UT-Komaba 12/2, 2009

Renormalizable 4D Quantum Gravity and its Cosmological Implications

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References

1. Conformal Field Theory on R x S^3 from Quantized Gravity, Int. J. Mod. Phys. A24 (2009) 3073—3110, arXiv:0811.1647[hep-th].
2. Renormalizable 4D Quantum Gravity as a Perturbed Theory from CFT, arXiv:0907.3969[hep-th].
3. From CFT Spectra to CMB Multipoles in Quantum Gravity Cosmology, arXiv:0908.0192[astro-ph] with S. Horata and T. Yukawa.

Introduction

The goal of quantum gravity is to understand beyond the Planck scale phenomena

Historical Background

Problems of quantizing Einstein gravity

- Coupling constant has dimension
- Space-time singularity exists
- Action is <u>not</u> bounded from below

⇨ Not renormalizable

In order to resolve these problems,
four-derivative actions are introduced

Four-derivative quantum gravity

- Coupling constant becomes dimensionless

  ➔power-counting renormalizable

- Action becomes bounded from below,
  but ghost mode appears

$$\frac{1}{m^2p^2+p^4} \to \frac{1}{p^2} - \frac{1}{p^2+m^2}$$

ghost mode

The models are classified by ideas how to tackle unitarity problem

- Lee-Wick-Tomboulis approach
- Horava approach
- CFT approach

- Lee-Wick-Tomboulis approach (1970s) :

Consider resummed propagator
for asymptotically free field theories ($\beta = -\beta_0 g^3$)

$$\frac{1}{m^2p^2 + \beta_0 p^4 \log(p^2/\Lambda^2)} = \frac{1}{p^2[m^2 + \beta_0 p^2 \log(p^2/\Lambda^2)]}$$

→real pole of ghost mode disappears
(This idea is still effective at IR, but <u>not</u> so at UV)

- Horava approach :

Give up Lorentz sym. → make ghosts non-dynamical

- CFT approach (our model) :

Use non-perturbative methods (CFT in UV limit)
Conformal sym. → make ghosts not gauge invariant and remove space-time singularity

Renomalizable 4D Quantum Gravity as a Perturbed Theory from CFT

Renormalizable 4D Quantum Gravity

The Action (Weyl + Euler + Einstein)

weight e^{iI}

$$I = \int d^4x \sqrt{-g} \left\{ \underbrace{-\frac{1}{t^2} C^2_{\mu\nu\lambda\sigma} - bG_4}_{\text{conformally invariant (no R^2)}} + \frac{1}{\hbar} \left(\frac{1}{16\pi G} R - \Lambda + \mathcal{L}_{\mathrm{M}} \right) \right\}$$

Planck constant

"t" is a <u>unique</u> dimensionless gravitational coupling constant indicating asymptotic freedom

At high energies $t \longrightarrow 0$ $C_{\mu\nu\lambda\sigma} \longrightarrow 0$ conformally flat

(Perturbation is defined about this config.)

"b" is not independent coupling, which is expanded by t

At $\hbar \longrightarrow 0$ the Einstein action dominates

Wess-Zumino Integrability Condition

Conformal variation of effective action (=path integral over conf. mode)

$$\delta_\omega \Gamma = \int d^4x \sqrt{-g}\,\omega \left\{ \eta_1 R^2_{\mu\nu\lambda\sigma} + \eta_2 R^2_{\mu\nu} + \eta_3 R^2 + \eta_4 \nabla^2 R + m_1 R + m_2 \right\}$$

conformal anomaly
(⇔ UV divergence)

WZ integrability condition

$$[\delta_{\omega_1}, \delta_{\omega_2}]\Gamma = 8(\eta_1 + \eta_2 + 3\eta_3) \times \int d^4x \sqrt{-g} R \omega_{[1} \nabla^2 \omega_{2]} = 0$$

(Last three terms are trivial)

➔ require Weyl and Euler combinations

$$\begin{cases} C^2_{\mu\nu\lambda\sigma} = R^2_{\mu\nu\lambda\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2 \\ G_4 = R^2_{\mu\nu\lambda\sigma} - 4R^2_{\mu\nu} + R^2 \end{cases}$$

Determine the type of UV divergences (= bare action)

R^2 is forbidden ➔ no R^2 divergences

(renormalizability is non-trivial !)

What Means No R^2 Action

In four-derivative models, R^2 action is commonly introduced as a kinetic term of the conformal mode

Therefore, no R^2 action means that there is no coupling constant for this mode

Non-perturbative treatment of the conformal mode is required

- Kinetic term of the conformal mode is induced from the measure quantum mechanically
- Renormalization factor of this mode is unity: $Z_\phi = 1$

and so on

I will show these properties explicitly using dimensional regularization

The Induced Action

$$
\begin{aligned}
Z &= \int [dg \cdots]_g \exp(iI) \\
&= \int [d\phi dh \cdots]_{\hat{g}} \exp\left(iS(\phi) + iI\right)
\end{aligned}
$$

$$g_{\mu\nu} = e^{2\phi} \bar{g}_{\mu\nu}$$

Practical measure defined on the background

Jacobian to preserve diff. inv.
= WZ action for conformal anomaly

Lowest term of S (=Riegert action) is coupling-independent

$$
\begin{aligned}
S_1(\phi, \bar{g}) &= -\frac{b_1}{(4\pi)^2} \int d^4x \int_0^\phi d\phi \sqrt{-g} E_4 \\
&= -\frac{b_1}{(4\pi)^2} \int d^4x \sqrt{-\bar{g}} \left(2\phi \bar{\Delta}_4 \phi + \bar{E}_4 \phi\right)
\end{aligned}
$$

$$E_4 = G_4 - \frac{2}{3} \nabla^2 R$$

cf. Liouville action

Dynamics of conformal mode is induced from the measure

$$b_1 = (N_X + 11N_D + 62N_A)/360 + 769/180$$

Relationship between 2 and 4 dim.

2DQG		4DQG
R	Euler density	$E_4 = G_4 - \frac{2}{3}\nabla^2 R$ (modified)
$\sqrt{-g}R = \sqrt{-\bar{g}}\left(2\bar{\Delta}_2\phi + \bar{R}\right)$	relation	$\sqrt{-g}E_4 = \sqrt{-\bar{g}}\left(4\bar{\Delta}_4\phi + \bar{E}_4\right)$
$\Delta_2 = -\nabla^2$	Conf. inv. diff. op.	$\Delta_4 = \nabla^4 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\nabla^2 + \frac{1}{3}\nabla^\mu R\nabla_\mu$
$-\frac{b_L}{4\pi}\int d^2x \int_0^\phi d\phi\sqrt{-g}R = -\frac{b_L}{4\pi}\int d^2x\sqrt{-\bar{g}}\left(\phi\bar{\Delta}_2\phi + \bar{R}\phi\right)$	WZ action	$-\frac{b_1}{(4\pi)^2}\int d^4x \int_0^\phi d\phi\sqrt{-g}E_4 = -\frac{b_1}{(4\pi)^2}\int d^4x\sqrt{-\bar{g}}\left(2\phi\bar{\Delta}_4\phi + \bar{E}_4\phi\right)$
Liouville action		Riegert action

Conformal Symmetry as Diffeomorphism Invariance

Background-metric independence

Diffeomorphism Invariance

$$\delta_\xi g_{\mu\nu} = g_{\mu\lambda}\nabla_\nu \xi^\lambda + g_{\nu\lambda}\nabla_\mu \xi^\lambda$$

ξ^μ : gauge parameter

Mode decomposition

traceless

$$g_{\mu\nu} = e^{2\phi}\bar{g}_{\mu\nu} \qquad \bar{g}_{\mu\nu} = (\hat{g}\, e^{th})_{\mu\nu} = \hat{g}_{\mu\lambda}\left(\delta^\lambda{}_\nu + t h^\lambda{}_\nu + \frac{t^2}{2}(h^2)^\lambda{}_\nu + \cdots\right)$$

no coupling const.

coupling const.

$$\begin{cases} \delta_\xi \phi = \xi^\lambda \partial_\lambda \phi + \frac{1}{4}\hat{\nabla}_\lambda \xi^\lambda \\ \delta_\xi h_{\mu\nu} = \frac{1}{t}\left(\hat{\nabla}_\mu \xi_\nu + \hat{\nabla}_\nu \xi_\mu - \frac{1}{2}\hat{g}_{\mu\nu}\hat{\nabla}_\lambda \xi^\lambda\right) + \xi^\lambda \hat{\nabla}_\lambda h_{\mu\nu} \\ \qquad + \frac{1}{2}h_{\mu\lambda}\left(\hat{\nabla}_\nu \xi^\lambda - \hat{\nabla}^\lambda \xi_\nu\right) + \frac{1}{2}h_{\nu\lambda}\left(\hat{\nabla}_\mu \xi^\lambda - \hat{\nabla}^\lambda \xi_\mu\right) + o(t\xi h^2) \end{cases}$$

Conformal mode and traceless tensor mode are completely decoupled

Gauge Symmetry at t = 0 (1)

Introduce the gauge parameter $\kappa^\mu = \xi^\mu / t$
and take the limit $t \to 0$ with leaving κ^μ finite

$$\delta_\kappa h_{\mu\nu} = \hat{\nabla}_\mu \kappa_\nu + \hat{\nabla}_\nu \kappa_\mu - \frac{1}{2} \hat{g}_{\mu\nu} \hat{\nabla}_\lambda \kappa^\lambda$$

$$[\delta_\kappa \phi = \delta_\kappa X = \delta_\kappa A_\mu = 0]$$

Usual gauge symmetry of the Weyl action

➔ gauge-fixed as usual

cf. This is similar to gauge symmetry of vector field

$$\delta_\lambda A_\mu = \hat{\nabla}_\mu \lambda$$

Gauge Symmetry at t = 0 (2)

Take the gauge parameter to be a conformal Killing vector:

$$\xi^\mu = \zeta^\mu \qquad \hat{\nabla}_\mu \zeta_\nu + \hat{\nabla}_\nu \zeta_\mu - \frac{1}{2}\hat{g}_{\mu\nu}\hat{\nabla}_\lambda \zeta^\lambda = 0$$

➔ Lowest term in the traceless-mode transformation vanishes!

