gravity (local,micro property)
- Property of the whole universe (global, macro property)

A Recent Trend

AdS/CFT :
Gravity Physics $\rightarrow$ Matter Physics
General Relativity (Space-Time Geom.)$\rightarrow$Fluid Dynamics, Visco-Elastic System
(Instability of Black Holes $\rightarrow$ Destruction of Solid and Liquid ???)
Renormalization Group Flow

Polyakov, 1983
The cosmological constant, just like QED coupling, flows to a small value in the IR region (screening phenomena).

1. 5 Dim Single Scalar Field $\Phi(X)$ on Background $G_{MN}(X)$

$$\mathcal{L}[\Phi; G_{MN}] = \sqrt{-G}\left(\frac{1}{2\kappa}(R-2\lambda) - \frac{1}{2}\nabla_M\Phi\nabla^M\Phi - \frac{m^2}{2}\Phi^2 - V(\Phi)\right)$$

Flat $\quad ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \equiv G_{MN}dX^M dX^N, \ \{X^M\} = \{x^\mu, y\}$

periodic : $\quad y \to y + 2l \quad , \quad$ Z_2 parity : $\quad y \leftrightarrow -y$

AdS$_5$ $\quad ds^2 = \frac{1}{\omega^2 z^2}\{\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\} \equiv G_{MN}dX^M dX^N, \ \{X^M\} = \{x^\mu, z\}$

line segment : $\quad \frac{1}{\omega} \leq |z| \leq \frac{1}{T} \quad , \quad$ Z_2 parity : $\quad z \leftrightarrow -z \quad$ (3)

Figure 1: IR-regularized geometry of 5D flat space (3).

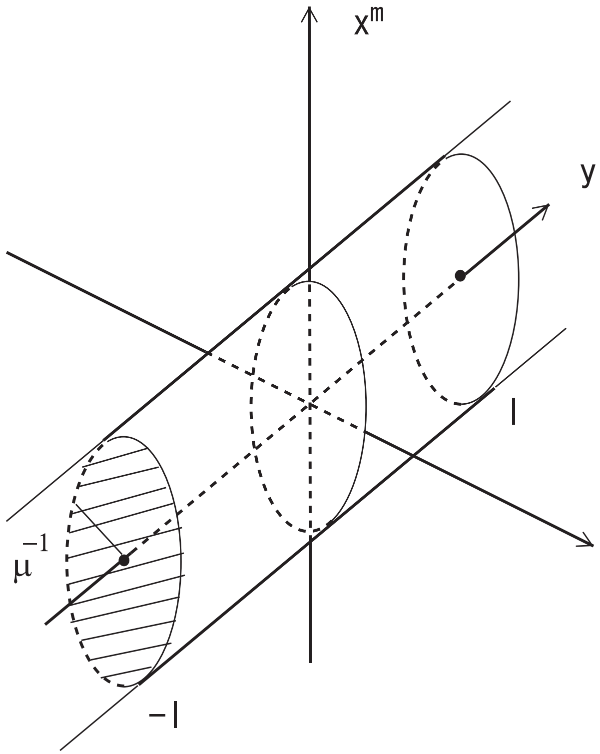
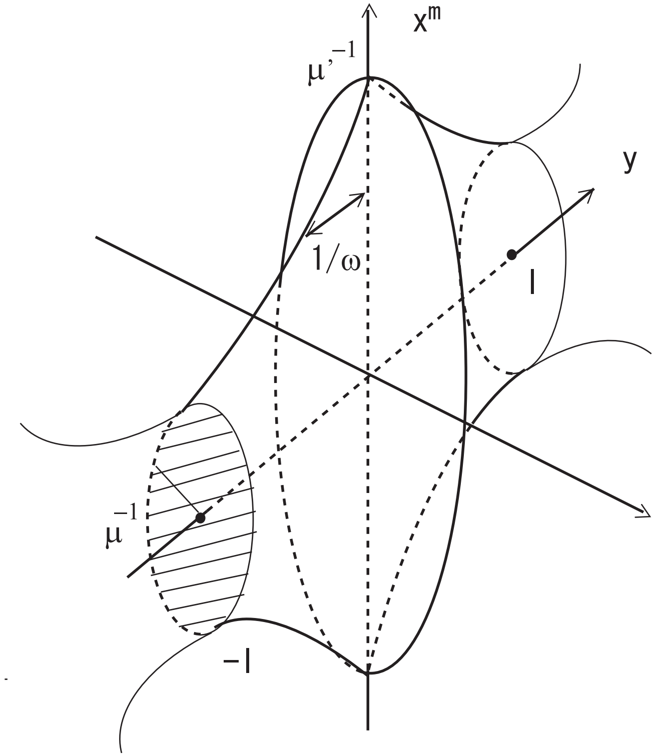


Figure 2: IR-regularized geometry of 5D warped space (3).



Background Expansion: $\Phi = \Phi_{cl} + \varphi$

$$\sqrt{-G}\left\{\nabla^2\Phi_{cl} - m^2\Phi_{cl}\right\} = \sqrt{-G}\frac{\delta V(\Phi_{cl})}{\delta\Phi_{cl}} \qquad (4)$$

NOT expand G_{MN}. Φ_{cl} is perturbatively solved as

$$\Phi_{cl}(X) = \Phi_0(X) + \int D(X - X')\ \sqrt{-G}\frac{\delta V(\Phi_{cl})}{\delta\Phi_{cl}}\bigg|_{X'} d^4X' ,$$

$$\sqrt{-G}(\nabla^2 - m^2)\Phi_0 = 0 \quad , \quad \text{asymptotic state}$$

$$\sqrt{-G}(\nabla^2 - m^2)D(X - X') = \delta^5(X - X') \quad , \quad \text{propagator} \quad , \qquad (5)$$

Effective Action (up to scalar 1-loop, $V(\Phi) = \frac{\sigma}{4!}\Phi^4$)

$$\mathrm{e}^{\Gamma^{eff}[\Phi_{cl};G]} = \exp \int d^5X \ \mathcal{L}[\Phi_{cl};G] \times$$

$$\int \mathcal{D}\varphi \exp \int d^5X \sqrt{-G}\{-\frac{1}{2}\nabla^M\varphi \cdot \nabla_M\varphi - \frac{1}{2}m^2\varphi^2 - \frac{\sigma}{4}\Phi_{cl}^2\varphi^2\} \quad (6)$$

n-point Generating Function (Areveva et al, 1973)

$$\frac{\delta}{\delta\Phi_0(X_{(1)})}\frac{\delta}{\delta\Phi_0(X_{(2)})}\cdots\frac{\delta}{\delta\Phi_0(X_{(n)})}\Gamma^{eff}[\Phi_{cl};G] \quad (7)$$

After integrating out the Gaussian part of (6),

$$\Gamma^{eff}[\Phi_{cl};G] = \int d^5X \left[\mathcal{L}[\Phi_{cl};G] + \log\frac{1}{\sqrt{-G}}\{\nabla^M(\sqrt{-G}\nabla_M) - m^2 - \frac{\sigma}{2}\Phi_{cl}^2\}\right] \quad (8)$$

(Log part gives, for $\sigma = 0$, Caimir energy .) Λ^5 UV-DIVERGENT !!

