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Gravitational Counterterms Near Four Dimensions and Conformal Anomalies

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References

- Determination of Gravitational Counterterms Near Four Dimensions from RG Equations, K. Hamada, Phys. Rev. D89 (2014) 104063, arXiv:1403.4354
- Trace Anomalies and QED in Curved Space, S. Hathrell, Ann. Phys. 142 (1982) 34.

1. Introduction and Motivation

Why Gravitational Counterterms

- Application to cosmology

  In 1979, Starobinsky indicated that inflation is induced by gravitational conformal anomaly

  His work suggests that we can construct inflation model without adding any phenomenological scalar fields

- Toward quantum gravity

  Curved space is the gate for quantum gravity

In any case, we must fix the gravitational action unambiguously to discuss these gravitational dynamics

What is Problem?

Usually, in flat space, counterterms of renormalizable quantum theory can be fixed by gauge and global symmetries at the classical level

However, going to curved space, some extra ambiguities arise:

- Coupling between scalar field and gravity

  $\xi R \varphi^2$ (conformal coupling is $\xi = 1/6$)

- Gravitational actions (=counterterms)

  There are three types of curvature-squared: $R^2_{\mu\nu\lambda\sigma}$, $R^2_{\mu\nu}$, R^2

  ➔ Ambiguities in choice of their combinations, or couplings

  (In this talk, only dimensionless couplings are considered)

Classically, we cannot fix these ambiguities by diffeomorphism inv.

⇨ To fix such ambiguities, we often use conformal invariance

There is a long-standing conjecture on conformal invariance

For conformally coupled theory, UV divergences are renormalized by two conformally invariant counterterms only:

- Square of Weyl tensor : $F_4 = C^2_{\mu\nu\lambda\sigma} = R^2_{\mu\nu\lambda\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2$
- Euler combination : $G_4 = R^2_{\mu\nu\lambda\sigma} - 4R^2_{\mu\nu} + R^2$

➔ No R^2 divergence

On the other hand, however, conformal invariance always breaks down for dynamical theory because dynamics induce a mass scale

➔ This conjecture is really correct !?

Furthermore, when using dimensional regularization, conformal invariance is explicitly violated in *D* dimensions, and further ambiguity appears

Dimensional Regularization

Nevertheless, I here use dimensional regularization in curved space

Advantages:

- It preserves gauge symmetries, including diffeomorphism invariance
- It is only one regularization method we can carry out higher loop calculations in curved space

Significant property:

In exactly 4 dimensional curved space, measure contributions such as conformal anomalies come from divergent quantity $\delta^4(0) = \langle x | x' \rangle |_{x' \to x}$

In dim. reg., however, it is regularized to zero as $\delta^D(0) = \int d^D k = 0$

→ Loop cals. are independent of how to choose path integral measure, and measure contributions (conformal anomalies) are contained between *D* and 4 dimensions

D-dep. of action is important !

The Goal of This Talk

I will show that ambiguities mentioned here can be fixed by imposing consistency conditions of renormalization:

$\left\{\begin{array}{l}\text{Finiteness of beta functions and anomalous dimensions}\\ \text{Finiteness of renormalized correlation functions}\end{array}\right.$

and conclude that long-standing conjecture is correct, namely

- Independent gravitational actions are "only two"
- Their *D*-dep. (= measure contributions) can be determined

➔ the forms of conformal anomalies can be fixed completely like $\underbrace{\frac{1}{D-4}}_{\text{from loop}} \times \underbrace{D-4}_{\text{in action}}$ ➔ finite (= conformal anomalies)

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2. Massless QED in Curved Space

- as a typical conformally coupled theory -

(We can do the similar argument for QCD)

The Action in Curved Space

The most general renormalizable QED action fixed by gauge invariance and diffeomorphism invariance is

Euclidean sgn.

$$S = \int d^D x \sqrt{g} \Big\{ \frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} + i\bar{\psi}_0 \not{D} \psi_0 + \frac{1}{2\xi_0} (\nabla^\mu A_{0\mu})^2 + \underbrace{a_0 F_D + b_0 G_4 + c_0 H^2}_{} \Big\}$$

uniquely fixed by gauge and diff. inv. gauge-fixing

For the moment, consider 3 types of gravitational actions

(➔ reduce them to 2 types later)

Weyl action: $F_D = R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} - \frac{4}{D-2} R_{\mu\nu} R^{\mu\nu} + \frac{2}{(D-1)(D-2)} R^2$

Euler density: $G_4 = R^2_{\mu\nu\lambda\sigma} - 4R^2_{\mu\nu} + R^2$

Rescaled scalar curvature: $H = \frac{R}{D-1}$

Renormalization factors

$$A_{0\mu} = Z_3^{1/2} A_\mu \qquad \psi_0 = Z_2^{1/2} \psi \qquad e_0 = \mu^{2-D/2} Z_3^{-1/2} e, \qquad \xi_0 = Z_3 \xi$$

(Ghost action is disregarded because we consider coupling-dependent parts only)

Familiar RG Equations

RGE is derived from the condition that bare quantities are independent of arbitrary mass scale introduced to make up for the loss of mass dimensions:

$$\mu \frac{d}{d\mu}(\mathrm{bare}) = 0 \qquad \mu \frac{d}{d\mu} = \mu \frac{\partial}{\partial \mu} + \mu \frac{d\alpha}{d\mu}\frac{\partial}{\partial \alpha} + \mu \frac{d\xi}{d\mu}\frac{\partial}{\partial \xi} + \cdots$$

because there is no such a scale in the original bare action

depend on α only

Ex. Beta function is defined by $\beta(\alpha, D) \equiv \frac{1}{\alpha}\mu \frac{d\alpha}{d\mu} = D - 4 + \bar{\beta}(\alpha)$