Conformal symmetry (on $\hat{g}_{\mu\nu}$)

$$\begin{cases} \delta_\zeta \phi = \zeta^\lambda \hat{\nabla}_\lambda \phi + \frac{1}{4}\hat{\nabla}_\lambda \zeta^\lambda \\ \delta_\zeta h_{\mu\nu} = \zeta^\lambda \hat{\nabla}_\lambda h_{\mu\nu} + \frac{1}{2} h_{\mu\lambda}\left(\hat{\nabla}_\nu \zeta^\lambda - \hat{\nabla}^\lambda \zeta_\nu\right) + \frac{1}{2} h_{\nu\lambda}\left(\hat{\nabla}_\mu \zeta^\lambda - \hat{\nabla}^\lambda \zeta_\mu\right) \end{cases}$$

$$Q_\zeta = \int d\Omega_3 \zeta^\mu : \hat{T}_{\mu 0} :$$ ←15 generators of conformal algebra

$$\delta_\zeta \phi = i[Q_\zeta, \phi] \qquad \delta_\zeta h_{\mu\nu} = i\left[Q_\zeta, h_{\mu\nu}\right] \qquad \hat{T}^{\mu\nu} = \frac{2}{\sqrt{-\hat{g}}}\frac{\delta I_{\mathrm{CFT}}}{\delta \hat{g}_{\mu\nu}}$$

generated from <u>Riegert</u> and <u>Weyl</u> actions quantum mechanically

➔ fixed by physical state conditions (=WdW eq.)

For example, conformally coupled scalar field action satisfies

$$\delta_\zeta X = \zeta^\lambda \hat{\nabla}_\lambda X + \frac{1}{4} X \hat{\nabla}_\lambda \zeta^\lambda$$

$$\begin{aligned}
\delta_\zeta I_X &= -\int d^4x \partial^\mu X \partial_\mu \left(\zeta^\lambda \partial_\lambda X + \frac{1}{4} X \partial_\lambda \zeta^\lambda \right) \\
&= \int d^4x \Big\{ -\frac{1}{4} \left(3\partial_\eta \zeta_0 + \partial_i \zeta^i \right) \partial_\eta X \partial_\eta X + \left(\partial_\eta \zeta_i + \partial_i \zeta_0 \right) \partial_\eta X \partial^i X \\
&\quad + \left[-\partial_i \zeta_j + \frac{1}{4} \delta_{ij} \left(-\partial_\eta \zeta_0 + \partial_k \zeta^k \right) \right] \partial^i X \partial^j X + \frac{1}{8} (\partial_\sigma \partial^\sigma \partial_\lambda \zeta^\lambda) X^2 \Big\} \\
&= 0
\end{aligned}$$

by conformal Killing vectors

on flat background

In the same way, the kinetic terms of the vector-field action, the Weyl action are invariant under the conformal transformations, respectively.

The Perturbation about CFT

This model :

CFT
Riegert + Weyl

+ perturbations
(by single "t")

Conformal symmetry mixes
positive- and negative-metric modes
➔ light on unitarity in strong gravity

See ref.1 in detail

Non-perturbative (conformal mode is treated exactly)

cf. Early 4-derivative models in 1970's

Free
R^2 + Weyl

+ perturbations
(by two couplings)

Gauge symmetry in UV limit does
not mix gravitational modes at all
➔ ghosts appear

perturbative (all modes are treated in perturbation)

graviton picture

[Remark: Lee and Wick's idea corresponds to mixing all modes by interactions.]

Dimensional Regularization and Renormalization

On Dimensional Regularization

Dimensional regularization

- manifestly diffeomorphism invariant
- can compute higher-loop corrections

guarantee

and

$$\delta^{(D)}(0) = \int d^D k = 0$$

conformal anomaly comes from DOF between D and 4 dimensions

cf. DeWitt-Schwinger method (exactly 4 dim. method)

one-loop order

$$\delta^{(4)}(0) = \langle x | e^{-\epsilon D} | x \rangle |_{\epsilon \to 0} \Longrightarrow \text{conformal anomaly}$$

heat kernel

Renormalizable action

Euclidean sign.

coupled with QED

Bare action ← D-dimensional WZ integrability

$$I = \int d^D x \sqrt{g} \left\{ \frac{1}{t^2} C^2_{\mu\nu\lambda\sigma} + b G_D + \frac{1}{4} F^2_{\mu\nu} + \sum_{j=1}^{n_F} i \bar{\psi}_j D\!\!\!\!/\, \psi_j - \frac{M_P^2}{2} R + \Lambda \right\}$$

$$\begin{cases} C^2_{\mu\nu\lambda\sigma} = R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} - \dfrac{4}{D-2} R_{\mu\nu} R^{\mu\nu} + \dfrac{2}{(D-1)(D-2)} R^2 \\ G_D = G_4 + \dfrac{(D-3)^2 (D-4)}{(D-1)^2 (D-2)} R^2 \end{cases}$$

Ambiguity is fixed !

Renormalization factors

$$\boxed{Z_\phi = 1}$$

conformal mode is not renormalized
because this mode has no its own coupling constant

$$A_\mu = Z_3^{1/2} A^r_\mu, \quad \psi_j = Z_2^{1/2} \psi^r_j, \quad h_{\mu\nu} = Z_h^{1/2} h^r_{\mu\nu}$$

$$e = Z_e e_r, \quad t = Z_t t_r \qquad (Z_e = Z_3^{-1/2})$$

Ward-Takahashi identity

Conformal Anomaly (WZ action)

$$Z_\phi = 1 \qquad Z_3 = 1 + \frac{x_1}{D-4} + \frac{x_2}{(D-4)^2} + \cdots$$

Residues are functions of renormalized coupling
➔ beta function

Bare action ➔ vertices and counterterms

$$\begin{aligned}
&\frac{1}{4}\int d^D x \sqrt{g} F_{\mu\nu} F^{\mu\nu} \\
&= \frac{1}{4} Z_3 \int d^D x e^{(D-4)\phi} F^r_{\mu\nu} F^r_{\lambda\sigma} \bar{g}^{\mu\lambda} \bar{g}^{\nu\sigma} \\
&= \frac{1}{4}\int d^D x \Big\{ \Big(1 + \frac{x_1}{D-4} + \frac{x_2}{(D-4)^2} + \cdots\Big) F^r_{\mu\nu} F^r_{\lambda\sigma} \bar{g}^{\mu\lambda} \bar{g}^{\nu\sigma} \\
&\qquad + \Big(D - 4 + x_1 + \frac{x_2}{D-4} + \cdots\Big) \phi F^r_{\mu\nu} F^r_{\lambda\sigma} \bar{g}^{\mu\lambda} \bar{g}^{\nu\sigma} \\
&\qquad + \frac{1}{2}\Big((D-4)^2 + (D-4)x_1 + x_2 + \cdots\Big) \phi^2 F^r_{\mu\nu} F^r_{\lambda\sigma} \bar{g}^{\mu\lambda} \bar{g}^{\nu\sigma} \\
&\qquad + \cdots \Big\}
\end{aligned}$$

← ordinary counterterms for gauge field

← new vertices and new counterterms

⇕

Bare Weyl action

$$\frac{1}{t^2}\int d^D x \sqrt{g} C^2_{\mu\nu\lambda\sigma} = \frac{1}{t^2}\int d^D x \sqrt{\bar{g}} e^{(D-4)\phi} \bar{C}^2_{\mu\nu\lambda\sigma}$$

Wess-Zumino action for conformal anomaly

Laurent expansion of b

$$b = \frac{1}{(4\pi)^2} \sum_{n=1}^{\infty} \frac{b_n}{(D-4)^n}$$

$$b_n = b_n(t_r, e_r)$$

$$b_1(t_r, e_r) = b_1 + b_1'(t_r, e_r)$$

Positive constant

$$b_1 = \frac{11 N_F}{360} + \frac{40}{9}$$

Euler term

$$b \int d^D x \sqrt{g} G_D$$
$$= \frac{1}{(4\pi)^2} \int d^D x \Big\{ \Big(\frac{b_1}{D-4} + \frac{b_2}{(D-4)^2} + \cdots \Big) \bar{G}_4$$
$$+ \Big(b_1 + \frac{b_2}{D-4} + \cdots \Big) \Big(2\phi \bar{\Delta}_4 \phi + \bar{E}_4 \phi + \frac{1}{18} \bar{R}^2 \Big)$$
$$+ \frac{1}{2} \Big((D-4) b_1 + b_2 + \cdots \Big) \Big(2\phi^2 \bar{\Delta}_4 \phi + \bar{E}_4 \phi^2 + \cdots \Big) + \cdots \Big\}$$

← counterterms

← new WZ actions and new counterterms

Kinetic term (Riegert action) is induced

Dynamics of conformal mode is induced

Beta functions

$$\beta_t = -\left(\frac{n_F}{40} + \underline{\frac{10}{3}}\right)\frac{t_r^3}{(4\pi)^2} - \frac{7n_F}{72}\frac{e_r^2 t_r^3}{(4\pi)^4} + o(t_r^5)$$

$$\beta_e = \frac{4n_F}{3}\frac{e_r^3}{(4\pi)^2} + \left(4n_F - \underline{\frac{8}{9}}\frac{n_F^2}{b_1}\right)\frac{e_r^5}{(4\pi)^4} + o(e_r^3 t_r^2)$$

no t^2 correction

Residues b_n

$$b_1(t_r, e_r) = b_1 + b_1'(t_r, e_r)$$

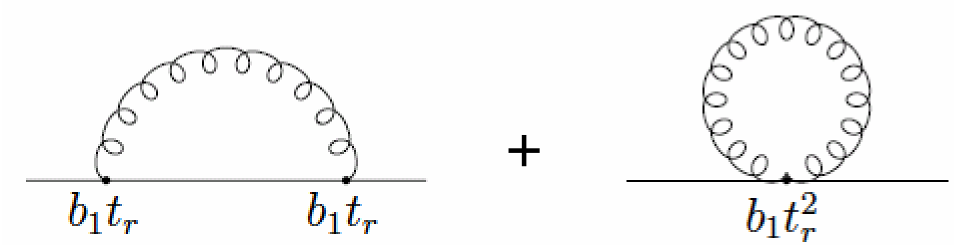
$$b_1 = \frac{11n_F}{360} + \underline{\frac{40}{9}}, \qquad b_1' = -\frac{n_F^2}{6}\frac{e_r^4}{(4\pi)^4} + o(t_r^2),$$

$$b_2 = \frac{2n_F^3}{9}\frac{e_r^6}{(4\pi)^6} + o(t_r^4)$$

Hathrell, Ann.Phys.142(1982)34

(corrections from diagrams with internal gravitational lines)

Non-renormalization of Conformal Mode $(Z_\phi = 1)$



= UV finite

$$\frac{2b_1}{(4\pi)^2}k^4\left[-3\frac{t_r^2}{(4\pi)^2}\left(\frac{1}{\bar{\epsilon}}-\log\frac{z^2}{\mu^2}+\frac{7}{6}\right)\right]$$

$$\frac{2b_1}{(4\pi)^2}k^4\left[3\frac{t_r^2}{(4\pi)^2}\left(\frac{1}{\bar{\epsilon}}-\log\frac{z^2}{\mu^2}+\frac{7}{12}\right)\right]$$

z: infinitesimal fictitious mass (IR regularization)

Not gauge invariant ➔ cancel out !

