Error of paraxial approximation

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- CSR impedance below cutoff frequency

$$k \lesssim \pi/h$$

- Not important in practical applications

$$Z_{(\mathrm{CSR})} \lesssim Z_{(\mathrm{RW})}$$

2010.11.8, KEK

Wave equation in the frequency domain

$$\left[\partial_s^2 + 2ik\partial_s + g^2\boldsymbol{\nabla}_\perp^2 + k^2\left(\beta^2 g^2 - 1\right)\right]E_x + \kappa_\rho\Big\{(g\partial_x - \kappa_\rho)E_x - 2(ik+\partial_s)E_s\Big\}$$
$$- g^{-1}(\partial_s\kappa_\rho)\Big\{x(ik+\partial_s)E_x + E_s\Big\} = g^2\partial_x J_0$$
$$\left[\partial_s^2 + 2ik\partial_s + g^2\boldsymbol{\nabla}_\perp^2 + k^2\left(\beta^2 g^2 - 1\right)\right]E_y + g\kappa_\rho\partial_x E_y - g^{-1}x(\partial_s\kappa_\rho)(ik+\partial_s)E_y = g^2\partial_y J_0$$
$$\left[\partial_s^2 + 2ik\partial_s + g^2\boldsymbol{\nabla}_\perp^2 + k^2\left(\beta^2 g^2 - 1\right)\right]E_s + \kappa_\rho\Big\{(g\partial_x - \kappa_\rho)E_s + 2(ik+\partial_s)E_x\Big\}$$
$$- g^{-1}(\partial_s\kappa_\rho)\Big\{x(ik+\partial_s)E_s - E_x\Big\} = ikg\left(1 - g\beta^2\right)J_0$$

Gauss' s law $\quad -(ik+\partial_s)E_s = g\left(\boldsymbol{\nabla}_\perp\cdot\boldsymbol{E}_\perp - J_0\right) + \kappa_\rho E_x$

Curvature $\quad \kappa_\rho = \dfrac{1}{\rho} \qquad g = 1 + \dfrac{x}{\rho}$

Parabolic equation

$$\left[2ik\partial_s + \boldsymbol{\nabla}_\perp^2 + k^2\left(\frac{2x}{\rho} - \frac{1}{\gamma^2}\right)\right]\boldsymbol{E}_\perp = \boldsymbol{\nabla}_\perp J_0$$

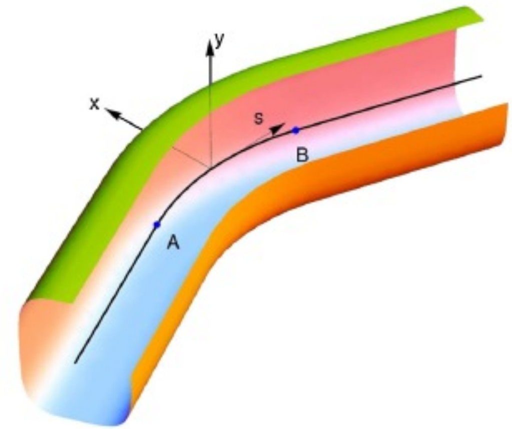
$$-ikE_s = \boldsymbol{\nabla}_\perp\cdot\boldsymbol{E}_\perp - J_0$$

Condition

$$k \gg \frac{\pi}{h}$$

cutoff frequency

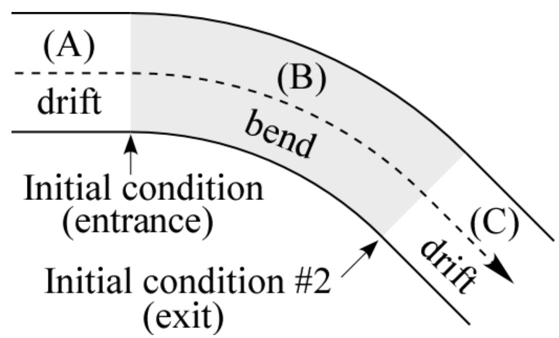
Transient field of CSR in a beam pipe



Eigenmode expansion

$$\hat{\boldsymbol{E}}_{\perp}(x,y,s) = \sum_{m,p} C_{mp}(s)\hat{\boldsymbol{E}}_{mp,\perp}(x,y,s)$$

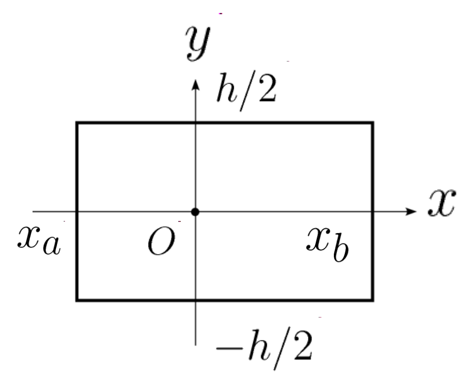
G. V. Stupakov and I. A. Kotelnikov,
PRST-AB 12, 104401 (2009)