From RGE $\mu \frac{d}{d\mu}\left(\frac{e_0^2}{4\pi}\right) = 0 = \frac{\mu^{4-D}}{Z_3}\alpha \left(4 - D - \mu \frac{d}{d\mu}\log Z_3 + \frac{\mu}{\alpha}\frac{d\alpha}{d\mu}\right)$

we obtain the expression $\bar{\beta} = \mu \frac{d}{d\mu}\log Z_3$ where $\log Z_3 = \sum_{n=1}^{\infty} \frac{f_n(\alpha)}{(D-4)^n}$

Finiteness of beta function then deduces

$$\bar{\beta} = \alpha \frac{\partial f_1}{\alpha} \quad \text{and} \quad \frac{\partial f_{n+1}}{\partial \alpha} + \bar{\beta}\frac{\partial f_n}{\partial \alpha} = 0$$

(to cancel out pole terms)

All residues can be derived from the simple pole residue

Bare couplings for gravitational counterterms are defined by

$$a_0 = \mu^{D-4}(a + L_a), \qquad L_a = \sum_{n=1}^{\infty} \frac{a_n(\alpha)}{(D-4)^n}$$

$$b_0 = \mu^{D-4}(b + L_b), \qquad L_b = \sum_{n=1}^{\infty} \frac{b_n(\alpha)}{(D-4)^n}$$

$$c_0 = \mu^{D-4}(c + L_c), \qquad L_c = \sum_{n=1}^{\infty} \frac{c_n(\alpha)}{(D-4)^n}$$

As similar to QED sector, beta functions are defined as

$$\beta_a(\alpha, D) \equiv \mu \frac{da}{d\mu} = -(D-4)a + \bar{\beta}_a(\alpha)$$

$$\mu \frac{d}{d\mu} a_0 = 0 \quad \rightarrow \quad \bar{\beta}_a = -\frac{\partial}{\alpha}(\alpha a_1) \quad \text{and} \quad \frac{\partial}{\partial \alpha}(\alpha a_{n+1}) + \bar{\beta}\alpha \frac{\partial a_n}{\partial \alpha} = 0$$

and similar eqs for *b_n* and *c_n*

For later importance,

Please mind that $1/\bar{\beta}$ is finite, but $\frac{1}{\beta} = \frac{1}{D-4}\left(1 + \sum_{n=1}^{\infty} \frac{(-\bar{\beta})^n}{(D-4)^n}\right)$ has poles

So, we cannot easily divide finite quantities by beta function

Consistency Conditions of Renormalization

Renormalizability:

All UV divergences can be renormalized by local counterterms

= No nonlocal divergences appear

(Intuitively, natural because UV implies local)

$$\cancel{\frac{\log^m(p^2/\mu^2)}{(D-4)^n}}$$

Furthermore,

Later, we consider renormalized composite fields, called normal products (ex. EM tensor)

Their correlation functions are not necessarily finite, but do not have nonlocal divergence as well

The goal of this talk is that the pole term *L_b* is related with *L_c* through nontrivial RGEs derived soon below on the basis of these consistency conditions, and then the number of gravitational couplings is reduced

3. Normal Products

(= Renormalized Composite Operators)

- Equation-of-motion operator
- Field-strength-squared operator
- Energy-momentum tensor operator

Normal Products [...]

(1) Equation-of-motion (EOM) operators

Similarly for gauge fields

$$E_{0A} = \frac{1}{\sqrt{g}} A_{0\mu} \frac{\delta S}{\delta A_{0\mu}}$$

$$E_{0\psi} = \frac{\delta S}{\delta \chi} \equiv \frac{1}{\sqrt{g}} \left(\bar{\psi}_0 \frac{\delta S}{\delta \bar{\psi}_0} + \psi_0 \frac{\delta S}{\delta \psi_0} \right) = 2i\bar{\psi}_0 \overleftrightarrow{\not{D}} \psi_0$$

Carrying out integration-by-part in correlation function $\langle \cdots \rangle = \int [dAd\psi\bar{\psi}] \cdots e^{-S}$, we obtain

$$\left\langle E_{0\psi}(x) \prod_{j=1}^{N_\psi} \left(\psi \text{ or } \bar{\psi}\right)(x_j) \right\rangle = \sum_{j=1}^{N_\psi} \frac{1}{\sqrt{g}} \delta^D(x - x_j) \left\langle \prod_{j=1}^{N_\psi} \left(\psi \text{ or } \bar{\psi}\right)(x_j) \right\rangle$$

Since r.h.s is aparently finite, EOM operator is finite, namely normal product

$E_{0\psi} = [E_\psi]$ and $\int d^D x \sqrt{g} [E_\psi] \Leftrightarrow N_\psi$ (inside of correlators)

The significant properties of EOM operator is

$$\langle [E_\psi(x)] P(y) \rangle_{\text{flat}} = \langle \delta P(y) / \delta \chi(x) \rangle_{\text{flat}} = 0$$

P(y) is any composite of fermions

tadpole-type and massless

Two-point functions with EOM are regularized to zero in flat space

(2) Field-strength-squared operator

In general, it has the following structure (reduce to bare one in free-field limit):

$$[F_{\mu\nu}F^{\mu\nu}] = (1+\sum \text{poles})F_{0\mu\nu}F_0^{\mu\nu} + (\sum \text{poles})(\text{other operators})$$