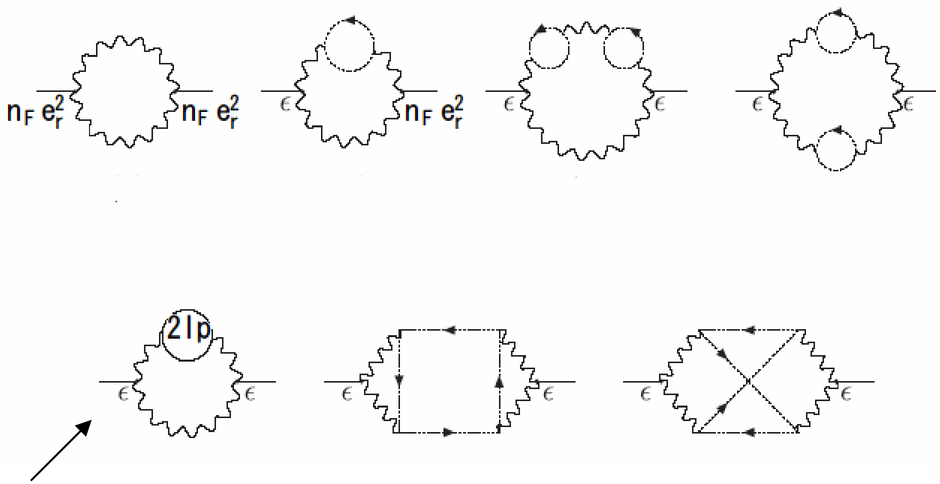
$$D = 4 - 2\epsilon$$

$$1/\bar{\epsilon} = 1/\epsilon - \gamma + \log 4\pi$$

propagator $1/k^4 \to 1/(k^2+z^2)^2$

[Remark : Einstein action cannot be considered as the mass term due to the existence of exponential factor of conformal mode]

Two-point function of e^4

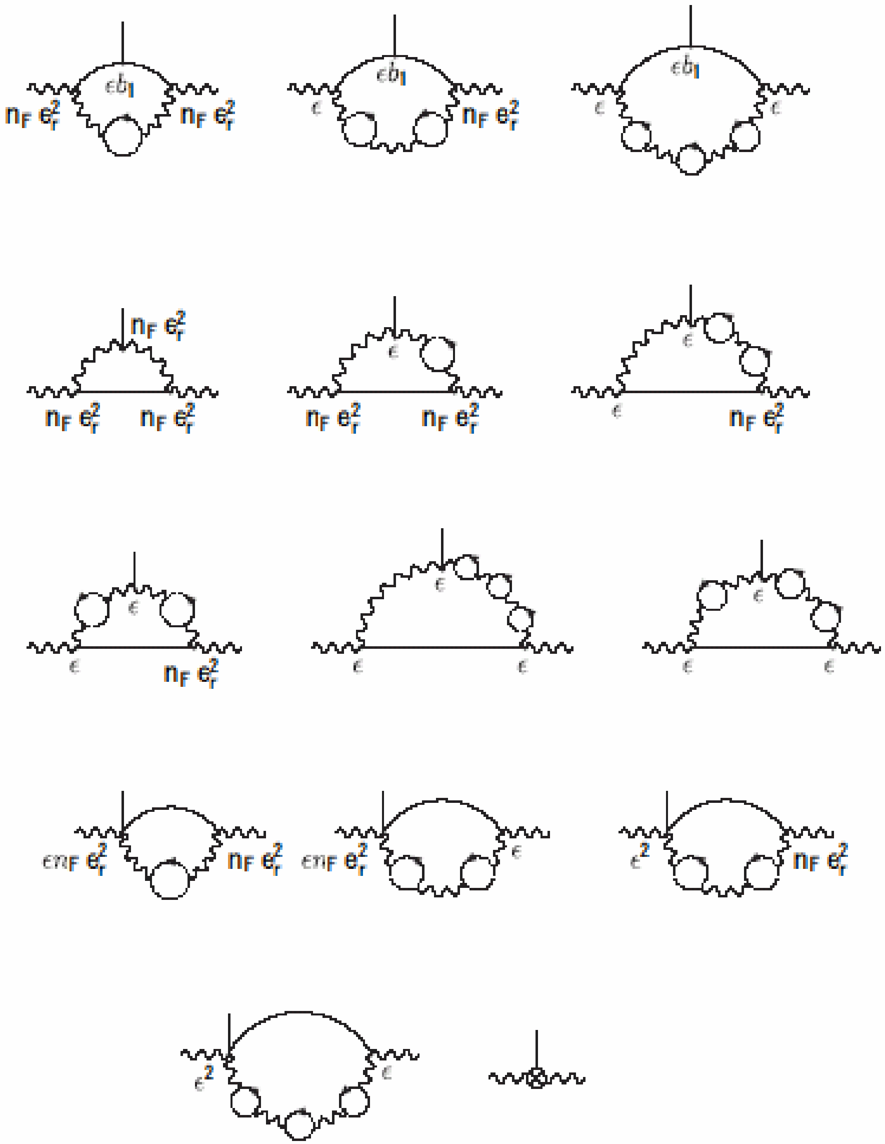


$\epsilon = (4 - D)/2$

Two-point function of e^6
was also checked.

These are renormalized by
the $Z_\phi = 1$ condition

Vertex function ($\phi F_{\mu\nu}^2$) of e^6



AF and Running Coupling Constant

$$\Gamma_{\mathrm{W}} = \left\{ \frac{1}{t_r^2} - 2\beta_0\phi + \beta_0 \log\left(\frac{k^2}{\mu^2}\right) \right\} \bar{C}^{r2}_{\mu\nu\lambda\sigma}$$
$$= \frac{1}{\bar{t}_r^2(p)} \sqrt{g_r} C^{r2}_{\mu\nu\lambda\sigma}.$$

$$\beta_t = -\beta_0 t_r^3$$

k : momentum defined on the flat background

where

$$\bar{t}_r^2(p) = \frac{1}{\beta_0 \log(p^2/\Lambda_{\mathrm{QG}}^2)}$$

Asymptotic Freedom

New dynamical scale: $\Lambda_{\mathrm{QG}} = \mu \exp(-1/2\beta_0 t_r^2)$

Physical momentum: $p^2 = k^2/a^2$ with $a = e^\phi$

(# Conf. anomaly is necessary to preserve diff. inv.)

In UV limit, CFT about $C_{\mu\nu\lambda\sigma} = 0$, and thus no singularity
In IR limit, conformal symmetry is completely broken at Λ_{QG}
➔ turn to Einstein gravity

Summary of Conformal Anomaly

Conformal anomalies are divided into two groups:

1. Coupling-independent part (= Riegert action)
   ⇔ guarantee conf. sym., or CFT at t = 0,
   against the name of 'anomaly'
   ⇔ quantum diff. inv. (= background-metric indep.)

2. Coupling-dependent part (= ordinary conf. anomaly)
   ⇔ beta function
   ⇔ violate conf. sym., as the name

The coupling 't' measures a degree of deviation from CFT

In any case, these conformal anomalies are necessary to preserve quantum diffeomorphism invariance

Reproduce Hathrell's Result of Conf. Anomaly

Hathrell, Ann.Phys.142(1982)34;
Ann.Phys.139(1982)136

Counterterms used by Hathrell

$$1/t^2 \longrightarrow aC^2_{\mu\nu\lambda\sigma} + bG_4 + cH^2 \qquad H = R/(D-1)$$

He carried out 3-loop computations of conformal anomaly in curved space for various matter fields
and found the following relationship between the residues b and c:

$$\boxed{b_2 = 2c_1}$$ (universal independent of matter contents)

D dim. Gravitational action

$$G_D = G_4 + \frac{(D-3)^2(D-4)}{(D-1)^2(D-2)}R^2 \Leftrightarrow c = \frac{(D-3)^2(D-4)}{(D-2)}b$$

$$\Rightarrow \quad c_1 = \frac{(D-3)^2}{D-2}b_2 = \frac{1}{2}b_2 + o(D-4)$$

Quantum Gravity Cosmology

From CFT spectra to CMB multipoles

From CFT to Einstein Theory

$$I = \int d^4x \sqrt{-g} \Big\{ \underbrace{-\frac{1}{t^2} C^2_{\mu\nu\lambda\sigma} - bG_4}_{\text{+ Riegert = CFT}} + \frac{1}{\hbar} \Big(\frac{1}{16\pi G} R - \Lambda + \mathcal{L}_{\mathrm{M}} \Big) \Big\}$$

+ Riegert = CFT ⇨ Einstein theory

Evolution of the universe is described as a function of running coupling

Time evolution ↓

At very high energies beyond the Planck scale

CFT, no space-time singularity $C_{\mu\nu\lambda\sigma} \to 0 \ (t_r \to 0)$

⇩

Running coupling increases according to inflationary expansion

At dynamical energy scale ($\Lambda_{\mathrm{QG}} \simeq 10^{17}\,\mathrm{GeV}$)