Laplace transform

Initial value problem with respect to s

$$\left(2ik\frac{\partial}{\partial s} + \boldsymbol{\nabla}_{\perp}^2 + 2k^2\frac{x}{\rho}\theta(s)\right)\boldsymbol{E}_{\perp} = \mu_0 \boldsymbol{\nabla}_{\perp} J_0$$



$$F(r) = \int_0^{\infty} f(\varsigma)e^{-\varsigma r}d\varsigma, \quad \text{where} \quad \varsigma = \kappa_s s, \quad \kappa_s = \left(\frac{k}{2\rho^2}\right)^{1/3}$$

Generally speaking about field analysis,

Eigenmode expansion

$$\psi = \sum_n c_n |n\rangle, \quad H|n\rangle = \lambda_n |n\rangle$$

Algebraic
$<$abstract, symbolic$>$

compact & beautiful formalism
(orthonormality, completeness)

easier to describe resonant field
= eigenstate of the structure

easier to deal with on computer
= useful in practical applications

Fourier, Laplace analysis

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\frac{\partial\psi}{\partial t}$$

Analytic
$<$concrete, graphic$>$

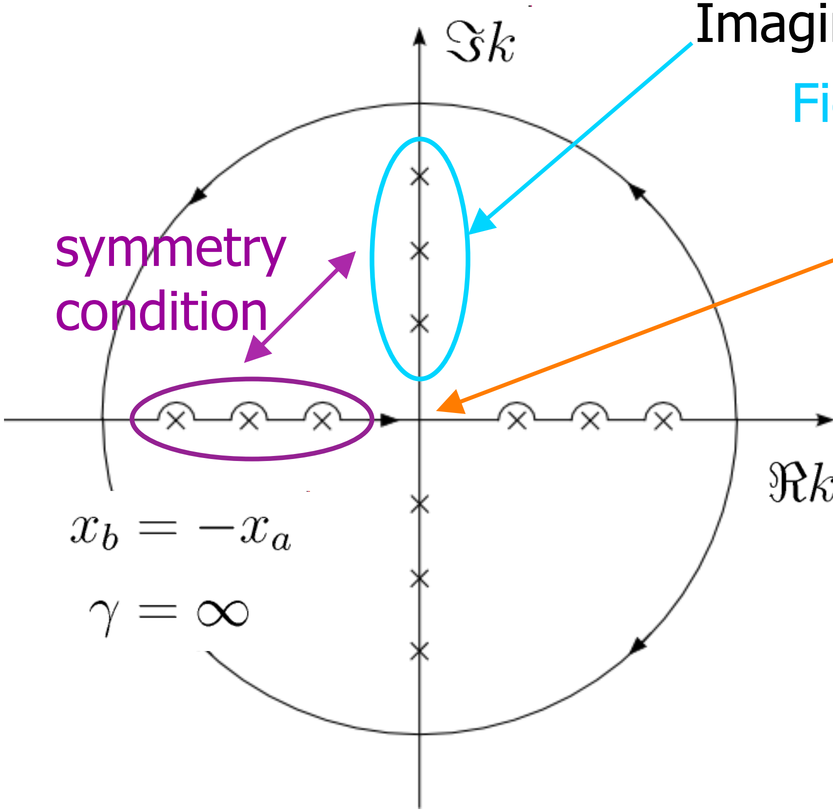
straightforward formulation
but often complex & intricate

resonance = singularity, δ -function
(special treatment needed)

easier to examine the structure
in non-resonant region

concrete and clear picture
to human brain

e.g.) Field structure of steady CSR in a perfectly conducting rectangular pipe



Imaginary poles

Field in front of the bunch $e^{-\bar{k}_m|z-z'|}$

symmetry condition

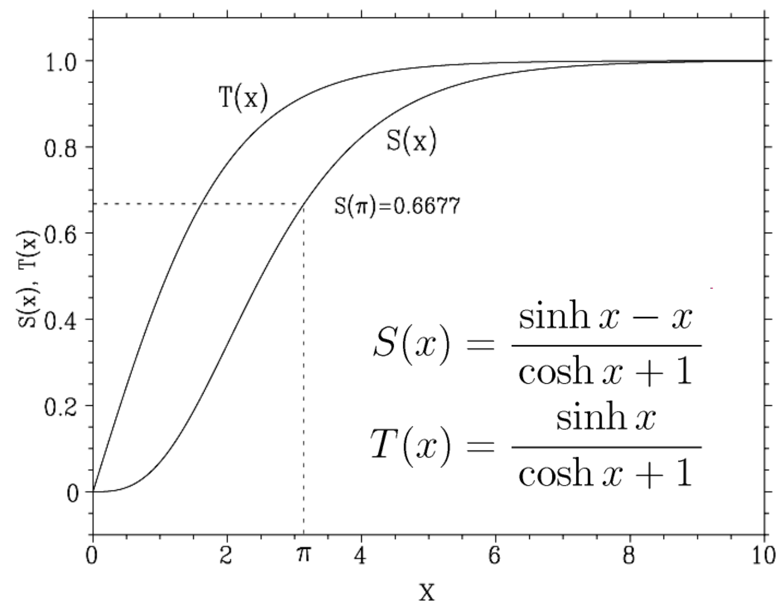
$x_b = -x_a$

$\gamma = \infty$

Regularity at the origin

No pole nor branch point

(byproduct) effect of sidewalls



$$Z_{\rm rad}(k) \simeq -iZ_0 \frac{3k^3}{2\pi\rho^2}\left(\frac{h}{\pi}\right)^4 S\left(\pi\frac{w}{h}\right) \quad (\text{if } w \gtrsim h)$$