Proposal (S.I. 2008)

Summation over 5D space $\{X^M\} \rightarrow$ Averaging with a weight

Flat:

$$\tilde{\Gamma}^{eff}[\Phi_{cl};\eta_{MN}] = \int_{r(0)=\rho, r(l)=\rho'} \prod_{m,y} \mathcal{D}x^m(y) \times$$

$$\Gamma^{eff}[\Phi_{cl}(x^m, y);\eta_{MN}] \exp\{-\frac{1}{2\alpha'}Area\} \ , \quad Area = \int_0^l \sqrt{r'^2 + 1}\, r^3 dy \tag{9}$$

Warped:

$$\tilde{\Gamma}^{eff}[\Phi_{cl};G] = \int_{r(1/\omega)=\rho, r(1/T)=\rho'} \prod_{m,z} \mathcal{D}x^m(z) \times$$

$$\Gamma^{eff}[\Phi_{cl}(x^m, z); G(x^m, z)] \exp\{-\frac{1}{2\alpha'} Area\} \ , \quad Area = \int_{1/\omega}^{1/T} \frac{1}{\omega^4 z^4} \sqrt{r'^2 + 1}\ r^3 dz \quad (10)$$

$1/\alpha'$ is a parameter which describes the surface tension. This value gives us the energy scale where the proposed path-integral quantization (of coordinates) effectively works.

As for the background metric $G_{MN}(X)$, it should be the solution of the minimum of $\tilde{\Gamma}^{eff}[\Phi_{cl}; G]$. At the *tree level*, and the zero surface tension limit ($\alpha' \to \infty$), it satisfies

$$R_{MN} - \frac{1}{2} R\ G_{MN} + \lambda\ G_{MN} = \kappa\ T_{MN} \quad ,$$

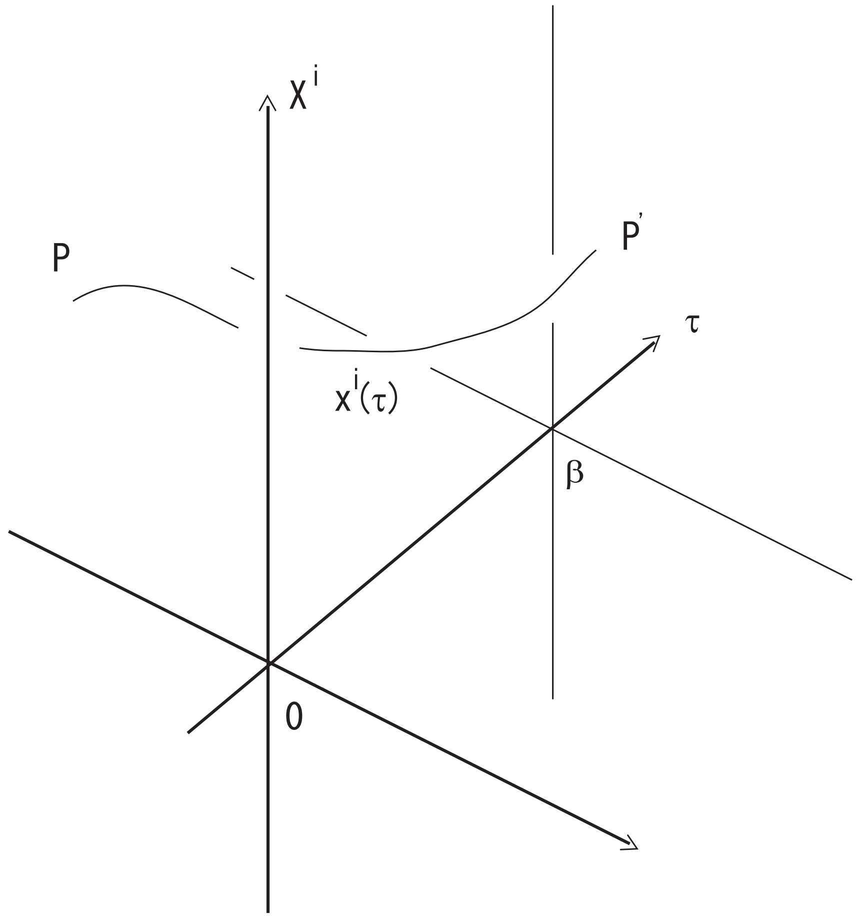
$$T^{MN} = \frac{2}{\sqrt{-G}} \frac{\delta(\sqrt{-G}\mathcal{L}_{mat})}{\delta G_{MN}} \quad ,$$

$$T_{MN} = \partial_M \Phi_{cl} \partial_N \Phi_{cl} - G_{MN} \left(\frac{1}{2} \partial^K \Phi_{cl} \partial_K \Phi_{cl} + \frac{m^2}{2} \Phi_{cl}^2 + V(\Phi_{cl}) \right) \quad . \tag{11}$$

3.Quantum Statistical System of N Harmonic Oscillators and O(N)-Nonlinear Generalization

'Dirac ' Type

Figure 3: A path $\{x^i(\tau)|i = 1, 2, \cdots, N\}$ in N(=2)+1 dim space. It starts at P=$(\rho_1, \rho_2, \cdots, \rho_N, 0)$ and ends at P′=$(\rho'_1, \rho'_2, \cdots, \rho'_N, \beta)$.