To determine unknown parts, we consider the following finite correlation

$$\alpha\frac{\partial}{\partial\alpha}\langle\prod_{j=1}^{N_A}A_{\mu_j}(x_j)\prod_{k=1}^{N_\psi}(\psi \text{ or } \bar{\psi})(x_k)\rangle = \text{finite}$$

using $\alpha\frac{\partial e_0}{\partial\alpha} = \frac{(D-4)e_0}{2\beta}$

and so on

In flat space, we obtain

$$\left\langle\int d^Dx \underbrace{\left\{\frac{D-4}{4\beta}F_{0\mu\nu}F_0^{\mu\nu} - \frac{\bar{\gamma}_2}{2\beta}E_{0\psi}\right\}}_{\frac{1}{4}[F_{\mu\nu}F^{\mu\nu}]}\prod_{j=1}^{N_A}A_{\mu_j}(x_j)\prod_{k=1}^{N_\psi}\left(\psi \text{ or } \bar{\psi}\right)(x_k)\right\rangle = \text{finite}$$

where $\bar{\gamma}_2 = \gamma_2 - (D-4)\xi\partial(\log Z_2)/\partial\xi$

[gauge-inv. combination]

In curved space

$$\begin{aligned}\frac{1}{4}[F_{\mu\nu}F^{\mu\nu}] &= \frac{D-4}{4\beta}F_{0\mu\nu}F_0^{\mu\nu} - \frac{\bar{\gamma}_2}{2\beta}E_{0\psi} + \frac{D-4}{\beta}\mu^{D-4}\left[\left(L_a + \frac{\bar{\beta}_a}{D-4}\right)F_D\right.\\ &\left. + \left(L_b + \frac{\bar{\beta}_b}{D-4}\right)G_4 + \left(L_c + \frac{\bar{\beta}_c}{D-4}\right)H^2 - \frac{4(\sigma+L_\sigma)}{D-4}\nabla^2 H\right]\end{aligned}$$

$$\frac{D-4}{\beta} = 1 + \sum_{n=1}^{\infty}\frac{(-\bar{\beta})^n}{(D-4)^n}$$

up to total divergence

(can be fixed by another condition)

(3) Energy-momentum (EM) tensor operator

EM tensor is defined by $\theta^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}$ The trace is $\theta = \frac{2}{\sqrt{g}} g_{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}} \equiv \frac{\delta S}{\delta \Omega}$

Therefore,

$$\frac{2}{\sqrt{g}} \frac{\delta}{\delta g_{\mu\nu}(x)} \langle \prod_{j=1}^{N_A} A_{\mu_j}(x_j) \prod_{k=1}^{N_\psi} (\psi \text{ or } \bar{\psi})(x_k) \rangle = -\langle \theta^{\mu\nu}(x) \prod_{j=1}^{N_A} A_{\mu_j}(x_j) \prod_{k=1}^{N_\psi} (\psi \text{ or } \bar{\psi})(x_k) \rangle$$

The l.h.s is apparently finite, and thus EM tensor is a finite operator

EM tensor, which is originally bare quantity, can be written using normal products as

$$\begin{aligned} \theta &= \frac{D-4}{4} F_{0\mu\nu} F_0^{\mu\nu} + \frac{D-1}{2} E_{0\psi} \\ &= \frac{\beta}{4} [F_{\mu\nu} F^{\mu\nu}] + \frac{1}{2} (D - 1 + \bar{\gamma}_2) [E_\psi] \end{aligned}$$

in flat space

conformal anomaly
(proportional to beta function)

gravitational parts are discussed later

4. Nontrivial RG Equations

from 2- and 3-Point Functions of Normal Products

Two-Point Functions of EM Tensor

Since the partition function is finite, its two-times variations is also finite

Therefore,
$$\langle\theta^{\mu\nu}(x)\theta^{\lambda\sigma}(y)\rangle - \frac{2}{\sqrt{g(y)}}\left\langle\frac{\delta\theta^{\mu\nu}(x)}{\delta g_{\lambda\sigma}(y)}\right\rangle = \text{finite}$$

After variations, we take flat-space limit, and thus obtain

$$\begin{cases} \langle\theta^{\mu\nu}(p)\theta_{\mu\nu}(-p)\rangle_{\text{flat}} - 4(D-3)(D+1)a_0p^4 - \dfrac{8}{D-1}c_0p^4 = \text{finite} \\ \langle\theta(p)\theta(-p)\rangle_{\text{flat}} - 8c_0p^4 = \text{finite} \end{cases}$$

from gravitational counterterms

$$a_0F_D + b_0G_4 + c_0H^2$$

From the second equation, we obtain the relation

$$\langle\theta(p)\theta(-p)\rangle_{\text{flat}} - 8p^4\mu^{D-4}L_c = \text{finite}$$

$$c_0 = \mu^{D-4}(c + L_c)$$

Notice: From the first eq. we can determine pole term *L_a* (not discussed here)
The pole term *L_b* will be determined from 3-point functions discussed later

Consider the following operator in flat space:

This operator is divergent, but the divergence is merely proportional to EOM operator

$$\{A^2\} \equiv \frac{D-4}{4\beta} F_{0\mu\nu} F_0^{\mu\nu} = \frac{1}{4}[F_{\mu\nu}F^{\mu\nu}] + \frac{\bar{\gamma}_2}{2\beta}[E_\psi]$$

finite

divergent $\frac{1}{\beta} = \frac{1}{D-4}\left(1 + \sum_{n=1}^{\infty} \frac{(-\bar{\beta})^n}{(D-4)^n}\right)$

As mentioned before, 2-point function with EOM vanishes in flat space

Thus, $\Gamma_{AA}(p^2) = \langle \{A^2(p)\}\{A^2(-p)\}\rangle_{\text{flat}}$ reduces to correlation of normal product