Transient field of CSR inside magnet

- Transverse field

$$\boldsymbol{E}_\perp(\xi,\eta,k;s) = \sum_{p=0}^{\infty} \begin{pmatrix} 2\mathcal{E}_x^p(\xi,k;s)\cos(\beta_p\eta) \\ 2i\mathcal{E}_y^p(\xi,k;s)\sin(\beta_p\eta) \end{pmatrix}$$

$$\frac{\mathcal{E}_x^p(\xi,k;s)}{i\pi A_0} = \frac{\mathcal{E}_x^p(\xi;0)}{i\pi A_0} + \sum_{j=1}^{\infty}\left(1-e^{i\varrho_{jp}^x\kappa_s s}\right)\frac{\hat{r}(v_{jp}^x, w_{jp(\xi)}^x)}{\partial_\varrho \hat{s}(v_{jp}^x, u_{jp}^x)}\int_{\xi_a}^{\xi_b} d\xi' \left(\frac{\psi_x(\xi')}{\varrho_{jp}^x} - \frac{\mathcal{E}_y^p(\xi';0)}{i\beta_p A_0}\right)\hat{s}(w_{jp(\xi')}^x, u_{jp}^x)$$

$$\frac{\mathcal{E}_y^p(\xi,k;s)}{i\pi A_0} = \frac{\mathcal{E}_y^p(\xi;0)}{i\pi A_0} - \sum_{j=1}^{\infty}\left(1-e^{i\varrho_{jp}^y\kappa_s s}\right)\frac{\hat{p}(v_{jp}^y, w_{jp(\xi)}^y)}{\partial_\varrho \hat{p}(v_{jp}^y, u_{jp}^y)}\int_{\xi_a}^{\xi_b} d\xi' \frac{\mathcal{E}_y^p(\xi';0)}{i\pi A_0}\hat{p}(w_{jp(\xi')}^y, u_{jp}^y)$$

- Longitudinal field

$$E_s(\xi,\eta,k;s) = \sum_{p=0}^{\infty} 2\mathcal{E}_s^p(\xi,k;s)\cos(\beta_p\eta)$$

$$\begin{aligned}\frac{\mathcal{E}_s^p(\xi,k;s)}{\pi A_0\kappa/k} = &\sum_{j=1}^{\infty}\boxed{\left(1-e^{i\varrho_{jp}^x\kappa_s s}\right)}\frac{\hat{s}(v_{jp}^x, w_{jp(\xi)}^x)}{\partial_\varrho \hat{s}(v_{jp}^x, u_{jp}^x)}\int_{\xi_a}^{\xi_b} d\xi'\,\hat{s}(w_{jp(\xi')}^x, u_{jp}^x)\left(\frac{\psi_x(\xi')}{\varrho_{jp}^x} - \frac{\mathcal{E}_y^p(\xi';0)}{i\beta_p A_0}\right)\\ &+\beta_p^2\sum_{j=1}^{\infty}\boxed{\left(1-e^{i\varrho_{jp}^y\kappa_s s}\right)}\frac{\hat{p}(v_{jp}^y, w_{jp(\xi)}^y)}{\partial_\varrho \hat{p}(v_{jp}^y, u_{jp}^y)}\int_{\xi_a}^{\xi_b} d\xi'\,\hat{p}(w_{jp(\xi')}^y, u_{jp}^y)\left(\frac{\psi_x(\xi')}{\varrho_{jp}^y} - \frac{\mathcal{E}_y^p(\xi';0)}{i\beta_p A_0}\right)\end{aligned}$$

s-dependence is only here

s-dependence of transient CSR inside magnet

$$F(r) = \int_0^\infty f(\varsigma) e^{-\varsigma r} d\varsigma, \quad \text{where} \quad \varsigma = \kappa_s s, \quad \kappa_s = \left(\frac{k}{2\rho^2}\right)^{1/3} = (\text{formation length})^{-1}$$

Poles of $F(r)$ on the Laplace plane $r \in \mathbb{C}$

$$r = i\varrho_{jp} \quad (\varrho \in \mathbb{R} \text{ for } k \in \mathbb{R})$$

$$e^{i\varrho_{jp}\kappa_s s} \sim e^{ia_{jp}s/k} \quad \text{for } k \to 0$$

$$\because \; \varrho_{jp}\kappa_s \sim -\left\{\left(\frac{\pi j}{\kappa w}\right)^2 + \left(\frac{\alpha_p}{\kappa}\right)^2\right\}\left(\frac{k}{2\rho^2}\right)^{1/3} = -\frac{1}{2k}\left\{\left(\frac{\pi j}{w}\right)^2 + \alpha_p^2\right\} \quad (\text{for } k \to 0)$$

That is,

$$k \to 0 \quad \Rightarrow \quad s/k \to \infty \quad \Rightarrow \quad s \to \infty$$

(low freq. limit) (steady field)

frozen

in the framework of the paraxial approximation.

