My Favorate Equations #006 :

"π" by beautiful and simple expression .

π = 3.141592653589…

$$\pi = \lim_{n\to\infty} 2^{n-2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}....}}}$$

πは不規則な数字の列で構成される典型的な無理数。しかしながら、数字ではなく、数式を用いると、非常に簡単で美しくもある式で表される。その一例を挙げてみる。

(例) 円を N 個の三角形で近似した場合：　N = 2^n として、三角関数の倍角の定理を用いると、πは、ルート、足し算・引き算・2・lim　だけで記述できる。

$$P_n \equiv \frac{N}{2}\sin(\frac{2\pi}{N})...;..N = 2^n...;...\sin 2\theta = 2\sin\theta\cos\theta... \Rightarrow \sin^2 2\theta = 4\sin^2\theta - 4\sin^4\theta$$

$$P_n \equiv \frac{2^n}{2}\sin(\frac{2\pi}{2^n}) = \frac{2^n}{2}\sin(2\frac{2\pi}{2^{n+1}}) = \frac{2^{n+1}}{2}\sin(\frac{2\pi}{2^{n+1}})\cos(\frac{2\pi}{2^{n+1}})$$

$$\frac{2^{n+1}}{2}P_n = \frac{2^{n+1}}{2}\sin(\frac{2\pi}{2^{n+1}})\frac{2^{n+1}}{2}\cos(\frac{2\pi}{2^{n+1}}) = P_{n+1}\sqrt{(\frac{2^{n+1}}{2})^2 - (\frac{2^{n+1}}{2})^2\sin^2(\frac{2\pi}{2^{n+1}})}$$

$$= P_{n+1}\sqrt{(\frac{2^{n+1}}{2})^2 - P_{n+1}^{\ 2}}$$

$$(\frac{2^{n+1}}{2})^2 P_n^{\ 2} = (\frac{2^{n+1}}{2})^2 P_{n+1}^{\ 2} - P_{n+1}^{\ 4}$$

$$P_{n+1}^{\ 4} - (\frac{2^{n+1}}{2})^2 P_{n+1}^{\ 2} + (\frac{2^{n+1}}{2})^2 P_n^{\ 2} = 0...:...P_n^{\ 2} \equiv (\frac{2^n}{2})^2 a_n...;...\sin(\frac{2\pi}{2^n}) = \sqrt{a_n}...:...P_n \equiv \frac{N}{2}\sqrt{a_n}$$

$$(\frac{2^{n+1}}{2})^4 a_{n+1}^{\ 2} - (\frac{2^{n+1}}{2})^4 a_{n+1} + (\frac{2^{n+1}}{2})^2(\frac{2^n}{2})^2 a_n = 0$$

$$a_{n+1}^{\ 2} - a_{n+1} + \frac{1}{4}a_n = 0$$

$$a_{n+1} = \frac{1-\sqrt{1-a_n}}{2}$$

$For...n \geq 2$

$$a_2 = \sin^2(\frac{2\pi}{2^4}) = 1...:...P_2 = 2$$

$$a_3 = \frac{1-\sqrt{1-a_2}}{2} = \frac{1}{2}...:...P_3 = 4\sqrt{\frac{1}{2}} = 2\sqrt{2}$$

$$a_4 = \frac{1-\sqrt{1-\frac{1}{2}}}{2} = \frac{2-\sqrt{2}}{4}...:...P_4 = 8\sqrt{\frac{2-\sqrt{2}}{4}} = 4\sqrt{2-\sqrt{2}}$$

$$a_5 = \frac{1-\sqrt{1-\frac{2-\sqrt{2}}{4}}}{2} = \frac{2-\sqrt{2+\sqrt{2}}}{4}...:...P_5 = 16\sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{4}} = 8\sqrt{2-\sqrt{2+\sqrt{2}}}$$

$$a_6 = \frac{1-\sqrt{1-\frac{2-\sqrt{2+\sqrt{2}}}{4}}}{2} = \frac{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}{4}...:...P_6 = 32\sqrt{\frac{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}{4}} = 16\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

$$\pi = \lim_{n\to\infty} 2^{n-1}\sqrt{a_n} = \lim_{n\to\infty} 2^{n-2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}....}}}$$

n	m	am	Sn	error (%)
3	2	0.7071068	2.828427125	-9.968368
4	3	0.3826834	3.061467459	-2.550464
5	4	0.1950903	3.121445152	-0.641315
6	5	0.0980171	3.136548491	-0.160561
7	6	0.0490677	3.140331157	-0.040155
8	7	0.0245412	3.141277251	-0.01004
9	8	0.0122715	3.141513801	-0.00251
10	9	0.0061359	3.14157294	-0.000627
11	10	0.003068	3.141587725	-0.000157
12	11	0.001534	3.141591422	-3.92E-05
13	12	0.000767	3.141592346	-9.8E-06
14	13	0.0003835	3.141592577	-2.45E-06
15	14	0.0001917	3.141592633	-6.41E-07
16	15	9.587E-05	3.141592655	3.876E-08
17	16	4.794E-05	3.141592645	-2.63E-07
18	17	2.397E-05	3.141592607	-1.47E-06
19	18	1.198E-05	3.141592911	8.192E-06
20	19	5.992E-06	3.141594125	4.684E-05
21	20	2.996E-06	3.141596554	0.0001241