UV divergences of such a correlation can be described by local poles as

$$\Gamma_{AA}(p^2) - p^4 \mu^{D-4} \left(\frac{D-4}{\beta}\right)^2 L_x = \text{finite} \qquad L_x = \sum_{n=1}^{\infty} \frac{x_n(\alpha)}{(D-4)^n}$$

Here, *L_x* is defined by this eq. and $(D-4)^2/\beta^2$ is for later convenience

Noting that EM tensor can be described as $\theta|_{\text{flat}} = \beta\{A^2\} + \frac{1}{2}(D-1)[E_\psi]$,

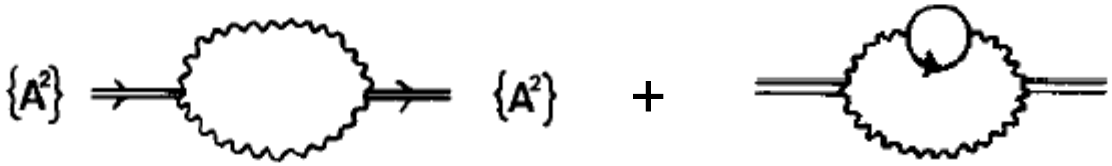
we obtain

$$\beta^2 \Gamma_{AA}(p^2) = \langle \theta(p)\theta(-p)\rangle_{\text{flat}} = 8p^4 \mu^{D-4} L_c + \text{finite}$$

This implies $(D-4)^2 L_x - 8L_c = \text{finite} \Rightarrow c_n = x_{n+2}/8$

Thus, *c_1* can be derived from *x_3*

The simple pole residue *x_1* can be easily calculated as



$$x_1 = -\frac{1}{2}\frac{1}{(4\pi)^2} + \frac{5}{3}\frac{\alpha}{(4\pi)^3} + o(\alpha^2)$$

The residue *x_3* can be calculated from *x_1* using RGE below

Here, note that if *F* is a finite quantity,

$$\frac{1}{\beta^n}\mu\frac{d}{d\mu}(\beta^n F) = \mu\frac{dF}{d\mu} + n\alpha\frac{\partial\bar{\beta}}{\partial\alpha}F = \text{finite}$$ (even though $1/\beta$ is divergent)

(frequently used below)

Paying attention to this fact, consider the following eq.

$$\frac{1}{\beta^2}\mu\frac{d}{d\mu}\left\{\beta^2\left[\underbrace{\Gamma_{AA}(p^2) - p^4\mu^{D-4}\frac{(D-4)^2}{\beta^2}L_x}\right]\right\} = \text{finite}$$

finite (= defining eq of *L_x*)

Since $\beta^2\Gamma_{AA}$ is <u>bare quantity</u> by definition, the first term vanishes due to $\mu\frac{d}{d\mu}(\mathrm{bare})=0$, and so we obtain

$$\frac{1}{\beta^2}\mu\frac{d}{d\mu}\left\{\mu^{D-4}(D-4)^2L_x\right\}=\text{finite}$$

From this eq., we finally obtain RGE

$$\begin{aligned}\frac{\partial}{\partial\alpha}(\alpha x_2)-\frac{\bar\beta}{\alpha}\frac{\partial}{\partial\alpha}\left(\alpha^2x_1\right) &= 0\\ \frac{\partial}{\partial\alpha}(\alpha x_3)-\frac{\bar\beta}{\alpha}\frac{\partial}{\partial\alpha}\left(\alpha^2x_2\right)+\frac{\bar\beta^2}{\alpha^2}\frac{\partial}{\partial\alpha}\left(\alpha^3x_1\right) &= 0\end{aligned}$$

In this way, from *x_1* we can calculate *x_3*, and thus *c_1* as

$$x_3=\frac{8}{9}\frac{\alpha^3}{(4\pi)^5}-\frac{40}{27}\frac{\alpha^4}{(4\pi)^6}+o(\alpha^5)$$

$$c_1=\frac{1}{8}x_3$$

where use 3-loop beta function

$$\bar\beta=\frac{8}{3}\frac{\alpha}{4\pi}+8\left(\frac{\alpha}{4\pi}\right)^2-\frac{124}{9}\left(\frac{\alpha}{4\pi}\right)^3$$

Note that from 1-loop cal., we obtain 3-loop result of *c_1*

Three-Point Functions

In general, 3-point function of $\{A^2\} = \frac{1}{4}[F_{\mu\nu}F^{\mu\nu}] + \underline{\frac{\bar{\gamma}_2}{2\beta}}[E_\psi]$ has the following form:

$$\Gamma_{AAA}(p_x^2, p_y^2, p_z^2) - \sum \text{poles} \times \left\{\Gamma_{AA}(p_x^2) + \Gamma_{AA}(p_y^2) + \Gamma_{AA}(p_z^2)\right\}$$
$$-\mu^{D-4} \sum \text{poles} \times \left\{\text{terms in } p_i^2 p_j^2\right\} = \text{finite}$$

To subtract nonlocal poles coming from div. term in {A^2}

For the special case with two on-shell momenta, we obtain

$$\Gamma_{AAA}(p^2, 0, 0) - \underline{\frac{\alpha^2}{\beta}\frac{\partial}{\partial\alpha}\left(\frac{\bar{\beta}}{\alpha}\right)}\Gamma_{AA}(p^2) - p^4\mu^{D-4}\left(\frac{D-4}{\beta}\right)^3 L_y = \text{finite}$$

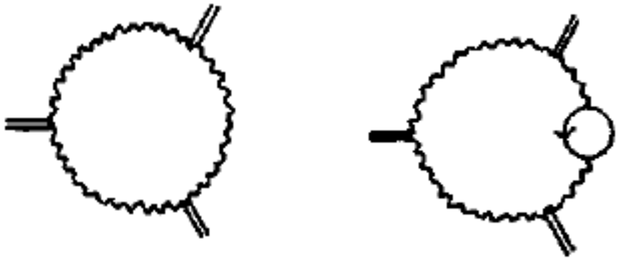
can be fixed directly

L_y is defined by this eq.