N+1 dim Euclidean space $(X^i, \tau), i = 1, 2, \cdots, N$.

$$ds^2 = \sum_{i=1}^{N} (dX^i)^2 + 2V(r) d\tau^2 = G_{AB} dX^A dX^B \quad , \tag{12}$$

$$A, B = 1, 2, \cdots, N, N+1; \quad X^{N+1} \equiv \tau \quad ,$$

$$(G_{AB}) = \text{diag}(1, 1, \cdots, 1, 2V(r)) \quad , \quad r^2 \equiv \sum_{i=1}^{N} (X^i)^2 \quad ,$$

$$V(r) = \frac{\omega^2}{2} r^2 = \frac{\omega^2}{2} \sum_{i=1}^{N} (X^i)^2 \quad \text{N harmonic oscillators;}$$

$$V(r) = \frac{1}{2} \qquad \text{N+1 dim Euclidean flat} \tag{13}$$

Curvatures are, for the elastic system (of N 'particles'): $V = \omega^2 r^2/2$,

$$R_{ij} = \frac{\delta_{ij}}{r^2} - \frac{X^i X^j}{(r^2)^2} \quad , \quad R_{\tau i} = 0 \quad , \quad R_{i\tau} = 0 \quad , \quad R_{\tau\tau} = (N-1)\omega^2 \quad ,$$

$$R = \frac{2(N-1)}{r^2} \quad , \quad \sqrt{G}R = 2(N-1)\frac{\omega}{r} \quad . \tag{14}$$

We require Periodicity : $\tau \to \tau + \beta$

A path $\{X^i = x^i(\tau) |\ 0 \leq \tau \leq \beta,\ i = 1, 2, \cdots, N\}$, Induced metric on this line:

$$X^i = x^i(\tau) \quad , \quad dX^i = \dot{x}^i d\tau \quad , \quad \dot{x}^i \equiv \frac{dx^i}{d\tau} \quad , \quad 0 \leq \tau \leq \beta \quad ,$$

$$ds^2 = \sum_{i=1}^{N} ((\dot{x}^i)^2 + 2V(r))d\tau^2 \quad . \tag{15}$$

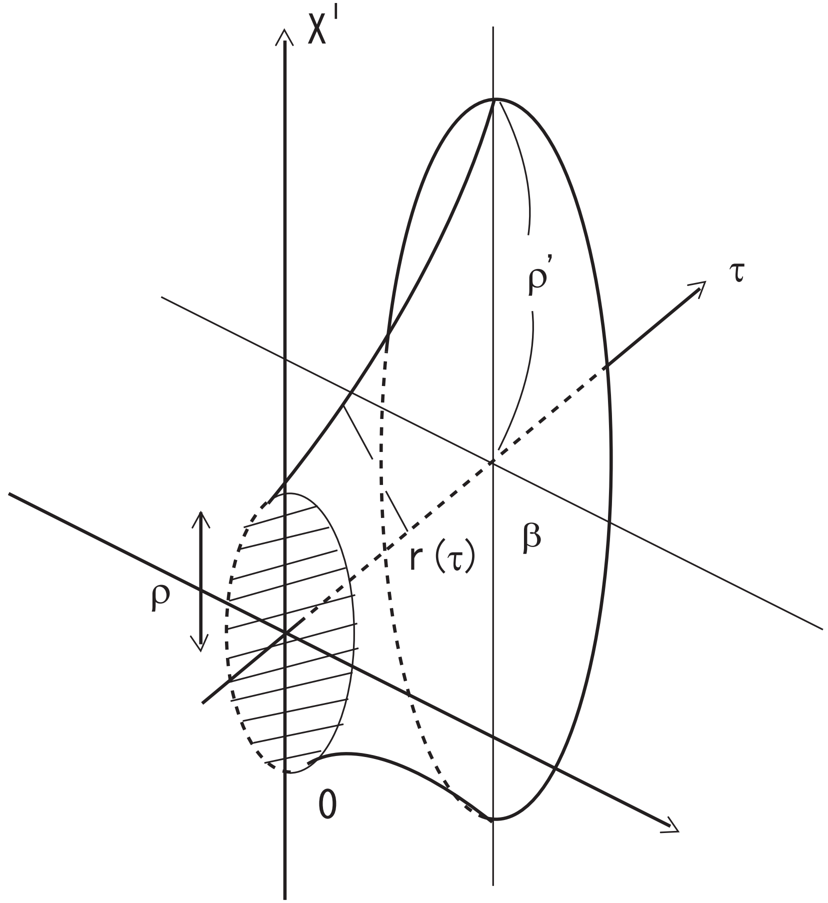
See Fig.3. Length L

$$L = \int ds = \int_0^\beta \sqrt{\sum_{i=1}^{N}((\dot{x}^i)^2 + 2V(r))}\ d\tau \quad . \tag{16}$$

Free energy F: (Hamiltonian $\frac{1}{2}L$, minimal length principle)

$$\mathrm{e}^{-\beta F} = (\prod_i \int_{-\infty}^{\infty} d\rho_i) \int_{\substack{x^i(0)=\rho_i \\ x^i(\beta)=\rho_i}} \prod_{\tau,i} \mathcal{D}x^i(\tau) \exp\left[-\frac{1}{2}\int_0^\beta \sqrt{\sum_{i=1}^{N}((\dot{x}^i)^2 + 2V(r))d\tau}\right]$$

Figure 4: N(=2) dim hypersurface in N+1 dim space $(X^1, X^2, \cdots, X^N, \tau)$. S^{N-1} radius $r(\tau)$ starts by $r(0) = \rho$ and ends by $r(\beta) = \rho'$.



Another geometrical quantity (,instead of the length L). N dim *hypersurface* in N+1 dim space (a closed string). See Fig.4.

$$\sum_{i=1}^{N}(X^i)^2 = r^2(\tau) \quad , \quad \sum_{i=1}^{N} X^i dX^i = r\dot{r}d\tau \quad . \tag{18}$$

$r(\tau)$ describes a path which is *isotropic* in the N dim 'brane' at τ. The induced metric on the N dim hypersurface:

$$ds^2 = \sum_{i,j}(\delta_{ij} + \frac{2V(r)}{r^2\dot{r}^2}X^iX^j)dX^idX^j \equiv \sum_{i,j} g_{ij}dX^idX^j \quad ,$$

$$g_{ij} = \delta_{ij} + \frac{2V(r)}{r^2\dot{r}^2}X^iX^j \quad , \quad r^2 = \sum_{i=1}^{N}(x^i)^2 \quad , \quad \det(g_{ij}) = 1 + \frac{2V(r)}{\dot{r}^2} \quad . \tag{19}$$

$- - - - - - >$ Metric of the O(N) nonlinear sigma model. Area of this N dim hypersurface:

$$A_N = \int \sqrt{\det g_{ij}} d^N X = \frac{N\pi^{N/2}}{\Gamma(\frac{N}{2}+1)} \int_0^\beta \sqrt{\dot{r}^2 + 2V(r)} r^{N-1} d\tau \quad . \tag{20}$$

Free energy F: (Hamiltonian $\frac{1}{2}A_N$, minimal area principle)

$$e^{-\beta F} = \int_0^\infty d\rho \int_{\substack{r(0)=\rho \\ r(\beta)=\rho}} \prod_{\tau,i} \mathcal{D}x^i(\tau) \exp\left[-\frac{1}{2}\frac{N\pi^{N/2}}{\Gamma(\frac{N}{2}+1)} \int_0^\beta \sqrt{\dot{r}^2 + 2V(r)} r^{N-1} d\tau\right] \quad .$$

If we take $2V(r) = 1$,

$$ds^2 = \sum_{i=1}^{N} (dX^i)^2 + d\tau^2 \quad \text{(N+1 dim Euclidean flat)} \quad , \tag{22}$$

, the integration measure (of N=4 case) becomes exactly the proposed one below.