Assumption: This fact must hold also in the exact Maxwell theory.

- Steady field a in perfectly conducting rectangular pipe

$$Z(k) = -iZ_0 \frac{2\pi}{\beta h} \sum_{p=0}^{\infty} \Lambda_p \left[\frac{\check{s}(v_p, w_p)\check{s}(w_p, u_p)}{\check{s}(v_p, u_p)} + \beta_p^2 \frac{\check{p}(v_p, w_p)\check{p}(w_p, u_p)}{\check{p}(v_p, u_p)} \right] \quad [\Omega/\mathrm{m}]$$

(no periodicity)

↓ Asymptotic expansion $k \to 0$

$$Z(k) = \frac{iZ_0}{\beta h} \sum_{p=0}^{\infty} \Lambda_p \left[\frac{k}{\alpha_p \gamma^2} T(\alpha_p w) - \frac{k^3}{2\rho^2 \alpha_p^5} S(\alpha_p w) \right]$$

contradiction

↑ Olver expansion $k > \alpha_p/\beta$ (para-approx.)

- Exact solution for periodic system

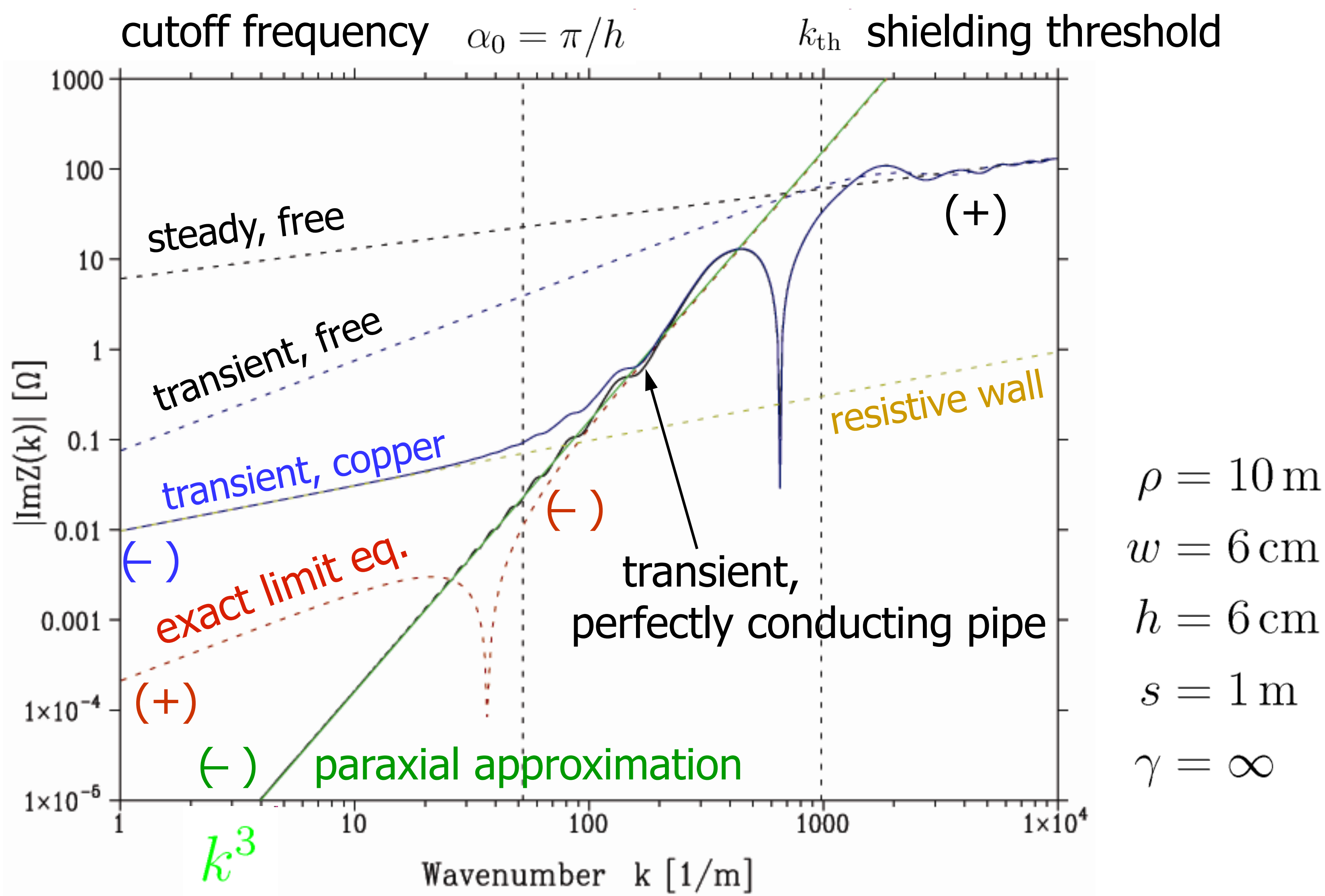
$$\frac{Z(n,\omega)}{n} = i \frac{\pi Z_0}{\beta h} \sum_{p=0}^{\infty} \Lambda_p \left[\beta^2 \frac{s_n(\gamma_p \rho_b, \gamma_p \rho) s_n(\gamma_p \rho, \gamma_p \rho_a)}{s_n(\gamma_p \rho_b, \gamma_p \rho_a)} + \frac{\alpha_p^2}{\gamma_p^2} \frac{p_n(\gamma_p \rho_b, \gamma_p \rho) p_n(\gamma_p \rho, \gamma_p \rho_a)}{p_n(\gamma_p \rho_b, \gamma_p \rho_a)} \right]$$