As similar for *L_x*, we can obtain RGE for *L_y* :

$$\left(\frac{D-4}{\beta}\right)^3 \left[(D-4)L_y + \beta\alpha\frac{\partial}{\partial\alpha}L_y\right] + \alpha^2\frac{\partial^2\bar{\beta}}{\partial\alpha^2}\left(\frac{D-4}{\beta}\right)^2 L_x = \text{finite}$$

From the defining eq. of *L_y*, we can directly calculate the simple pole *y_1* :



$$y_1 = -\frac{1}{2}\frac{1}{(4\pi)^2} + \frac{11}{6}\frac{\alpha}{(4\pi)^3} + o(\alpha^2)$$

$$L_y = \sum_{n=1}^{\infty} \frac{y_n(\alpha)}{(D-4)^n}$$

By solving RGE for residues

$$\frac{\partial}{\partial\alpha}(\alpha y_{n+1}) + \bar{\beta}\alpha\frac{\partial y_n}{\partial\alpha} + \sum_{m=1}^{n-1}(-1)^m \frac{(m+1)(m+2)}{2}\bar{\beta}^m \left[\frac{\partial}{\partial\alpha}(\alpha y_{n-m+1}) + \bar{\beta}\alpha\frac{\partial y_{n-m}}{\partial\alpha}\right] + (-1)^n\frac{(n+1)(n+2)}{2}\bar{\beta}^n\frac{\partial}{\partial\alpha}(\alpha y_1) - \alpha^2\frac{\partial^2\bar{\beta}}{\partial\alpha^2}\sum_{m=1}^{n}(-1)^m m\bar{\beta}^{m-1}x_{n-m+1} = 0$$

we can obtain all *y_n* from simple poles of *x_1* and *y_1*

Furthermore, three-point function of EM tensor is given by

$$\langle \theta(x)\theta(y)\theta(z)\rangle - \left\langle \frac{\delta\theta(x)}{\delta\Omega(y)}\theta(z)\right\rangle - \left\langle \frac{\delta\theta(y)}{\delta\Omega(z)}\theta(x)\right\rangle - \left\langle \frac{\delta\theta(z)}{\delta\Omega(x)}\theta(y)\right\rangle$$
$$+\left\langle \frac{\delta\theta(x)}{\delta\Omega(y)\delta\Omega(z)}\right\rangle = \text{finite}$$

Using $\theta|_{\text{flat}} = \beta\{A^2\} + \frac{1}{2}(D-1)[E_\psi]$ and expression of Γ_{AA} , we obtain

$$\beta^3\Gamma_{AAA}(p^2,0,0) - p^4\mu^{D-4}\left[2(D-2)(D-3)(D-4)L_b + 4(D-6)L_c\right] = \text{finite}$$

Combining with *L_y-equation to remove* Γ_{AAA} , we obtain

$$2(D-2)(D-3)(D-4)L_b + 4\left[D-6-2\alpha^2\frac{\partial}{\partial\alpha}\left(\frac{\bar\beta}{\alpha}\right)\right]L_c - (D-4)^3 L_y = \text{finite}$$

given

$$\Rightarrow \quad 4b_{n+1} + 6b_{n+2} + 2b_{n+3} - 8\left[1+\alpha^2\frac{\partial}{\partial\alpha}\left(\frac{\bar\beta}{\alpha}\right)\right]c_n + 4c_{n+1} - y_{n+3} = 0$$

This RGE gives the relationship between *L_b* and *L_c completely*

5. Solutions of RG Equations

Determination of gravitational counterterms
and conformal anomalies

Novel Gravitational Counterterms

RG equations indicate that two counterterms $b_0 G_4 + c_0 H^2$ should be unified as

$$S_g = \int d^D x \sqrt{g} \, \{a_0 F_D + \underline{b_0 G_D}\}$$

$$a_0 = \mu^{D-4}(a + L_a)$$

$$b_0 = \mu^{D-4}(b + L_b)$$

where $\quad G_D = G_4 + (D-4)\chi(D)H^2$

finite function of *D* only

This implies that

- The coupling *c* is eliminated and

$$L_c - (D-4)\chi(D)L_b = \text{finite}$$

$$c_0 = \mu^{D-4}(\cancel{c} + L_c)$$

not indep.

- The *D*-dependence of action are slightly changed because of $\chi(D)$ ➔ affects the form of conformal anomalies later

Let us expand $\chi(D)$ in series of *D-4 as*

$$\chi(D)=\sum_{n=1}^{\infty}\chi_n(D-4)^{n-1}=\chi_1+\chi_2(D-4)+\chi_3(D-4)^2+\cdots$$

Numbers (independent of couplings)

Then, we obtain the relation between the residues

$$L_c-(D-4)\chi(D)L_b=\text{finite} \quad \Rightarrow \quad c_n=\sum_{k=1}^{\infty}\chi_k b_{n+k}$$