Proposed Casimir Energy of 5D Systems: Flat Geometry

$$-\mathcal{E}_{Cas}(l, \Lambda) = \int_{1/\Lambda}^{l} d\rho \int_{r(0)=r(l)=\rho} \prod_{a,y} \mathcal{D}x^a(y)$$
$$\times \{ \int_0^l dy' F_1(\frac{1}{r(y')}, y') \} \ \exp\left[\frac{-1}{2\alpha'} \int_0^l \sqrt{r'^2 + 1}\, r^3 dy \right] \tag{23}$$

where $r = \sqrt{\sum_{a=1}^{4}(x^a)^2}$ and α' is the string tension parameter. l is the compactification radius parameter.

Standard Type (go back, from hypersurface-path, to line-path)

Another type of N+1 dim Euclidean space (X^i, τ); $i = 1, 2, \cdots N$:

$$ds^2 = d\tau^{-2}\{\sum_{i=1}^{N}(dX^i)^2\}^2 + 4V(r)^2 d\tau^2 + 4V(r)\{\sum_{j=1}^{N}(dX^j)^2\}$$

$$= \frac{1}{d\tau^2}\{\sum_{i=1}^{N}(dX^i)^2 + 2V(r)d\tau^2\}^2 \quad , \tag{24}$$

where we assume

$$d\tau^2 \sim O(\epsilon^2) \quad , \quad (dX^i)^2 \sim O(\epsilon^2) \quad , \quad \frac{1}{d\tau^2}\{\sum_{i=1}^{N}(dX^i)^2\} \sim O(1) \quad , \qquad (25)$$

In this case, we do not have N+1 dim (bulk) metric ('primordial' geometry). Imposing again periodicity: $\tau \to \tau + \beta$, and on the path $\{x^i(\tau)\}$,

$$\text{Length:} \quad L = \int ds = \int_0^{\beta} \{\sum_{i=1}^{N}((\dot{x}^i)^2 + 2V(r))\}\ d\tau \quad . \qquad (26)$$

$- - - - - >$ N harmonic oscillators

Middle Type

Instead of (24), a slightly modified metric.

$$ds^2 = 4V(r)^2 d\tau^2 + 4\kappa V(r)\{\sum_{j=1}^{N}(dX^j)^2\}$$
$$= 4V(r)\{V(r)d\tau^2 + \kappa\sum_{j=1}^{N}(dX^j)^2\} \quad (27)$$

where we need not the condition (25). We have the bulk metric G_{AB} in this case (ordinary geometry). Explicitly for $N = 2$, $V(r) = \frac{\omega^2}{2}r^2 = \frac{\omega^2}{2}(x^2 + y^2)$,

$$ds^2 = \omega^4(x^2 + y^2)^2 d\tau^2 + 2\omega^2\kappa(x^2 + y^2)(dx^2 + dy^2) \quad ,$$

$$(R_{AB}) = \frac{1}{(r^2)^2} \begin{pmatrix} 4y^2 & -4xy & 0 \\ -4yx & 4x^2 & 0 \\ 0 & 0 & \frac{2\omega^2}{\kappa}(r^2)^2 \end{pmatrix}, R = -\frac{4}{\kappa\omega^2(r^2)^2} \quad , \quad r^2 = x^2 + y^2,$$

$$\sqrt{G} = 2\kappa\omega^4 r^4 \quad , \quad \sqrt{G}R = 8\omega^2 \quad , \quad (28)$$

Take the N dim hypersurface (18) as the path. Then the Induced metric is given as

$$ds^2 = \sum_{i,j=1}^{N} 4V(r)(\kappa\delta_{ij} + \frac{1}{2}\frac{\omega^2}{\dot{r}^2}X^i X^j)dX^i dX^j \equiv \sum_{i,j} g_{ij} dX^i dX^j \quad . \quad (29)$$

Area:

$$A_N = \int \sqrt{\det g_{ij}} d^N X = \frac{(2\pi|\kappa|)^{N/2}}{\Gamma(\frac{N}{2}+1)} \int_0^\beta V(r)^{N/2} \sqrt{\dot{r}^2 + \frac{V(r)}{|\kappa|}}\, r^{N-1} d\tau \quad . \tag{30}$$

Free energy: (Hamiltonian $\frac{1}{2}A_N$ minimal area principle)

$$\mathrm{e}^{-\beta F} = \int_0^\infty d\rho \int_{\substack{r(0)=\rho \\ r(\beta)=\rho}} \prod_{\tau,i} \mathcal{D}x^i(\tau)$$

$$\exp\left[-\frac{1}{2}\frac{(2\pi|\kappa|)^{N/2}}{\Gamma(\frac{N}{2}+1)} \int_0^\beta V(r)^{N/2} \sqrt{\dot{r}^2 + \frac{V(r)}{|\kappa|}}\, r^{N-1} d\tau\right] \quad . \tag{31}$$

Compare this result with the proposed 5D Casimir energy for the *warped* (AdS_5)

geometry. We recognize, if we start with

Modified Type

$$\text{Euclidean } (\text{AdS})_{N+1}: \quad ds^2 = \frac{1}{\tau^2}\{d\tau^2 + \sum_{j=1}^{N}(dX^j)^2\}$$

$$= W(\tau)\{V(r)d\tau^2 + \sum_{j=1}^{N}(dX^j)^2\} \quad , \qquad (32)$$

instead of (27), the integration measure exactly becomes the same as the proposed one.