↓ Debye expansion $k < \alpha_p/\beta$

$$Z(k) = \frac{iZ_0}{\beta h} \sum_{p=0}^{\infty} \Lambda_p \left[\left(\frac{k}{\alpha_p \gamma^2} + \frac{k}{2\rho^2 \alpha_p^3} \right) T(\alpha_p w) - \frac{k^3}{2\rho^2 \alpha_p^5} S(\alpha_p w) \right] \quad [\Omega/\mathrm{m}]$$

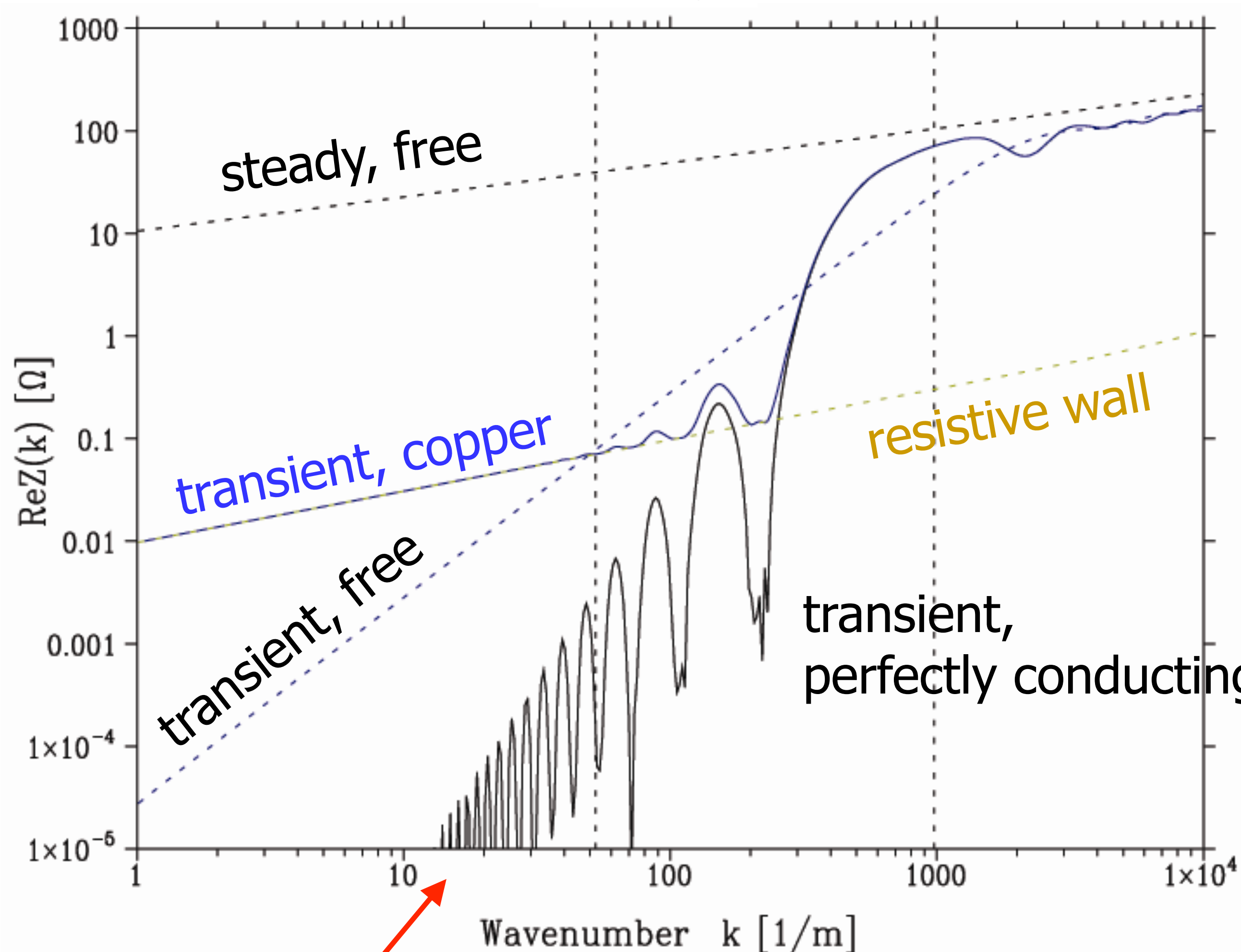
no longer periodic

Imaginary impedance



Real impedance

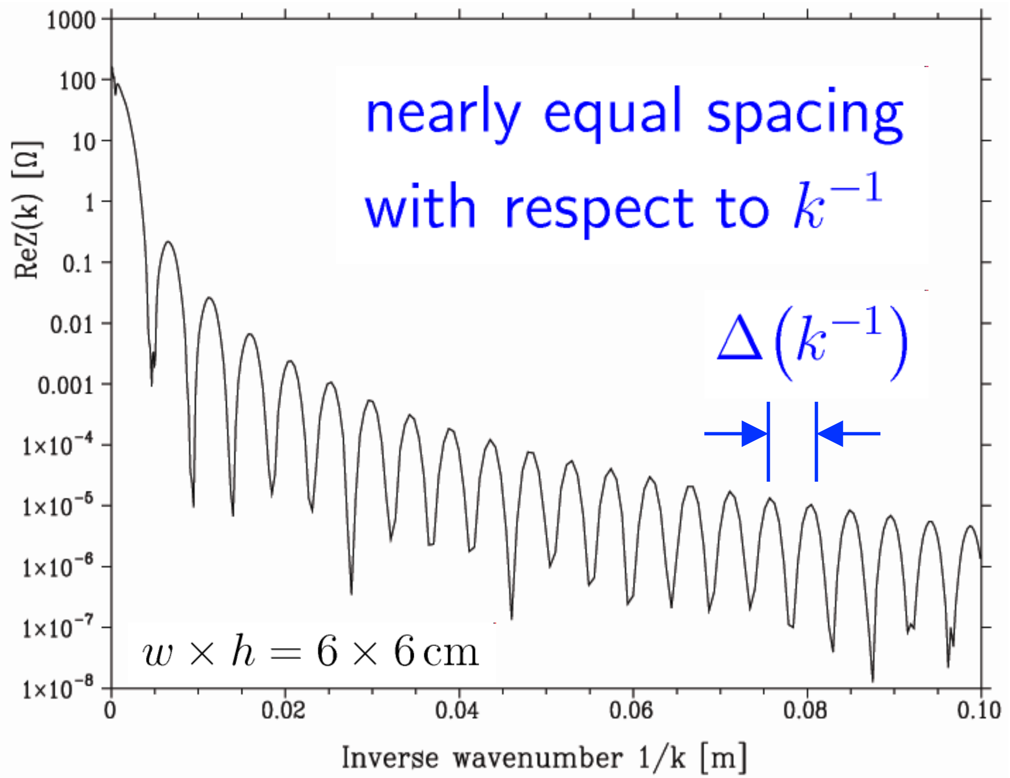