Space-time transition (breaking of conformal invariance)

$$\bar{t}_r^2(p) = \frac{1}{\beta_0 \log(p^2/\Lambda_{\mathrm{QG}}^2)}$$

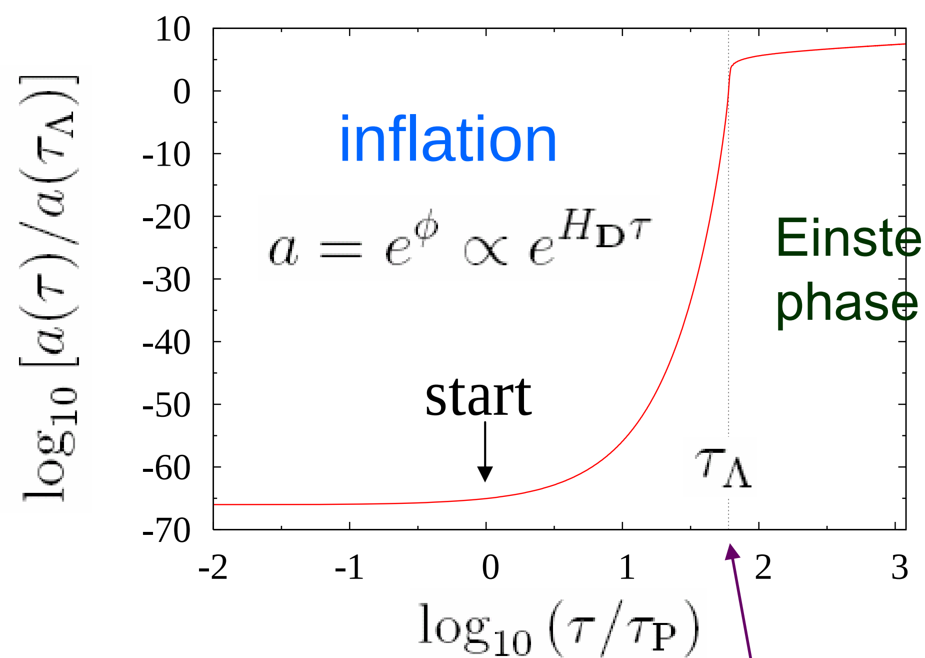
$\Lambda_{\mathrm{QG}} \ (\ll M_{\mathrm{P}})$

Inflation induced by Quantum Gravity

Inflation is driven by Riegert + Einstein system

$$-\frac{b_1}{8\pi^2} B_0(\tau)\partial_\eta^4\phi + 3M_{\mathrm{P}}^2 e^{2\phi}\left(\partial_\eta^2\phi + \partial_\eta\phi\partial_\eta\phi\right) = 0$$

$$\begin{aligned} B_0(\bar{t}_r) &= 1 - a_1\bar{t}_r^2(\tau) + \cdots \\ &= \frac{1}{1 + a_1\bar{t}_r^2(\tau)} \end{aligned}$$



$$d\tau = a\,d\eta$$

RGE : $-\tau\frac{d}{d\tau}\bar{t}_r = \beta_t(\bar{t}_r)$

Inflation starts at Planck time

$$\tau_{\mathrm{P}} = 1/H_{\mathrm{D}} \qquad H_{\mathrm{D}} = M_{\mathrm{P}}\sqrt{\frac{8\pi^2}{b_1}}$$

$(\simeq m_{\mathrm{pl}})$

End at dynamical time

$$\tau_\Lambda = 1/\Lambda_{\mathrm{QG}}$$

Running coupling diverges

K. H, S. Horata, and T. Yukawa,
Phys. Rev. D74 (2006) 123502

Einstein Phase ($E < \Lambda_{\rm QG}$)

Low energy effective theory of gravity
(=derivative expansion about Einstein theory)

$$I_{\rm low} = \int d^4x \sqrt{-g}\,\{\underset{\uparrow\atop\text{tree + 1-loop}}{\mathcal{L}_2} + \underset{\nwarrow\atop\text{tree}}{\mathcal{L}_4} + \cdots\}$$

$$\mathcal{L}_2 = \frac{M_{\rm P}^2}{2} R + \mathcal{L}_2^{\rm M}$$

cf. chiral perturbation theory

Using Einstein Equation $M_{\rm P}^2 R_{\mu\nu} = T_{\mu\nu}^{\rm M}$
higher-derivative terms are reduced to be one

$$\mathcal{L}_4 = \frac{\alpha}{(4\pi)^2} R^{\mu\nu} R_{\mu\nu}$$

1-loop correction : $\alpha(E) = \alpha_0 + \zeta \log(E^2/\Lambda_{\rm QG}^2), \quad (\zeta > 0)$

If one taks phenomenological parameter α_0 to be positive
this term becomes irrelevant at low energies

Evolutional Scenario

$$\xi_\Lambda = 1/\Lambda_{QG} \ (\gg l_{pl})$$

Number of e-foldings

$$\mathcal{N}_e = \log \frac{a(\tau_\Lambda)}{a(\tau_P)} \simeq \frac{m_{pl}}{\Lambda_{QG}}$$

➔ $\boxed{\Lambda_{QG} \simeq 10^{17} \ \mathrm{GeV}}$

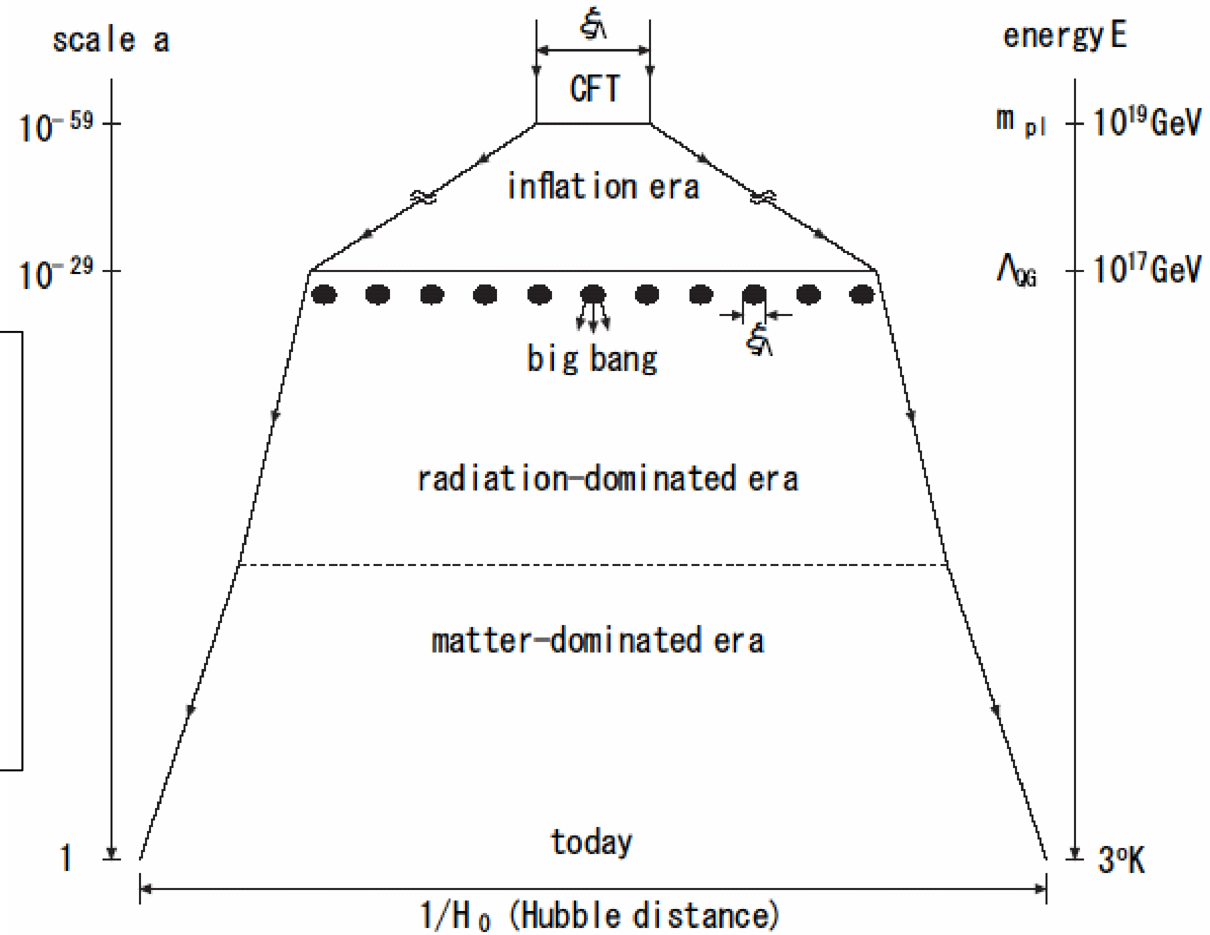
Expansion of the universe 10^{59}

Inflation era

$10^{30} \ (\Leftrightarrow \mathcal{N}_e = 70)$

Friedmann era

$10^{29} \ (\Leftrightarrow 10^{17}\mathrm{GeV}/2.7\mathrm{K})$



$$1/H_0 \simeq 10^{59}\xi_\Lambda$$

Resolve the horizon problem

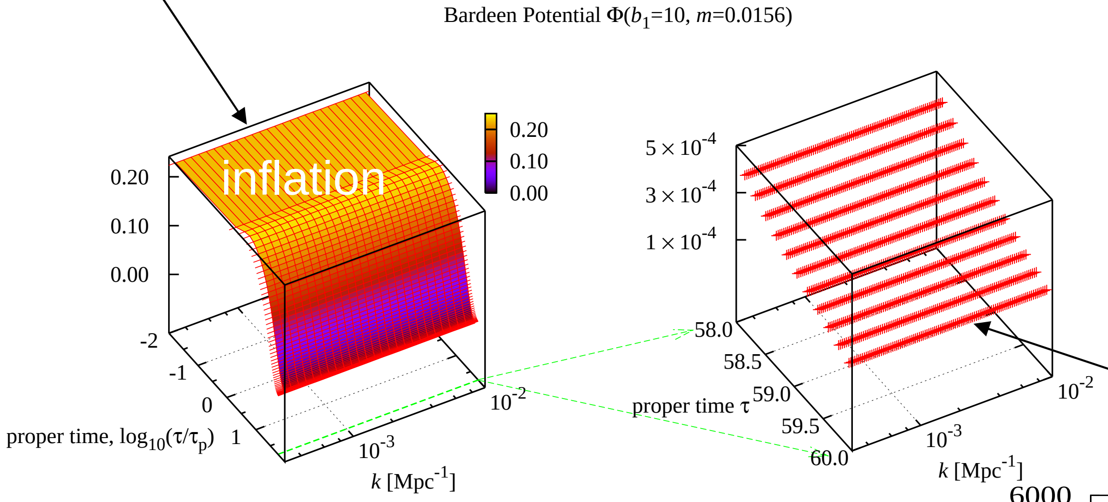
(~4000Mpc)

can be observed through CMB

Evolution of fluctuation (From CFT to CMB)