Proposed Casimir Energy of 5D Systems: Warped

(AdS_5) Geometry

$$-\mathcal{E}_{Cas}(\omega, T, \Lambda) = \int_{1/\Lambda}^{1/\mu} d\rho \int_{r(1/\omega)=r(1/T)=\rho} \prod_{a,z} \mathcal{D}x^a(z)$$

$$\times \{ \int_{1/\omega}^{1/T} dz' F_2(\frac{1}{r(z')}, z') \} \exp\left[-\frac{1}{2\alpha'} \int_{1/\omega}^{1/T} \frac{1}{\omega^4 z^4} \sqrt{r'^2 + 1}\, r^3 dz \right], \qquad (33)$$

RG-flow of the Cosm. Const. and Conclusion

We have numerically evaluated the proposed averaging method using some trial weight functions.

$$E^W_{Cas}/\Lambda T^{-1} = -\alpha\omega^4\,(1-4c\ln(\Lambda/\omega)-4c'\ln(\Lambda/T)) = -\alpha\omega'^4 \quad ,$$
$$\omega' = \omega\sqrt[4]{1-4c\ln(\Lambda/\omega)-4c'\ln(\Lambda/T)} \quad , \qquad (34)$$

where $(-4c,-4c') = \quad (0.11,-0.10)$ for the elliptic averaging; $(0.07,-0.10)$ for the parabolic-1 averaging; $(0.07,-0.07)$ for the parabolic-2 averaging; $(0.09,-0.10)$ for the reciprocal averaging; $(0.06,-0.08)$ for the higher-derivative averaging; .

We identify $-\alpha\omega'^4$ as the cosmo. part: $\int d^4x\sqrt{-g}(1/16\pi G_N)2\lambda$

The renormalization group function for the warp factor ω is given as

$$|c| \ll 1 \quad , \quad |c'| \ll 1 \quad , \quad \omega' = \omega(1 - c\ln(\Lambda/\omega) - c'\ln(\Lambda/T)) \quad ,$$
$$\beta(\beta\text{-function}) \equiv \frac{\partial}{\partial(\ln\Lambda)} \ln\frac{\omega'}{\omega} = -c - c' \quad . \qquad (35)$$

We should notice that, in the flat geometry case, the IR parameter (extra-space size) l is renormalized . In the present warped case, however, the corresponding parameter T is not renormalized, but the warp parameter ω is renormalized. Depending on the sign of $c + c'$, the 5D bulk curvature ω flows as follows. When $c + c' > 0$, the bulk curvature ω decreases (increases) as the the measurement energy scale Λ increases (decreases). When $c + c' < 0$, the flow goes in the opposite way.

$$\frac{1}{G_N}\lambda_{obs} \sim \frac{1}{G_N {R_{cos}}^2} \sim m_\nu^4 \sim (10^{-3}eV)^4 \quad , \tag{36}$$

where R_{cos} is the cosmological size (Hubble length), m_ν is the neutrino mass.

$$\frac{1}{G_N}\lambda_{th} \sim \frac{1}{{G_N}^2} = {M_{pl}}^4 \sim (10^{28}eV)^4 \quad . \tag{37}$$

The famous huge discrepancy factor: $\lambda_{th}/\lambda_{obs} \sim 10^{124}$. If we apply the present approach, we have the warp factor ω, and the result (34) strongly suggests the following choice:

$$\text{INPUT 1} \quad \Lambda = M_{pl} \quad ,$$

INPUT 2(Newton's law exp.) $\omega \sim \frac{1}{\sqrt[4]{G_N {R_{cos}}^2}} = \sqrt{\frac{M_{pl}}{R_{cos}}} \sim m_\nu \sim 10^{-3}$eV

FACT $\quad S \sim \int d^4x\sqrt{-g}\frac{1}{G_N}\lambda_{obs} \sim R^4_{COS}\omega^4$

Result(34)requires $\quad \mathrm{e}^{-S} \leftrightarrow \mathrm{e}^{-E_{Cas}/T^4} = \exp\{-T^{-4}\Lambda T^{-1}\omega^4\}$

$$\Longrightarrow \text{ OUTPUT } \quad T^5 = \frac{M_{pl}}{R^4_{cos}} = (10^{-20}\text{eV})^5 = (10^{-12}R_{cos})^{-5} \quad . \quad (38)$$

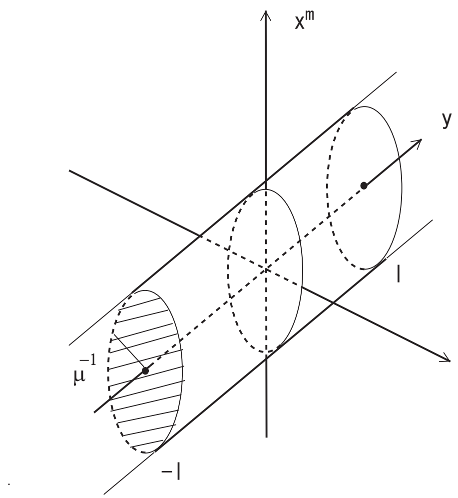
We do not yet succeed in obtaining the right sign, but succeed in obtaining the finiteness and its gross absolute value of the cosmological constant. Now we understand that the smallness of the cosmological constant comes from the renormalization flow for the 'non asymptotic-free' case ($c + c' < 0$ in (35)).

END

0. Introduction

5D Electromagnetism on the *flat* geometry
The extra space is *periodic* (periodicity $2l$) and Z_2-parity

Figure 5: IR-regularized geometry of 5D flat space (39).



$$
ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \quad , \quad -\infty < x^\mu, y < \infty \quad , \quad y \to y + 2l,\ y \leftrightarrow -y \quad ,
$$
$$
(\eta_{\mu\nu}) = \mathsf{diag}(-1,1,1,1)\ ,(X^M) = (X^\mu = x^\mu, X^5 = y) \equiv (x,y)\ ,
$$
$$
M,N = 0,1,2,3,5;\ \ \mu,\nu = 0,1,2,3. \qquad (39)
$$

The Casimir energy

$$
E_{Cas}(\Lambda, l) = \frac{2\pi^2}{(2\pi)^4} \int_{1/l}^{\Lambda} d\tilde{p} \int_{1/\Lambda}^{l} dy\ \tilde{p}^3 W(\tilde{p}, y) F(\tilde{p}, y) \quad ,
$$
$$
F(\tilde{p}, y) \equiv F^-(\tilde{p}, y) + 4F^+(\tilde{p}, y) = \int_{\tilde{p}}^{\Lambda} d\tilde{k} \frac{-3\cosh \tilde{k}(2y-l) - 5\cosh \tilde{k}l}{2\sinh(\tilde{k}l)} \quad . \qquad (40)
$$

Λ the 4D-momentum cutoff; $W(\tilde{p}, y)$ the *weight function*

1) Un-weighted case: $W = 1$

Un-restricted integral region :

$$E_{Cas}(\Lambda, l) = \frac{1}{8\pi^2}\left[-0.1249 l\Lambda^5 - (1.41, 0.706, 0.353) \times 10^{-5}\ l\Lambda^5 \ln(l\Lambda)\right] \quad ,$$