cutoff frequency $\alpha_0 = \pi/h$ k_{th} shielding threshold

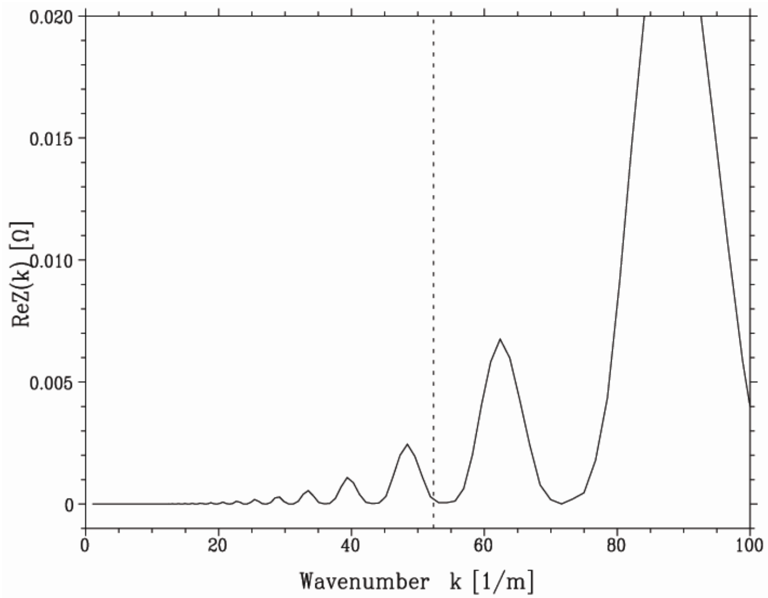


strange structure

The lowest resonance $k_0 = 1029\,[1/\mathrm{m}]$

Fluctuation of impedance at low frequency





$$e^{i\varrho_{jp}\kappa_s s} \sim e^{ia_{jp}s/k} \quad \text{for } k \to 0$$

$$a_{jp} = -\frac{1}{2}\left\{\left(\frac{\pi j}{w}\right)^2 + \alpha_p^2\right\}$$