Planck phenomena (CFT) ➔ space-time transition (big bang) ➔ today

Scale-inv. spectrum at Planck time



From Planck length to cosmological distance

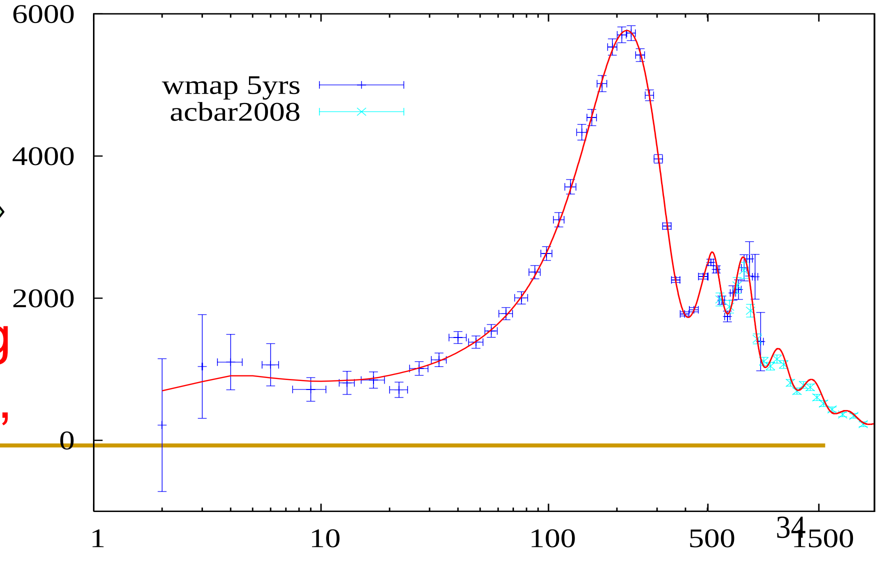
$$10^{59} = 10^{30} + 10^{29}$$

inflation Friedmann

Spectrum at transition point

Amplitude decreases during inflation

➔ Resolve the flatness problem



CMB spectrum is computed using cosmological perturbation theory, which is consistent with WMAP

Conclusion and Discussion

- Renormalizable 4D quantum gravity was formulated as a perturbed theory from CFT, in which it is essential that there is no R^2 action.

- The conformal mode is treated non-perturbatively so that conformal symmetry becomes exact quantum mechanically at the vanishing coupling limit.

- Using dimensional regularization, I computed higher order corrections, and then showed that the conformal mode is not renormalized.

- Quantum gravity scenario of inflation is constructed, in which the conformal mode serves for the scalar field (=inflaton).

- It is proposed that the primordial spectrum of the universe originates from conformal invariance.

The conformal invariance forces us change the aspect of space-time at very high energies above the Planck scale, where a traditional S-matrix description is not adequate at all. Consequently, this requires a new prescription to deal with negative-metric modes in the context of CFT.

on R x S^3

Conformal Algebra = SO(4,2) ($[Q_M, Q_N^\dagger] = 2\delta_{MN} H + 2R_{MN}$ etc.)

$$Q_M^\phi = \left(\sqrt{2b_1} - i\hat{p}\right) a_{\frac{1}{2}M} + \sum_{J\geq 0}\sum_{M_1}\sum_{M_2} \mathbf{C}^{\frac{1}{2}M}_{JM_1, J+\frac{1}{2}M_2} \Big\{ \alpha(J)\epsilon_{M_1} a^\dagger_{J-M_1} a_{J+\frac{1}{2}M_2} + \beta(J)\epsilon_{M_1} b^\dagger_{J-M_1} b_{J+\frac{1}{2}M_2} + \underline{\epsilon_{M_2} a^\dagger_{J+\frac{1}{2}-M_2} b_{JM_1}} \Big\}$$

Special conf. transf.

Negative-metric creation mode mixes with positive-metric creation mode through special conformal transformation

➔ negative-metric mode does not appear independently as a gauge-inv. state

$$\left[Q_M^\phi, b^\dagger_{JM_1}\right] = -\sum_{M_2} \epsilon_{M_2} \mathbf{C}^{\frac{1}{2}M}_{JM_1, J+\frac{1}{2}-M_2} a^\dagger_{J+\frac{1}{2}M_2} - \beta\left(J - \frac{1}{2}\right) \sum_{M_2} \epsilon_{M_2} \mathbf{C}^{\frac{1}{2}M}_{JM_1, J-\frac{1}{2}-M_2} b^\dagger_{J-\frac{1}{2}M_2}$$

See ref. 1 in detail

Since conformal symmetry mixes positive-metric and negative-metric modes of the field, we cannot consider these modes separately and thus the field acts as a whole in physical quantities.

Physical states ⇔ diffeomorphism invariant fields (not each modes)

Ex. scalar curvature

"Real fields"

This suggests that the correctness of the overall sign of the gravitational action (not the sign of each mode) is significant for unitarity.

Naively, two-point function of the "real field" is expected to be positive, because the Riegert and Weyl actions have the correct sign bounded from below and thus the path integral is well-defined ➔ future problem

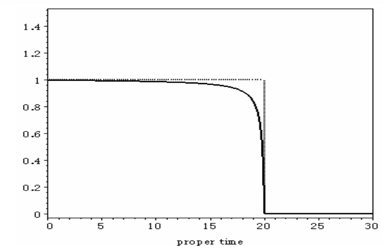
Appendix I
Cosmology

Evolution equation for inflation

$$-\frac{b_1}{8\pi^2}B_0(\tau)\partial_\eta^4\phi + 3M_P^2 e^{2\phi}\left(\partial_\eta^2\phi + \partial_\eta\phi\partial_\eta\phi\right) = 0$$

Dynamical factor

$$B_0(t_r^2) = 1 - a_1 t_r^2 + \cdots = \frac{1}{1 + a_1 t_r^2(\tau)}$$

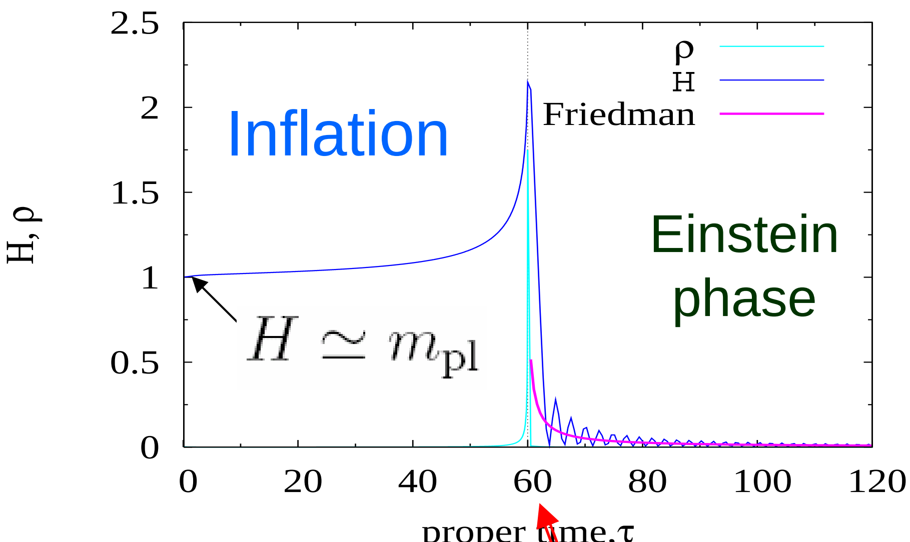


Energy conservation

matter density

$$\frac{b_1}{8\pi^2}B_0(\tau)\left\{2\partial_\eta^3\phi\partial_\eta\phi - \partial_\eta^2\phi\partial_\eta^2\phi\right\} - 3M_P^2 e^{2\phi}\partial_\eta\phi\partial_\eta\phi + e^{4\phi}\rho = 0$$

Energy shift = big bang



Inflation starts at Planck time

$$a = e^{\phi} \propto e^{m_{\rm pl}\tau}$$

$$H = \dot{a}/a$$

End at dynamical time

$$\tau_\Lambda = 1/\Lambda_{QG}$$

Coupling is small

Running coupling diverges

Bardeen's gravitational potentials

$$ds^2 = a^2[-(1+2\Psi)d\eta^2 + (1+2\Phi)d\mathbf{x}^2]$$

Evolution equation for gravitational potentials

$$\begin{aligned}
&\frac{b_1}{8\pi^2}B_0(\tau)\Big\{-2\partial_\eta^4\Phi - 2\partial_\eta\phi\partial_\eta^3\Phi + \left(-8\partial_\eta^2\phi + \frac{10}{3}\not{\partial}^2\right)\partial_\eta^2\Phi \\
&\qquad + \left(-12\partial_\eta^3\phi + \frac{10}{3}\partial_\eta\phi\not{\partial}^2\right)\partial_\eta\Phi + \left(\frac{16}{3}\partial_\eta^2\phi - \frac{4}{3}\not{\partial}^2\right)\not{\partial}^2\Phi \\
&\qquad +2\partial_\eta\phi\partial_\eta^3\Psi + \left(8\partial_\eta^2\phi + \frac{2}{3}\not{\partial}^2\right)\partial_\eta^2\Psi + \left(12\partial_\eta^3\phi - \frac{10}{3}\partial_\eta\phi\not{\partial}^2\right)\partial_\eta\Psi \\
&\qquad + \left(-\frac{16}{3}\partial_\eta^2\phi - \frac{2}{3}\not{\partial}^2\right)\not{\partial}^2\Psi\Big\} \\
&+M_{\mathrm{P}}^2 e^{2\phi}\Big\{6\partial_\eta^2\Phi + 18\partial_\eta\phi\partial_\eta\Phi - 4\not{\partial}^2\Phi - 6\partial_\eta\phi\partial_\eta\Psi \\
&\qquad\qquad + (12\partial_\eta^2\phi + 12\partial_\eta\phi\partial_\eta\phi - 2\not{\partial}^2)\Psi\Big\} = 0.
\end{aligned}$$