Randall-Schwartz integral region :

$$E^{RS}_{Cas} = \frac{1}{8\pi^2}[-0.0893\ \Lambda^4] \quad . \text{(41)}$$

2) Weighted case

$$E^{W}_{Cas} =$$

$$
\begin{cases}
-2.50\frac{\Lambda}{l^3} + (-0.142, 1.09, 1.13) \times 10^{-4}\frac{\Lambda \ln(l\Lambda)}{l^3} & \text{for} \;\; W_1 = (1/N_1)\mathrm{e}^{-(1/2)l^2\tilde{p}^2 - (1/2)y^2/l^2} \\
-6.03 \times 10^{-2}\frac{\Lambda}{l^3} & \text{for} \;\; W_2 = (1/N_2)\mathrm{e}^{-\tilde{p}y} \\
-2.51\frac{\Lambda}{l^3} + (19.5, 11.6, 6.68) \times 10^{-4}\frac{\Lambda \ln(l\Lambda)}{l^3} & \text{for} \;\; W_8 = (1/N_8)\mathrm{e}^{-(l^2/2)(\tilde{p}^2 + 1/y^2)}
\end{cases}
$$

(W_1: elliptic, W_2: hyperbolic, W_8: reciprocal).

The renormalization of the compactification size l.

$$
E^W_{Cas}/\Lambda l = -\frac{\alpha}{l^4}(1 - 4c\ln(l\Lambda)) = -\frac{\alpha}{l'^4} \quad , \tag{43}
$$

The quantity Λl is the normalization factor.

Casimir Energy of 4D Electromagnetism

Figure 6: Graph of Planck's radiation formula. $\mathcal{P}(\beta, k) = \frac{1}{(c\hbar)^3}\frac{1}{\pi^2}k^3/(\mathrm{e}^{\beta k} - 1) \quad (1 \leq \beta \leq 2,\ 0.01 \leq k \leq 10)$.

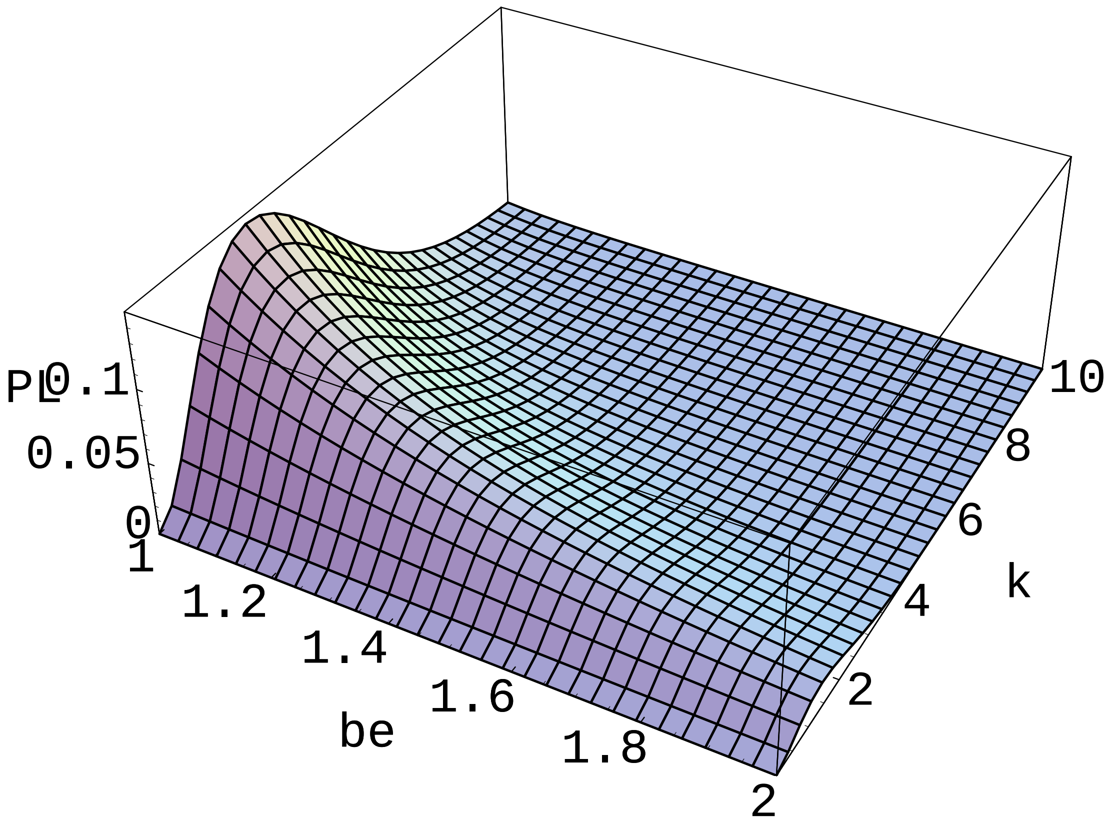
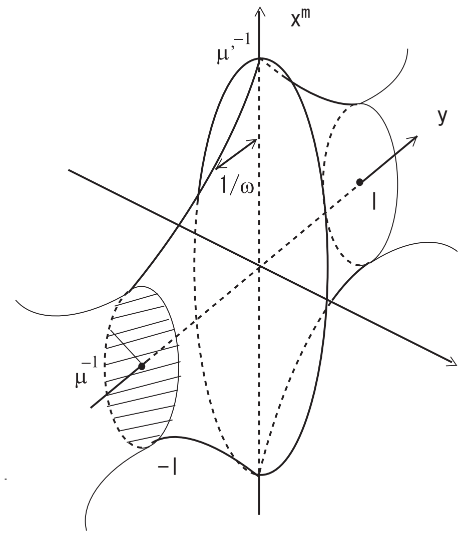


Figure 7: IR-regularized geometry of 5D warped space (44).



$$ds^2 = \frac{1}{\omega^2 z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2) = \mathrm{e}^{-2\omega|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \ , \ |z| = \frac{1}{\omega}\mathrm{e}^{\omega|y|} \quad . \tag{44}$$

Heat-Kernel Approach and Position/Momentum Propagator

$$G_p^{\mp}(z,z') = \mp\frac{\omega^3}{2}z^2{z'}^2\frac{\{\mathbf{I}_0(\frac{\tilde{p}}{\omega})\mathbf{K}_0(\tilde{p}z)\mp\mathbf{K}_0(\frac{\tilde{p}}{\omega})\mathbf{I}_0(\tilde{p}z)\}\{\mathbf{I}_0(\frac{\tilde{p}}{T})\mathbf{K}_0(\tilde{p}z')\mp\mathbf{K}_0(\frac{\tilde{p}}{T})\mathbf{I}_0(\tilde{p}z')\}}{\mathbf{I}_0(\frac{\tilde{p}}{T})\mathbf{K}_0(\frac{\tilde{p}}{\omega})-\mathbf{K}_0(\frac{\tilde{p}}{T})\mathbf{I}_0(\frac{\tilde{p}}{\omega})} \quad ,$$

$$\tilde{p}\equiv\sqrt{p^2} \quad , \quad p^2\geq 0 \text{ (space-like)} \quad .$$