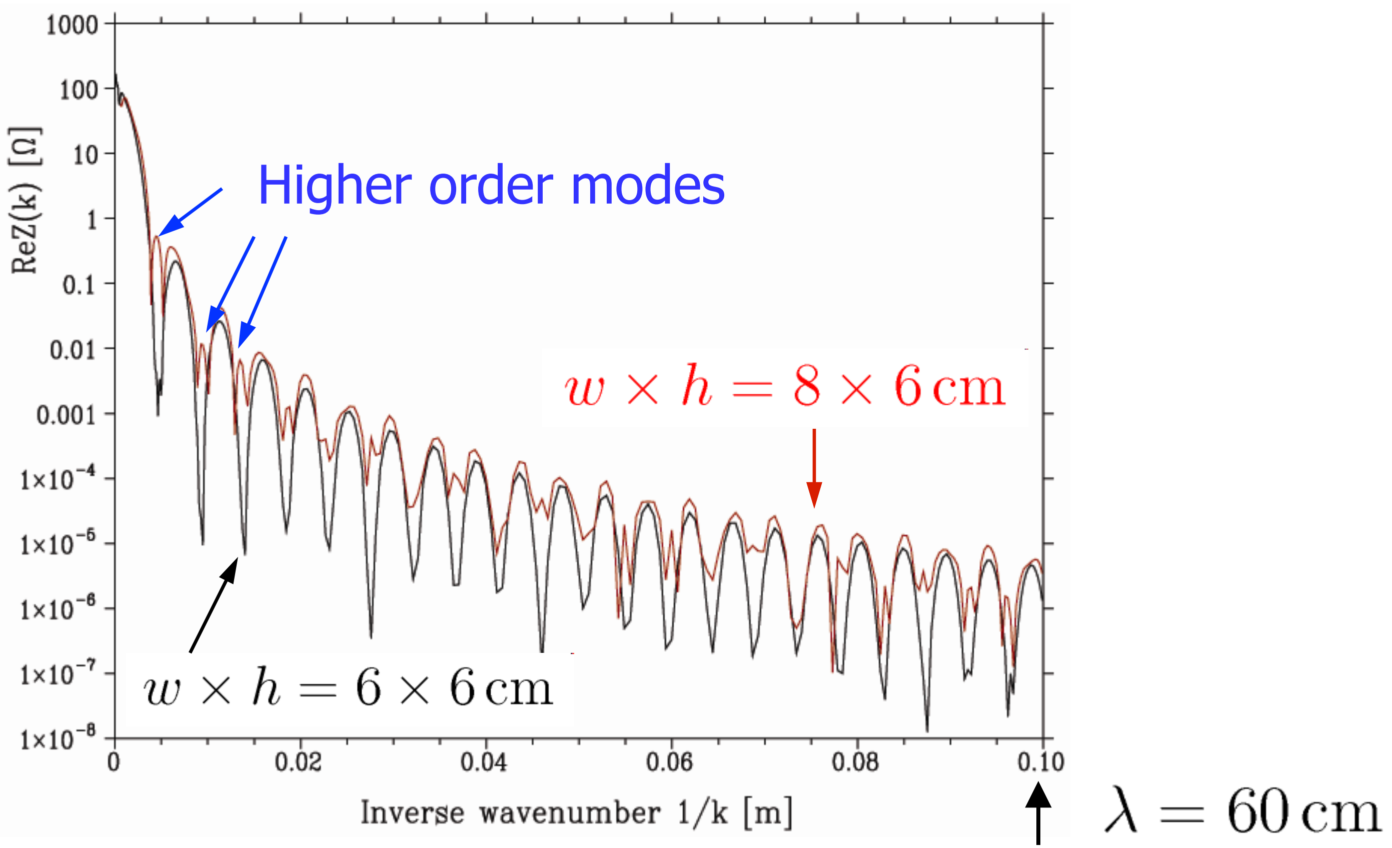
Fundamental mode $(j,p) = 0$ is dominant

Fluctuation period w.r.t. k^{-1}

$$\Delta\left(k^{-1}\right) = \frac{4\pi}{s\alpha_0^2} = \frac{4h^2}{\pi s}$$

agreement with grid calculation

$$\Delta\left(k^{-1}\right) = \begin{cases} 4.58 \times 10^{-3} & \text{(theory)} \\ 4.61 \times 10^{-3} & \text{(simulation)} \end{cases}$$



Q) What's the physical sense of this structure?