Dynamical factor

$$\begin{aligned}
B_0(t_r^2) &= 1 - a_1 t_r^2 + \cdots \\
&= \frac{1}{1 + a_1 t_r^2(\tau)}
\end{aligned}$$

Constraint equation

$$\begin{aligned}
&\frac{b_1}{8\pi^2}B_0(\tau)\Big\{\frac{4}{3}\partial_\eta^2\Phi + 4\partial_\eta\phi\partial_\eta\Phi + \left(\frac{28}{3}\partial_\eta^2\phi - \frac{8}{3}\partial_\eta\phi\partial_\eta\phi - \frac{8}{9}\not{\partial}^2\right)\Phi \\
&\qquad -\frac{4}{3}\partial_\eta\phi\partial_\eta\Psi + \left(-\frac{4}{3}\partial_\eta^2\phi + \frac{8}{3}\partial_\eta\phi\partial_\eta\phi - \frac{4}{9}\not{\partial}^2\right)\Psi\Big\} \\
&+\frac{2}{t_r^2(\tau)}\Big\{4\partial_\eta^2\Phi - \frac{4}{3}\not{\partial}^2\Phi - 4\partial_\eta^2\Psi + \frac{4}{3}\not{\partial}^2\Psi\Big\} \\
&+M_{\mathrm{P}}^2 e^{2\phi}\{-2\Phi - 2\Psi\} = 0.
\end{aligned}$$

$$\Rightarrow \begin{cases} \text{initially } (t_r = 0) & \Phi = \Psi \\ \text{finally } (t_r = \infty) & \Phi = -\Psi \end{cases}$$

Scale Invariant Spectrum

Initial condition = two-point function of conformal mode

$$\langle \varphi(\tau_i, \mathbf{x}) \varphi(\tau_i, \mathbf{x}') \rangle = -\frac{1}{4b_1} \log\left(m^2 |\mathbf{x} - \mathbf{x}'|^2 \right)$$

$\tau_i = 1/E_i$

$(E_i \geq H_{\mathbf{D}})$

$m = a(\tau_i) H_{\mathbf{D}}$

In Fourier space

$$-\log\left(m^2 |\mathbf{x}|^2\right) = \int_{k>\epsilon} \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{4\pi^2}{k^3} e^{i\mathbf{k}\cdot\mathbf{x}} \cancel{- \log\left(\frac{m^2}{\epsilon^2 e^{2\gamma - 2}}\right)}$$

Delta function
In Fourier space

$$P(\tau_i, k) = \frac{k^3}{2\pi^2} \langle |\tilde{\varphi}(\tau_i, \mathbf{k})|^2 \rangle = \frac{1}{2b_1}$$

$b_1 \simeq 10$

for GUT models

Harrison-Zel'dovich-Peebles spectrum

Appendix II
Conformal Algebra

Canonical Quantization on R x S^3

R x S^3 background metric (➔ mode-expansions become simple)

$$d\hat{s}^2_{R\times S^3} = \hat{g}_{\mu\nu}dx^\mu dx^\nu = -d\eta^2 + \hat{\gamma}_{ij}dx^i dx^j$$
$$= -d\eta^2 + \frac{1}{4}(d\alpha^2 + d\beta^2 + d\gamma^2 + 2\cos\beta d\alpha d\gamma)$$

$$\hat{R}_{ijkl} = (\hat{\gamma}_{ik}\hat{\gamma}_{jl} - \hat{\gamma}_{il}\hat{\gamma}_{jk}), \quad \hat{R}_{ij} = 2\hat{\gamma}_{ij}, \quad \hat{R} = 6 \qquad \left[\hat{C}^2_{\mu\nu\lambda\sigma} = \hat{G}_4 = 0\right]$$

$$d\Omega_3 = d^3x\sqrt{\hat{\gamma}} = \frac{1}{8}\sin\beta d\alpha d\beta d\gamma \qquad \mathrm{V}_3 = \int d\Omega_3 = 2\pi^2$$

Isometry of S^3 = SU(2)xSU(2)

Tensor harmonics that belongs to rep. $(J+\varepsilon_n, J-\varepsilon_n)$ with $\varepsilon_n = \pm n/2$

$$\Box_3 Y^{i_1\cdots i_n}_{J(M\varepsilon_n)} = \{-2J(2J+2)+n\}Y^{i_1\cdots i_n}_{J(M\varepsilon_n)}$$

Laplacian on S^3

$M = (m, m')$

$\epsilon_M = (-1)^{m-m'}$

$$Y^{i_1\cdots i_n *}_{J(M\varepsilon_n)} = (-1)^n \epsilon_M Y^{i_1\cdots i_n}_{J(-M\varepsilon_n)},$$
$$\int_{S^3} d\Omega_3 Y^{i_1\cdots i_n *}_{J_1(M_1\varepsilon^1_n)} Y_{i_1\cdots i_n J_2(M_2\varepsilon^2_n)} = \delta_{J_1 J_2}\delta_{M_1 M_2}\delta_{\varepsilon^1_n\varepsilon^2_n}$$

Conformally Coupled Scalar Field

The action on R x S^3

$$I_X = \int d\eta \int_{S^3} d\Omega_3 \frac{1}{2} X \left(-\partial_\eta^2 + \Box_3 - 1 \right) X$$

dispersion relation ➔ $E^2 - (2J+1)^2 = 0$

Mode expansion

$$X = \sum_{J \geq 0} \sum_M \frac{1}{\sqrt{2(2J+1)}} \left\{ \varphi_{JM} e^{-i(2J+1)\eta} Y_{JM} + \varphi^\dagger_{JM} e^{i(2J+1)\eta} Y^*_{JM} \right\}$$

Scalar harmonics $Y_{JM} = \sqrt{\frac{2J+1}{V_3}} D^J_{mm'}, \quad M = (m, m')$

Wigner D function

Quantization

$[X(\eta, \mathbf{x}), P_X(\eta, \mathbf{y})] = i\delta_3(\mathbf{x} - \mathbf{y})$ ➔ $[\varphi_{J_1 M_1}, \varphi^\dagger_{J_2 M_2}] = \delta_{J_1 J_2} \delta_{M_1 M_2}$

Conformal Algebra on R x S^3

The generator of conformal algebra

$$\hat{T}^{\mu\nu} = \frac{2}{\sqrt{-\hat{g}}} \frac{\delta I_{\rm CFT}}{\delta \hat{g}_{\mu\nu}}$$

$$Q_\zeta = \int_{S^3} d\Omega_3 \zeta^\mu : \hat{T}_{\mu 0} : \qquad \delta_\zeta f = i[Q_\zeta, f] \qquad f = \phi,\ h_{\mu\nu},\ X, \cdots$$

15 conformal Killing vectors on R x S^3

Time translation: $\zeta_{\rm T}^\mu = (1, 0, 0, 0)$

Rotation on S^3: $\zeta_{\rm R}^\mu = (0, \zeta_{\rm R}^i)$ $\quad (\zeta_{\rm R}^i)_{MN} = i\frac{\mathrm{V}_3}{4}\left\{ Y^*_{\frac{1}{2}M} \hat{\nabla}^i Y_{\frac{1}{2}N} - Y_{\frac{1}{2}N} \hat{\nabla}^i Y^*_{\frac{1}{2}M} \right\}$

Special conformal: $\zeta_{\rm S}^\mu = (\zeta_{\rm S}^0, \zeta_{\rm S}^i)$

$$(\zeta_{\rm S}^0)_M = \frac{1}{2}\sqrt{\mathrm{V}_3}\, e^{i\eta} Y^*_{\frac{1}{2}M}, \quad (\zeta_{\rm S}^i)_M = -\frac{i}{2}\sqrt{\mathrm{V}_3}\, e^{i\eta} \hat{\nabla}^i Y^*_{\frac{1}{2}M}$$

15 generators

H	Hamiltonian	1
R_{MN}	S^3 rotation	6
$Q_M \quad Q^\dagger_M$	Special conf. + dilatation transf. [=4 vectors of SO(4)]	4+4=8

Conformal algebra on R x S^3

$$
\begin{aligned}
\left[Q_M, Q^\dagger_N\right] &= 2\delta_{MN} H + 2R_{MN}, \\
[H, Q_M] &= -Q_M, \\
[H, R_{MN}] &= [Q_M, Q_N] = 0, \\
[Q_M, R_{M_1M_2}] &= \delta_{MM_2} Q_{M_1} - \epsilon_{M_1}\epsilon_{M_2}\delta_{M-M_1} Q_{-M_2}, \\
[R_{M_1M_2}, R_{M_3M_4}] &= \delta_{M_1M_4} R_{M_3M_2} - \epsilon_{M_1}\epsilon_{M_2}\delta_{-M_2M_4} R_{M_3-M_1} \\
&\quad -\delta_{M_2M_3} R_{M_1M_4} + \epsilon_{M_1}\epsilon_{M_2}\delta_{-M_1M_3} R_{-M_2M_4}
\end{aligned}
$$

$R_{MN} = -\epsilon_M \epsilon_N R_{-N-M}, \qquad R^\dagger_{MN} = R_{NM}$ ➔ 6 generators of SU(2)xSU(2)

Stress-tensor

$$\hat{T}^X_{\mu\nu} = \frac{2}{3}\hat{\nabla}_\mu X \hat{\nabla}_\nu X - \frac{1}{3} X \hat{\nabla}_\mu \hat{\nabla}_\nu X - \frac{1}{6}\hat{g}_{\mu\nu}\left\{\hat{\nabla}_\lambda X \hat{\nabla}^\lambda X + \frac{1}{6}\hat{R}X^2\right\} + \frac{1}{6}\hat{R}_{\mu\nu}X^2$$