Λ-regularized Casimir energy.

$$E_{Cas}^{\Lambda,\mp}(\omega,T) = \int \frac{d^4p}{(2\pi)^4}\bigg|_{\tilde{p}\leq\Lambda}\int_{1/\omega}^{1/T} dz\ F^{\mp}(\tilde{p},z) \quad ,$$

$$F^{\mp}(\tilde{p}, z) = \frac{2}{(\omega z)^3} \int_{\tilde{p}}^{\Lambda} \tilde{k}\ G_k^{\mp}(z, z) d\tilde{k} \equiv \int_{\tilde{p}}^{\Lambda} \mathcal{F}^{\mp}(\tilde{k}, z) d\tilde{k} \quad , \tag{46}$$

Figure 8: Space of (z,$\tilde{p}$) for the integration.

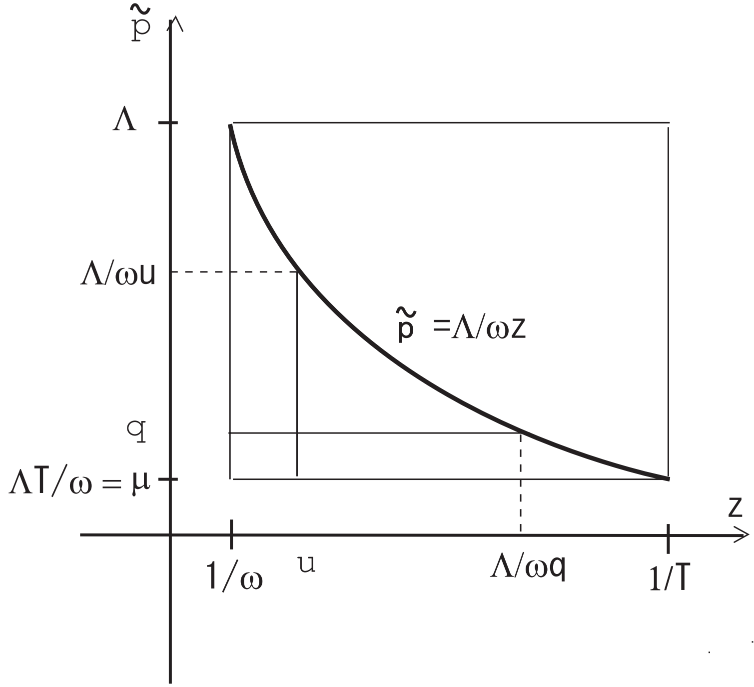
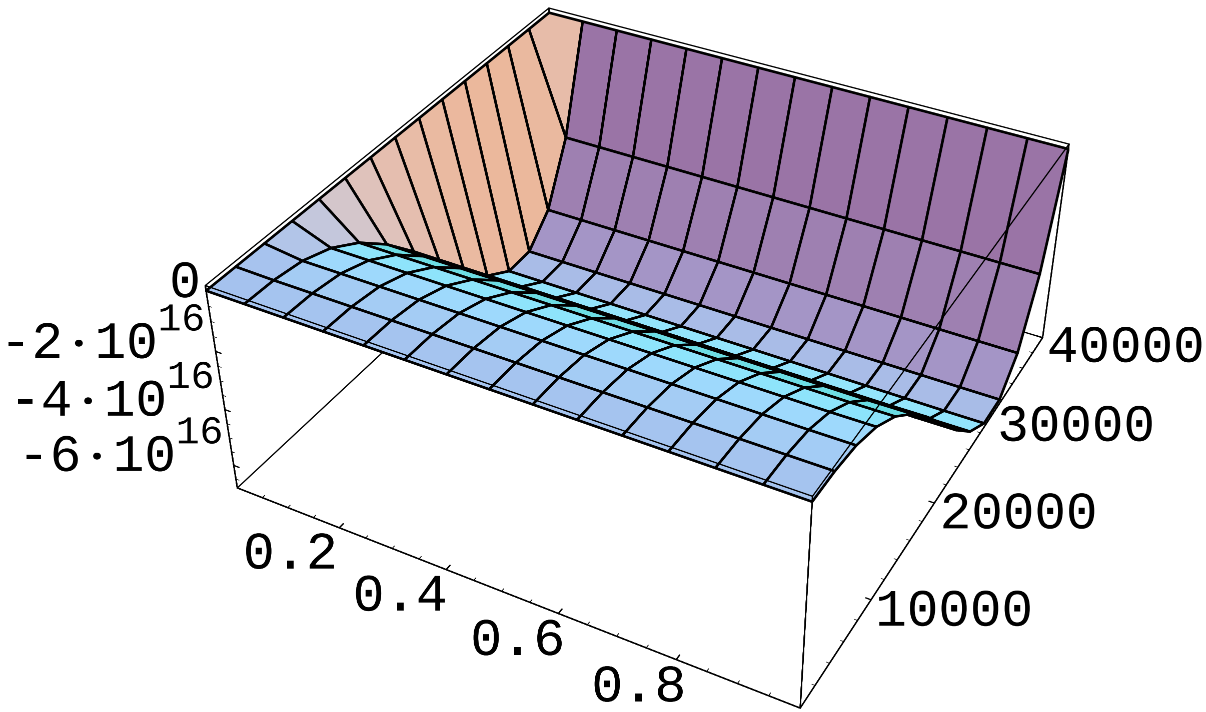


Figure 9: Behaviour of $(-1/2)\tilde{p}^3 F^-(\tilde{p}, z)$ (**??**). $T = 1, \omega = 10^4, \Lambda = 4 \cdot 10^4$. $1.0001/\omega \leq z < 0.9999/T$, $\Lambda T/\omega \leq \tilde{p} \leq \Lambda$.