Therefore, we can determine $\chi(D)$ order by order completely by solving RGE

already given

$$4b_{n+1}+6b_{n+2}+2b_{n+3}-8\left[1+\alpha^2\frac{\partial}{\partial\alpha}\left(\frac{\bar{\beta}}{\alpha}\right)\right]c_n+4c_{n+1}-y_{n+3}=0$$

coupled with $\frac{\partial}{\partial\alpha}(\alpha b_{n+1})+\bar{\beta}\alpha\frac{\partial b_n}{\partial\alpha}=0$ and using QED beta function

The first three terms are explicitly calculated as

$$\chi(D)=\frac{1}{2}+\frac{3}{4}(D-4)+\frac{1}{3}(D-4)^2+o((D-4)^3)$$

where using $\bar{\beta}=\frac{8}{3}\frac{\alpha}{4\pi}+8\left(\frac{\alpha}{4\pi}\right)^2-\frac{124}{9}\left(\frac{\alpha}{4\pi}\right)^3$

Residue of *L_b*

The simple-pole residue of *L_b is calculated as*

$$b_1 = \frac{73}{360}\frac{1}{(4\pi)^2} - \underbrace{\frac{1}{6}\frac{\alpha^2}{(4\pi)^4} + \frac{25}{108}\frac{\alpha^3}{(4\pi)^5}}_{\text{Determined by RGE}} + o(\alpha^4)$$

From direct loop calculation in curved space (see Duff)

For completion,

the simple-pole residue of *L_a is given by*

$$a_1 = -\frac{3}{20}\frac{1}{(4\pi)^2} - \frac{7}{72}\frac{\alpha}{(4\pi)^3} + o(\alpha^2)$$

Gravitational Conformal Anomalies

As mentioned before, EM tensor is bare quantity, but finite

Its trace is now described using normal products as

$$\begin{aligned}\theta &= \frac{D-4}{4}F_{0\mu\nu}F_0^{\mu\nu} + \frac{D-1}{2}E_{0\psi} + (D-4)\left[a_0 F_D + b_0 E_D\right] \\ &= \frac{\beta}{4}\left[F_{\mu\nu}F^{\mu\nu}\right] + \frac{1}{2}(D-1+\bar{\gamma}_2)\left[E_\psi\right] - \underline{\mu^{D-4}(\beta_a F_D + \beta_b E_D)}\end{aligned}$$

where

$$E_D = G_D - \underline{4\chi(D)\nabla^2 H}$$ where $H = R/(D-1)$

- GCA are unified into two forms only at all orders of perturbation
- CA are proportional to beta functions
- Familiar ambiguous $\nabla^2 R$ term is fixed completely

known as trivial conformal anomaly

6. Summary and Physical Implications

One of significant observations in this work is as follows

At the classical level,

there is some uncertainty in how to choose the combinations of 4-th order gravitational actions and their dimensionless coupling constants

At the quantum level, however,

we can fix uncertainty by imposing finiteness conditions of renormalization

This reminds us the following fact:

respect symmetry

Quantum field theory = symmetry + regularization

The action is also restricted by renormalizability!

Necessary conditions

Now a case, dimensional regularization respects diffeomorphism invariance, but not conformal invariance precisely

Nevertheless, the counterterm respects classical conformal invariance at *D=4*

Summary of The Results

Here, I consider massless QED in curved space as a prototype of conformally coupled theories

From RG analysis using dimensional regularization, the gravitational actions are determined to be only two forms: the square of Weyl tensor *F_D* and

$$G_D = G_4 + (D-4)\chi(D)H^2$$

(These reduce conf. inv. ones at *D*=4)

The conformal anomaly related to Euler term are determined to be

$$E_D = G_D - 4\chi(D)\nabla^2 H$$

↖fixed unambiguously

$$H = R/(D-1)$$

The function $\chi(D)$ can be determined order by order by solving RGE

$$\chi(D) = \frac{1}{2} + \frac{3}{4}(D-4) + \frac{1}{3}(D-4)^2 + \cdots$$

It is expected to be theory-independent, especially first two terms are confirmed for massless QCD in the same way, while for conformally-coupled scalar theory, the first term is confirmed, but not complete because of less symmetry

Physical Implications

At the D → 4 limit, the conformal anomaly E_D reduces to the form:

$$E_4 = G_4 - \underline{\frac{2}{3}\nabla^2 R}$$ where $\chi(4) = 1/2$ is used

This is just the combination proposed by Riegert in1984 in analogy with Polyakov's work. We here find that it really appears in quantum theory, which satisfies $\sqrt{g}E_4 = \sqrt{\bar{g}}(4\bar{\Delta}_4\phi + \bar{E}_4)$ where $g_{\mu\nu} = e^{2\phi}\bar{g}_{\mu\nu}$

conf. inv. 4-th order op: $\Delta_4 = \nabla^4 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\nabla^2 + \frac{1}{3}\nabla^\mu R\nabla_\mu$

Integrating *E_4* over the conformal factor, we obtain Riegert effective action

$$S_{\rm Riegert} = \int d^4x \int_0^\phi d\phi \sqrt{g}E_4 = \int d^4x \sqrt{\bar{g}}\left(\underline{2\phi\bar{\Delta}_4\phi} + \bar{E}_4\phi\right)$$

Kinetic term is induced

This is the 4-dim. version of Liouville action

⇨ Dynamics of conformal factor is induced quantum mechanically

Ex. BRST conformal invariance, Inflation

Nonlocal form is $\frac{1}{8}\int d^4x\sqrt{g}E_4\frac{1}{\Delta_4}E_4$

My Final Goal

K.H., PTP108(2002)399; arXiv:1407.xxxx

Renormalizable quantum gravity action is now determined to be

$$S = \int d^D x \sqrt{g} \left\{ \underbrace{\frac{1}{t_0^2} C_{\mu\nu\lambda\sigma}^2 + \underline{b_0 G_D}} + \frac{1}{4} F_{0\mu\nu}^2 + i\bar{\psi}_0 \gamma^\mu D_\mu \psi_0 - \frac{M_{P0}^2}{2} R + \Lambda_0 \right\}$$