The 15 generators of conformal algebra $\delta_\zeta X = i[Q_\zeta, X]$

$$H^X = \sum_{J\geq 0}\sum_M (2J+1)\varphi^\dagger_{JM}\varphi_{JM}$$

$$R^X_{MN} = -\frac{1}{2}\sum_{J\geq 0}\sum_{S_1}\sum_{S_2}\sum_{V,y}(-\epsilon_V)\mathbf{G}^{\frac{1}{2}M}_{\frac{1}{2}(-Vy);\frac{1}{2}N}\mathbf{G}^{JS_1}_{\frac{1}{2}(Vy);JS_2}\varphi^\dagger_{JS_1}\varphi_{JS_2}$$

$$Q^X_M = \sum_{J\geq 0}\sum_{M_1,M_2}\mathbf{C}^{\frac{1}{2}M}_{JM_1,J+\frac{1}{2}M_2}\sqrt{(2J+1)(2J+2)}\epsilon_{M_1}\varphi^\dagger_{J-M_1}\varphi_{J+\frac{1}{2}M_2}$$

SU(2)xSU(2) Clebsch-Gordan coeff. of SSS type

$$\begin{aligned}\mathbf{C}^{JM}_{J_1M_1,J_2M_2} &= \sqrt{V_3}\int_{S^3} d\Omega_3 Y^*_{JM} Y_{J_1M_1} Y_{J_2M_2} \\ &= \sqrt{\frac{(2J_1+1)(2J_2+1)}{2J+1}} C^{Jm}_{J_1m_1,J_2m_2} C^{Jm'}_{J_1m'_1,J_2m'_2}\end{aligned}$$

Quantization of Conformal Mode

(quantized on $R \times S^3$)

Riegert action

$$I_\phi = \int d\eta \int_{S^3} d\Omega_3 \left\{ -\frac{2b_1}{(4\pi)^2} \phi \left(\partial_\eta^4 - 2\Box_3 \partial_\eta^2 + \Box_3^2 + 4\partial_\eta^2 \right) \phi \right\}$$

Reduce to second order action by introducing new variable $\chi = \partial_\eta \phi$

$$I_\phi = \int d\eta \int_{S^3} d\Omega_3 \left\{ -\frac{b_1}{8\pi^2} \left[(\partial_\eta \chi)^2 + 2\chi \Box_3 \chi - 4\chi^2 + (\Box_3 \phi)^2 \right] + v(\partial_\eta \phi - \chi) \right\}$$

➔ Dirac quantization

Scalar harmonics on S^3

$$Y_{JM} = \sqrt{\frac{2J+1}{V_3}} D^J_{mm'}, \quad M = (m, m')$$

Mode expansion

$$\phi = \frac{\pi}{2\sqrt{b_1}} \Bigg\{ 2(\hat{q} + \hat{p}\eta) Y_{00} + \sum_{J \geq \frac{1}{2}} \sum_M \frac{1}{\sqrt{J(2J+1)}} \left(a_{JM} e^{-i2J\eta} Y_{JM} + a^\dagger_{JM} e^{i2J\eta} Y^*_{JM} \right)$$
$$+ \sum_{J \geq 0} \sum_M \frac{1}{\sqrt{(J+1)(2J+1)}} \left(b_{JM} e^{-i(2J+2)\eta} Y_{JM} + b^\dagger_{JM} e^{i(2J+2)\eta} Y^*_{JM} \right) \Bigg\}$$

where

$$[\hat{q}, \hat{p}] = i, \quad [a_{J_1 M_1}, a^\dagger_{J_2 M_2}] = \delta_{J_1 J_2} \delta_{M_1 M_2}, \quad [b_{J_1 M_1}, b^\dagger_{J_2 M_2}] = -\delta_{J_1 J_2} \delta_{M_1 M_2}$$

Hamiltonian

Casimir effect on R x S^3

$$H^{\phi} = \frac{1}{2}\hat{p}^2 + b_1 + \sum_{J\geq 0}\sum_{M}\{2J a^{\dagger}_{JM} a_{JM} - (2J+2) b^{\dagger}_{JM} b_{JM}\}$$

4+4 generators of special conf. transf.

SU(2)xSU(2) Clebsch-Gordan

$$Q^{\phi}_{M} = \left(\sqrt{2b_1} - i\hat{p}\right) a_{\frac{1}{2}M} + \sum_{J\geq 0}\sum_{M_1}\sum_{M_2} \mathrm{C}^{\frac{1}{2}M}_{JM_1,J+\frac{1}{2}M_2}\Big\{\alpha(J)\epsilon_{M_1} a^{\dagger}_{J-M_1} a_{J+\frac{1}{2}M_2} + \beta(J)\epsilon_{M_1} b^{\dagger}_{J-M_1} b_{J+\frac{1}{2}M_2} + \epsilon_{M_2} a^{\dagger}_{J+\frac{1}{2}-M_2} b_{JM_1}\Big\}$$

Conformal Algebra SO(4,2)

6 rotation generator on S^3

$$[Q_M, Q^{\dagger}_N] = 2\delta_{MN} H + 2R_{MN}$$

etc.

Special conformal transformation mixes positive- and negative-metric creation modes

$$\alpha(J) = \sqrt{2J(2J+2)}$$

$$\beta(J) = -\sqrt{(2J+1)(2J+3)}$$

$$\left[Q^{\phi}_{M}, b^{\dagger}_{JM_1}\right] = -\sum_{M_2}\epsilon_{M_2}\mathrm{C}^{\frac{1}{2}M}_{JM_1,J+\frac{1}{2}-M_2} a^{\dagger}_{J+\frac{1}{2}M_2} - \beta\left(J-\frac{1}{2}\right)\sum_{M_2}\epsilon_{M_2}\mathrm{C}^{\frac{1}{2}M}_{JM_1,J-\frac{1}{2}-M_2} b^{\dagger}_{J-\frac{1}{2}M_2}$$

Traceless Tensor Fields

Traceless tensor mode is decomposed as

$$h_{00} = h, \qquad h_{0i} = h_i, \qquad h_{ij} = h^{\mathbf{tr}}_{ij} + \frac{1}{3}\hat{\gamma}_{ij} h$$

Take transverse gauge by using the four κ^μ gauge parameters

$$\hat{\nabla}^i h_i = \hat{\nabla}^i h^{\mathbf{tr}}_{ij} = 0 \quad \Rightarrow \quad h_i = h^{\mathrm{T}}_i, \qquad h^{\mathbf{tr}}_{ij} = h^{\mathrm{TT}}_{ij}$$

Gauge-fixed Weyl action at t=0

$$I_h = \int d\eta \int_{S^3} d\Omega_3 \Big\{ -\frac{1}{2} h^{\mathrm{TT}}_{ij} \left(\partial^4_\eta - 2\Box_3 \partial^2_\eta + \Box^2_3 + 8\partial^2_\eta - 4\Box_3 + 4\right) h^{ij}_{\mathrm{TT}} + h^{\mathrm{T}}_i \underline{(\Box_3 + 2)} \left(-\partial^2_\eta + \Box_3 - 2\right) h^i_{\mathrm{T}} - \frac{1}{27} h \underline{(16\Box_3 + 27)\,\Box_3} h \Big\}$$

Furthermore, we take radiation gauge+ $h = h^{\mathrm{T}}_i|_{J=\frac{1}{2}} = 0$

➔ residual gauge DOF = conformal symmetry

Vector harmonics = $(J+y, J-y)$ rep. with $y=\pm 1/2$
Tensor harmonics = $(J+x, J-x)$ rep. with $x=\pm 1$ (polarizations)

Transverse-traceless tensor mode

$$h_{\mathrm{TT}}^{ij} = \frac{1}{4}\sum_{J\geq 1}\sum_{M,x}\frac{1}{\sqrt{J(2J+1)}}\left\{c_{J(Mx)}e^{-i2J\eta}Y^{ij}_{J(Mx)} + c^{\dagger}_{J(Mx)}e^{i2J\eta}Y^{ij*}_{J(Mx)}\right\}$$
$$+\frac{1}{4}\sum_{J\geq 1}\sum_{M,x}\frac{1}{\sqrt{(J+1)(2J+1)}}\left\{d_{J(Mx)}e^{-i(2J+2)\eta}Y^{ij}_{J(Mx)} + d^{\dagger}_{J(Mx)}e^{i(2J+2)\eta}Y^{ij*}_{J(Mx)}\right\}$$

Transverse vector mode

$$h_{\mathrm{T}}^{i} = \frac{1}{2}\sum_{J\geq 1}\sum_{M,y}\frac{i}{\sqrt{(2J-1)(2J+1)(2J+3)}}\left\{e_{J(My)}e^{-i(2J+1)\eta}Y^{i}_{J(My)} - e^{\dagger}_{J(My)}e^{i(2J+1)\eta}Y^{i*}_{J(My)}\right\}$$

Commutators

$$\left[c_{J_1(M_1x_1)}, c^{\dagger}_{J_2(M_2x_2)}\right] = -\left[d_{J_1(M_1x_1)}, d^{\dagger}_{J_2(M_2x_2)}\right] = \delta_{J_1J_2}\delta_{M_1M_2}\delta_{x_1x_2},$$
$$\left[e_{J_1(M_1y_1)}, e^{\dagger}_{J_2(M_2y_2)}\right] = -\delta_{J_1J_2}\delta_{M_1M_2}\delta_{y_1y_2}$$

The generators of conformal algebra $\delta_\zeta h_{\mu\nu} = i[Q^h_\zeta, h_{\mu\nu}]$

up to field-dep. gauge transf.

$$
\begin{aligned}
H^h &= \sum_{J\geq 1}\sum_{M,x}\{2Jc^\dagger_{J(Mx)}c_{J(Mx)} - (2J+2)d^\dagger_{J(Mx)}d_{J(Mx)}\} \\
&\quad -\sum_{J\geq 1}\sum_{M,y}(2J+1)e^\dagger_{J(My)}e_{J(My)}
\end{aligned}
$$

$$
\begin{aligned}
A(J) &= \sqrt{\frac{4J}{(2J-1)(2J+3)}}, \\
B(J) &= \sqrt{\frac{2(2J+2)}{(2J-1)(2J+3)}}, \\
C(J) &= \sqrt{\frac{(2J-1)(2J+1)(2J+2)(2J+4)}{2J(2J+3)}}
\end{aligned}
$$

$$
\begin{aligned}
Q^h_M &= \sum_{J\geq 1}\sum_{M_1,x_1,M_2,x_2}\mathbf{E}^{\frac{1}{2}M}_{J(M_1x_1),J+\frac{1}{2}(M_2x_2)}\Big\{\alpha(J)\epsilon_{M_1}c^\dagger_{J(-M_1x_1)}c_{J+\frac{1}{2}(M_2x_2)} \\
&\qquad +\beta(J)\epsilon_{M_1}d^\dagger_{J(-M_1x_1)}d_{J+\frac{1}{2}(M_2x_2)} + \gamma(J)\epsilon_{M_2}c^\dagger_{J+\frac{1}{2}(-M_2x_2)}d_{J(M_1x_1)}\Big\} \\
&\quad +\sum_{J\geq 1}\sum_{M_1,x_1,M_2,y_2}\mathbf{H}^{\frac{1}{2}M}_{J(M_1x_1);J(M_2y_2)}\Big\{A(J)\epsilon_{M_1}c^\dagger_{J(-M_1x_1)}e_{J(M_2y_2)} \\
&\qquad\qquad +B(J)\epsilon_{M_2}e^\dagger_{J(-M_2y_2)}d_{J(M_1x_1)}\Big\} \\
&\quad +\sum_{J\geq 1}\sum_{M_1,y_1,M_2,y_2}\mathbf{D}^{\frac{1}{2}M}_{J(M_1y_1),J+\frac{1}{2}(M_2y_2)}C(J)\epsilon_{M_1}e^\dagger_{J(-M_1y_1)}e_{J+\frac{1}{2}(M_2y_2)}
\end{aligned}
$$

SU(2)^2 CG coeff.