Weight Function and Casimir Energy Evaluation

$$E^{\mp\ W}_{Cas}(\omega,T) \equiv \int \frac{d^4p}{(2\pi)^4} \int_{1/\omega}^{1/T} dz\ W(\tilde{p},z) F^{\mp}(\tilde{p},z)$$

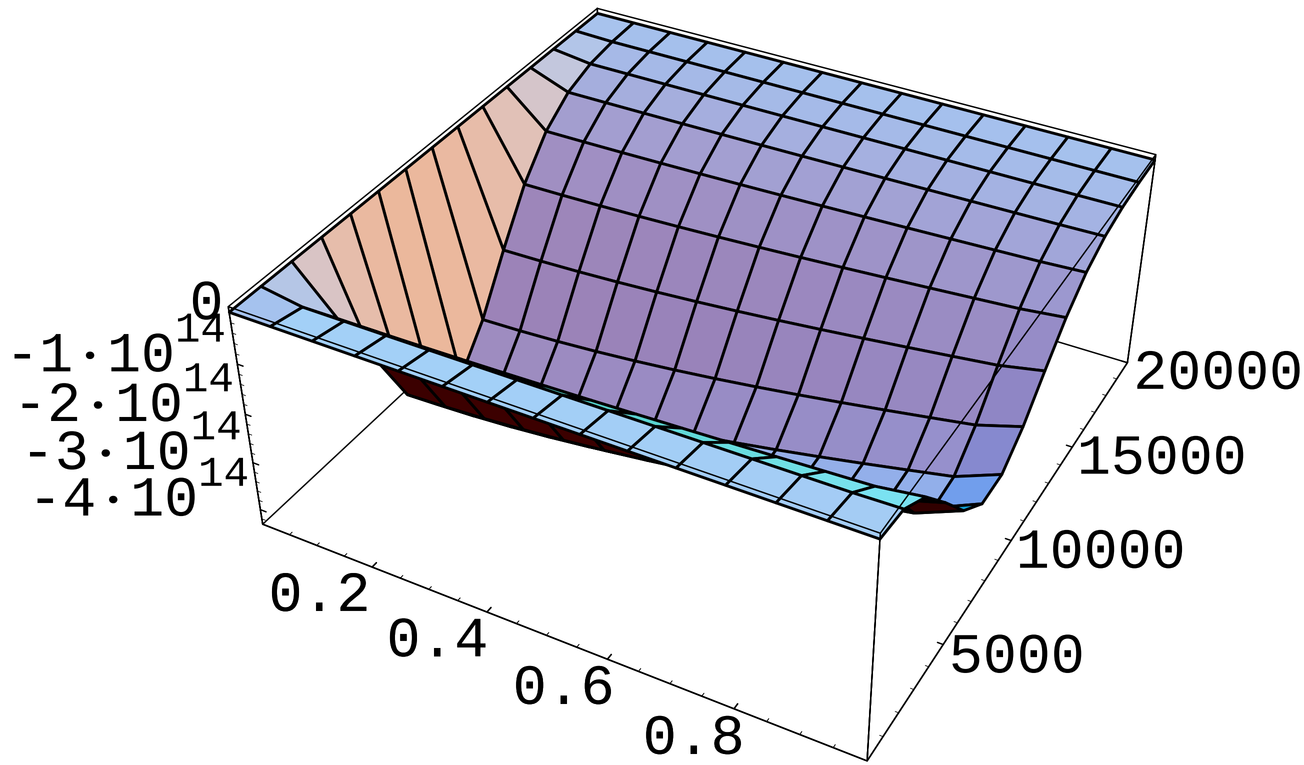
$$F^{\mp}(\tilde{p},z) = s(z)\int_{p^2}^{\infty} \{G_k^{\mp}(z,z)\} dk^2 = \frac{2}{(\omega z)^3}\int_{\tilde{p}}^{\infty} \tilde{k}\ G_k^{\mp}(z,z) d\tilde{k}$$

Examples of $W(\tilde{p},z)$: $\quad W(\tilde{p},z) =$

$$\left\{ \begin{array}{ll} (N_1)^{-1} e^{-(1/2)\tilde{p}^2/\omega^2-(1/2)z^2T^2} \equiv W_1(\tilde{p},z),\ N_1 = 1.711/8\pi^2 & \text{elliptic suppression} \\ (N_2)^{-1} e^{-\tilde{p}zT/\omega} \equiv W_2(\tilde{p},z),\ N_2 = 2\frac{\omega^3}{T^3}/8\pi^2 & \text{hyperbolic suppression1} \\ (N_8)^{-1} e^{-1/2(\tilde{p}^2/\omega^2+1/z^2T^2)} \equiv W_8(\tilde{p},z),\ N_8 = 0.4177/8\pi^2 & \text{reciprocal suppression1} \end{array} \right.$$

where $G_k^{\mp}(z,z)$ are defined in (45). N_i are normalization constants. We show the shape of the energy integrand $(-1/2)\tilde{p}^3 W_1(\tilde{p},z)F^-(\tilde{p},z)$ in Fig.10.

Figure 10: Behavior of $(-1/2)\tilde{p}^3 W_1(\tilde{p},z)F^-(\tilde{p},z)$(elliptic suppression). $\Lambda = 20000,\ \omega = 5000,\ T = 1$. $1.0001/\omega \le z \le 0.9999/T,\ \mu = \Lambda T/\omega \le \tilde{p} \le \Lambda$.



We can check the divergence (scaling) behavior of $E^{\mp\ W}_{Cas}$ by $numerically$ evaluating the $(\tilde{p}, z)$-integral (47) for the rectangle region of Fig.8.

$$-E^W_{Cas} =$$

$$\begin{cases} \frac{\omega^4}{T}\Lambda \cdot 1.2\left\{1 + 0.11\ \ln\frac{\Lambda}{\omega} - 0.10\ \ln\frac{\Lambda}{T}\right\} & \text{for} \quad W_1 \\ \frac{T^2}{\omega^2}\Lambda^4 \cdot 0.062\left\{1 + 0.03\ \ln\frac{\Lambda}{\omega} - 0.08\ \ln\frac{\Lambda}{T}\right\} & \text{for} \quad W_2 \\ \frac{\omega^4}{T}\Lambda \cdot 1.6\left\{1 + 0.09\ \ln\frac{\Lambda}{\omega} - 0.10\ \ln\frac{\Lambda}{T}\right\} & \text{for} \quad W_8 \end{cases} \tag{48}$$

They give, after normalizing the factor Λ/T, only the log-divergence.

$$E^W_{Cas}/\Lambda T^{-1} = -\alpha\omega^4\left(1 - 4c\ln(\Lambda/\omega) - 4c'\ln(\Lambda/T)\right) \quad , \tag{49}$$

This means the 5D Casimir energy is $finitely$ obtained by the ordinary renormalization of the warp factor ω. In the above result of the warped case, the IR parameter l in the flat result (43) is replaced by the inverse of the warp factor ω.

Figure 11: UV regularization surface in 5D coordinate space.