Classically, no dynamics of conformal factor

(*G_D* is topological at *D=4*)

Furthermore, *b_0* is not dynamical coupling

The metric is expanded about $C_{\mu\nu\lambda\sigma} = 0$ as $g_{\mu\nu} = e^{2\phi} \left(\hat{g}_{\mu\nu} + t_0 h_{0\mu\nu} + \cdots \right)$

exactly

perturbatively

(ruled by Weyl action)

Renormalization can be carried out as follows:

$$b_0 = \frac{\mu^{D-4}}{(4\pi)^{D/2}} \sum_{n=1}^{\infty} \frac{b_n}{(D-4)^n}$$

(←pure-pole term)

Riegert action then appears in the expansion:

novel vertices and counterterms

$$b_0 \int d^D x \sqrt{g} G_D = \frac{\mu^{D-4}}{(4\pi)^{D/2}} \int d^D x \left\{ \underline{2 b_1 \phi \partial^4 \phi} + (D-4) b_1 \phi^2 \partial^4 \phi + \cdots \right\} \qquad (\hat{g}_{\mu\nu} = \text{flat})$$

Riegert action (= kinetic term of conf. mode)

Summary of Quantum Gravity Dynamics

K.H, PRD85(2012)024028; PRD86(2012)124006

- BRST conformal symmetry arises in UV limit ($t \to 0$ limit)

  = all spacetimes connecting each other under conformal transformations

  become <u>gauge-equivalent:</u> $g_{\mu\nu} \approx \Omega g_{\mu\nu}$

  = a representation of background-metric independence

  (like BRST Virasoro sym. in 2DQG)

Notice:
not a free theory
(a certain CFT)

We can go beyond the Planck scale !!

guaranteed by asymptotic-free behavior of the coupling t

- Physical (BRST inv.) states are "primary scalars" only

cf. vertex op.

  → Primordial spectrum is a scale-invariant and scalar-like

  (tensor is small of order t^2 in UV limit)

- Predicts stable inflation of Starobinsky-type

  It starts about Planck scale and ends at the IR scale $\Lambda_{QG} \simeq 10^{17}\text{Gev}$

  (This scale also explains the sharp-falloff at low multipoles of CMB)

K.H., S. Horata and T.Yukawa, PRD74(2006)123502; PRD81(2010)083533

Evolution of Universe

correlation length of RQG

$\xi_\Lambda = 1/\Lambda_{QG} \ (\gg l_{pl})$

Number of e-foldings

$$\mathcal{N}_e = \log \frac{a(\tau_\Lambda)}{a(\tau_P)} \simeq \frac{m_{pl}}{\Lambda_{QG}}$$