- $\mathbf{E}$: STT type
- $\mathbf{H}$: STV type
- $\mathbf{D}$: SVV type

Conformal symmetry mixes all modes in tensor field

Emphasize that negative-metric modes are necessary to form the close algebra of conformal symmetry quantum mechanically

Physical State Conditions

Confomal symmetry = diffeomorphism invariance

➔Physical state condition = Wheeler-DeWitt equation

$$\boxed{(H-4)|\mathrm{phys}\rangle = Q_M|\mathrm{phys}\rangle = R_{MN}|\mathrm{phys}\rangle = 0}$$

Consider composite creation op. R_n satisfying

$$[Q_M, \mathcal{R}_n] = [R_{MN}, \mathcal{R}_n] = 0$$

then

$$\begin{aligned} |\mathrm{phys}\rangle &= \mathcal{R}_n\left(a^\dagger_{JM}, b^\dagger_{JM}, \cdots\right) e^{ip\sqrt{2b_1}\phi}|\Omega\rangle \\ &= \mathcal{R}_n\left(a^\dagger_{JM}, b^\dagger_{JM}, \cdots\right) e^{\gamma_n\phi}|\Omega\rangle \end{aligned}$$

vacuum state

$$|\Omega\rangle = e^{-2b_1\phi}|0\rangle$$

$$Q^\dagger_M|\Omega\rangle = 0$$

$$H = 4 \Rightarrow p = -i\gamma_n/\sqrt{2b_1}, \quad \gamma_n = 2b_1\left(1-\sqrt{1-(4-n)/b_1}\right)$$

pure imaginary

$$= 4 - n + o(1/b_1)$$

➔ “Real” states, such as scalar curvature

Building Block R_n for Scalar Field

Commutator of Q_M and creation mode

$$[Q^X_M, \varphi^\dagger_{JM_1}] = \sqrt{2J(2J+1)} \sum_{M_2} \epsilon_{M_2} \mathrm{C}^{\frac{1}{2}M}_{JM_1, J-\frac{1}{2}-M_2} \varphi^\dagger_{J-\frac{1}{2}M_2}$$

Q_M-invariant creation operator is only $\varphi^\dagger_{00}$

Consider a bilinear form

$$\Phi^{[L]\dagger}_{JN} = \sum_{K=0}^{L} \sum_{M_1,M_2} f(L,K) \mathrm{C}^{JN}_{L-KM_1,KM_2} \varphi^\dagger_{L-KM_1} \varphi^\dagger_{KM_2}$$

Q_M invariant ➔ J=L and $f(L,K) = \dfrac{(-1)^{2K}}{\sqrt{(2L-2K+1)(2K+1)}} \begin{pmatrix} 2L \\ 2K \end{pmatrix}$

Thus, Q_M invariant operator in scalar field sector is given by

$$\Phi^\dagger_{LN} = \Phi^{[L]\dagger}_{LN}$$

Here, $\Phi^\dagger_{00} = (\varphi^\dagger_{00})^2$

Z_2 symmetry
X ↔ -X

Building Blocks for Conformal Mode

No creation mode that commute with Q_M

➔ Consider Q_M invariant bilinear forms, which are given by

$$
\begin{aligned}
S^{\dagger}_{LN} &= \chi(\hat{p}) a^{\dagger}_{LN} + \sum_{K=\frac{1}{2}}^{L-\frac{1}{2}} \sum_{M_1,M_2} x(L,K) \mathrm{C}^{LN}_{L-KM_1,KM_2} a^{\dagger}_{L-KM_1} a^{\dagger}_{KM_2}, \\
S^{\dagger}_{L-1N} &= \psi(\hat{p}) b^{\dagger}_{L-1N} + \sum_{K=\frac{1}{2}}^{L-\frac{1}{2}} \sum_{M_1,M_2} x(L,K) \mathrm{C}^{L-1N}_{L-KM_1,KM_2} a^{\dagger}_{L-KM_1} a^{\dagger}_{KM_2} \\
&\quad + \sum_{K=\frac{1}{2}}^{L-1} \sum_{M_1,M_2} y(L,K) \mathrm{C}^{L-1N}_{L-K-1M_1,KM_2} b^{\dagger}_{L-K-1M_1} a^{\dagger}_{KM_2}
\end{aligned}
$$

where

$$
\begin{aligned}
x(L,K) &= \frac{(-1)^{2K}}{\sqrt{(2L-2K+1)(2K+1)}} \sqrt{\begin{pmatrix} 2L \\ 2K \end{pmatrix} \begin{pmatrix} 2L-2 \\ 2K-1 \end{pmatrix}}, \\
y(L,K) &= -2\sqrt{(2L-2K-1)(2L-2K+1)}\, x(L,K)
\end{aligned}
$$

$$
\begin{aligned}
\chi(\hat{p}) &= \frac{1}{\sqrt{2(2L-1)(2L+1)}} \left(\sqrt{2b_1} - i\hat{p}\right), \\
\psi(\hat{p}) &= -\sqrt{2}\left(\sqrt{2b_1} - i\hat{p}\right)
\end{aligned}
$$

Building Blocks of Physical States

rank of tensor index	0
creation op.	$\Phi^\dagger_{LN}$
level ($L \in \mathbf{Z}_{\geq 0}$)	$2L+2$

Scalar fields

rank of tensor index	0	1	2
creation op.	$\Psi^\dagger_{LN}$	$q^\dagger_{\frac{1}{2}(Ny)}$	$\Upsilon^\dagger_{L(Nx)}$
level ($L \in \mathbf{Z}_{\geq 2}$)	$2L+2$	2	$2L+2$

U(1) gauge fields

rank of tensor index	0
creation op.	$S^\dagger_{LN}$ $\mathcal{S}^\dagger_{L-1N}$
level ($L \in \mathbf{Z}_{\geq 1}$)	$2L$

Conformal mode

rank of tensor index	0	1	2	3	4
creation op.	$A^\dagger_{LN}$ $\mathcal{A}^\dagger_{L-1N}$	$B^\dagger_{L-\frac{1}{2}(Ny)}$	$c^\dagger_{1(Nx)}$	$D^\dagger_{L-\frac{1}{2}(Nz)}$	$E^\dagger_{L(Nw)}$ $\mathcal{E}^\dagger_{L-1(Nw)}$
level ($L \in \mathbf{Z}_{\geq 3}$)	$2L$	$2L$	2	$2L$	$2L$

Traceless tensor fields

Classified by using crossing symmetry

Physical states ⟷ Diffeomorphism invariant fields

$$|\mathrm{phys}\rangle = \lim_{\eta\to i\infty} e^{-i4\eta}\mathcal{O}(\eta,\mathbf{x})|\Omega\rangle$$

Conformal fields = “real fields” with even derivatives

Level n=0 $e^{\gamma_0\phi_0}|\Omega\rangle \Longleftrightarrow \sqrt{-g}$ (= dressed identity operator)

n=2 $\mathcal{S}^\dagger_{00}e^{\gamma_2\phi_0}|\Omega\rangle, \quad \Phi^\dagger_{00}e^{\gamma_2\phi_0}|\Omega\rangle \Longleftrightarrow \sqrt{-g}R, \quad \sqrt{-g}X^2$

n=4 $\sum_{N,x}\epsilon_N c^\dagger_{1(-Nx)}c^\dagger_{1(Nx)}|\Omega\rangle, \quad (\mathcal{S}^\dagger_{00})^2|\Omega\rangle, \quad \sum_N \epsilon_N S^\dagger_{1-N}S^\dagger_{1N}|\Omega\rangle,$

$\Phi^\dagger_{00}\mathcal{S}^\dagger_{00}|\Omega\rangle, \quad (\Phi^\dagger_{00})^2|\Omega\rangle,$

$\Updownarrow$

$$\sqrt{-g}C^2_{\mu\nu\lambda\sigma}, \ \sqrt{-g}R^2, \ \sqrt{-g}G_4, \ \sqrt{-g}RX^2, \ \sqrt{-g}X^4$$

On Positivity of Two-Point Function

Physical states ←→ Diffeomorphism invariant fields

$$|\mathrm{phys}\rangle = \lim_{\eta \to i\infty} e^{-i4\eta} \mathcal{O}(\eta, \mathbf{x}) |\Omega\rangle$$

Conformal fields = "real fields" with even derivatives
(level of building blocks are even)

At large b_1 limit, scalar curvature operator can be written by

$$\sqrt{-g}R \simeq e^{2\phi}(-\partial^2\phi)$$

Using the correlation function $\langle \phi(x)\phi(y) \rangle = -\frac{1}{4b_1}\log(|x-y|^2)$

$$\langle \sqrt{-g}R(x)\sqrt{-g}R(y) \rangle = \frac{A}{|x-y|^{2\Delta_R}} \qquad \Delta_R = 2 + \frac{1}{b_1}$$

Positivity means b_1 > 0 (right sign of WZ action)

→ Initial spectrum of the universe