➔ $\boxed{\Lambda_{QG} \simeq 10^{17} \ \mathrm{GeV}}$

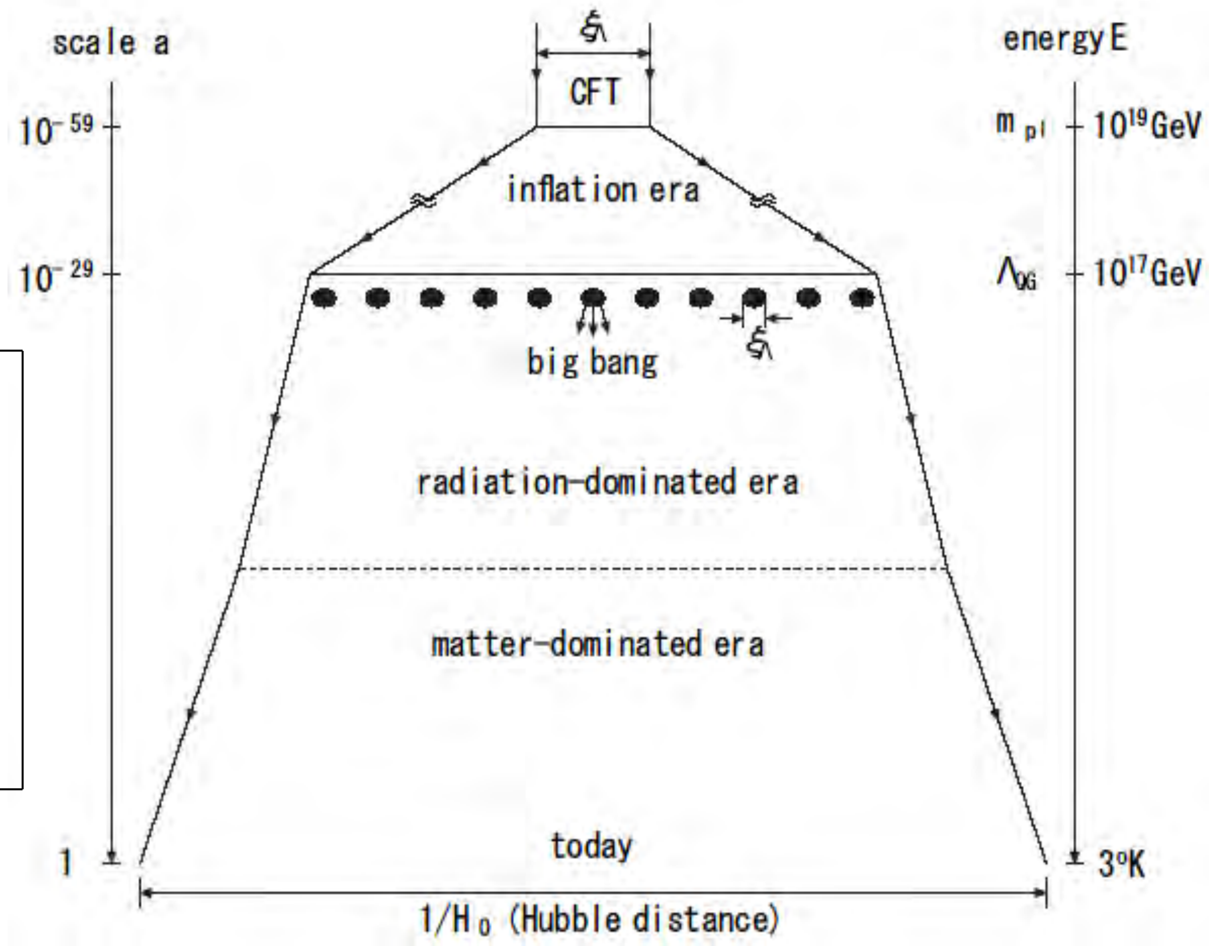
Expansion of the universe 10^{59}

Inflation era

$10^{30} \ (\Leftrightarrow \mathcal{N}_e = 70)$

Friedmann era

$10^{29} \ (\Leftrightarrow 10^{17}\mathrm{GeV}/2.7\mathrm{K})$

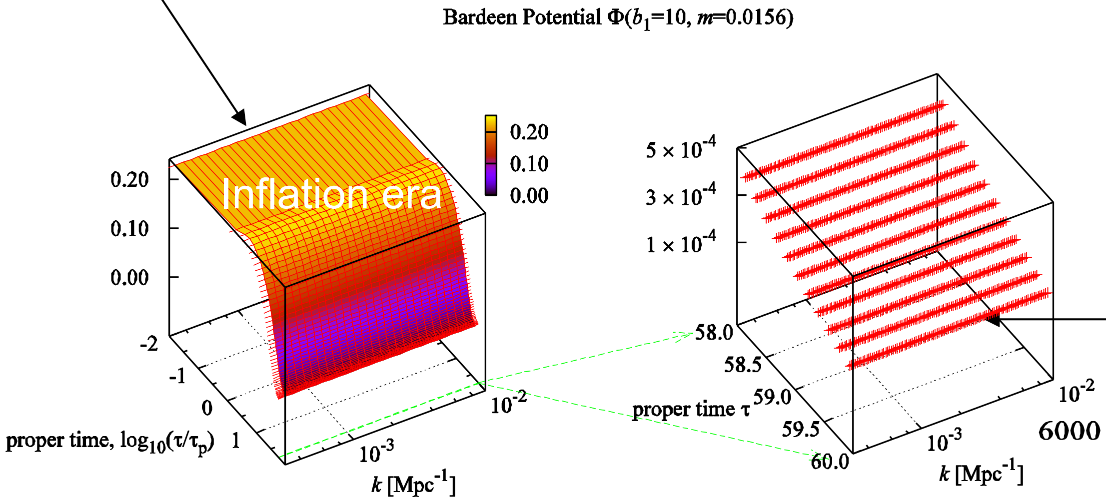


$\boxed{1/H_0 \simeq 10^{59}\xi_\Lambda}$ (~4000Mpc)

Evolution of Fluctuation (From CFT to CMB)

Planck phenomena (CFT) ➔ space-time transition（big bang）➔ today

Scale-inv. spectrum at Planck time



From Planck length to Hubble distance

$$10^{59} = 10^{30} + 10^{29}$$

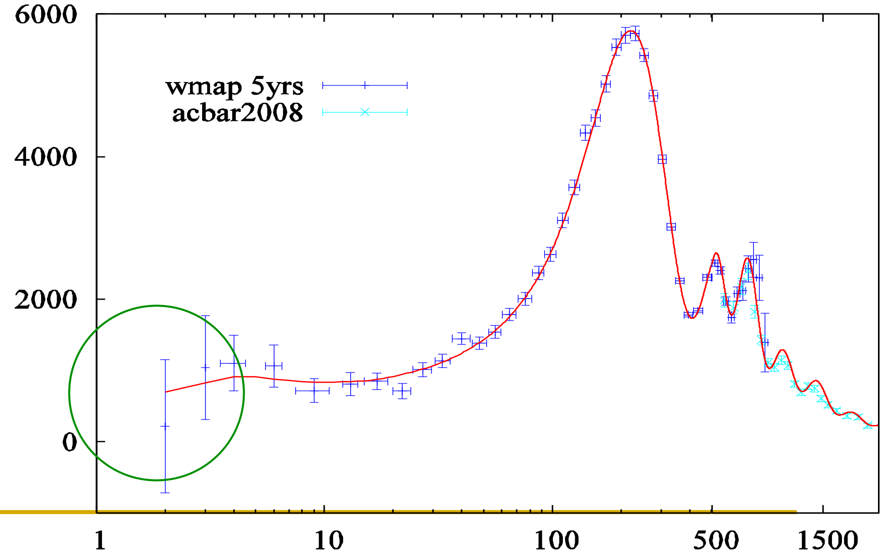
inflation　　Friedmann

Spectrum at transition point
(= almost HZ spectrum)

Linear Cosmological perturbation
for Riegert+Weyl+Einstein

Amplitude decreases, which is consistent with rough estimation: $\delta R/R \sim (\Lambda_{\rm QG}/m_{\rm pl})^2$

To fit higher multipoles, nonlinear effects may be significant (future problem)



The END