or, error of the paraxial approximation? $\partial_s^2 + 2ik\partial_s$

or, due to some assumption? $1/\rho(s) = \theta(s)/\rho_0$

Conventional parabolic equation

$$\left[2ik\partial_s + \boldsymbol{\nabla}_\perp^2 + k^2\left(\frac{2x}{\rho} - \frac{1}{\gamma^2}\right)\right]\boldsymbol{E}_\perp = \boldsymbol{\nabla}_\perp J_0$$

$$-ikE_s = \boldsymbol{\nabla}_\perp \cdot \boldsymbol{E}_\perp - J_0$$

Modified parabolic equation

$$\left[2ik\partial_s + g^2\boldsymbol{\nabla}_\perp^2 + k^2\left(\beta^2 g^2 - 1\right)\right]\boldsymbol{E}_\perp - \frac{2ik}{\rho}E_s\boldsymbol{e}_x = g^2\boldsymbol{\nabla}_\perp J_0$$

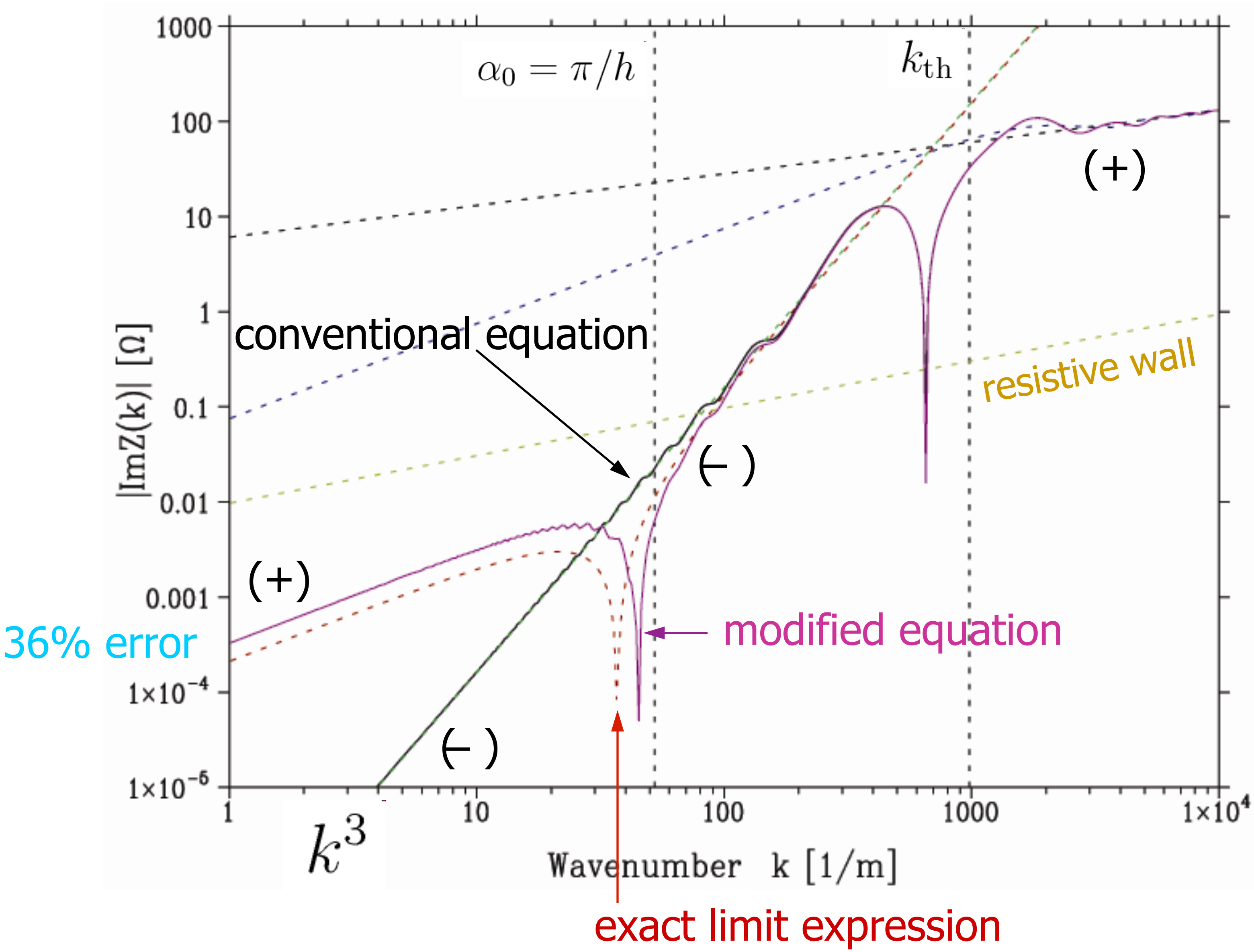
$$-ikE_s = g\left(\boldsymbol{\nabla}_\perp \cdot \boldsymbol{E}_\perp - J_0\right)$$

Curvature factor $g = 1 + \dfrac{x}{\rho}$

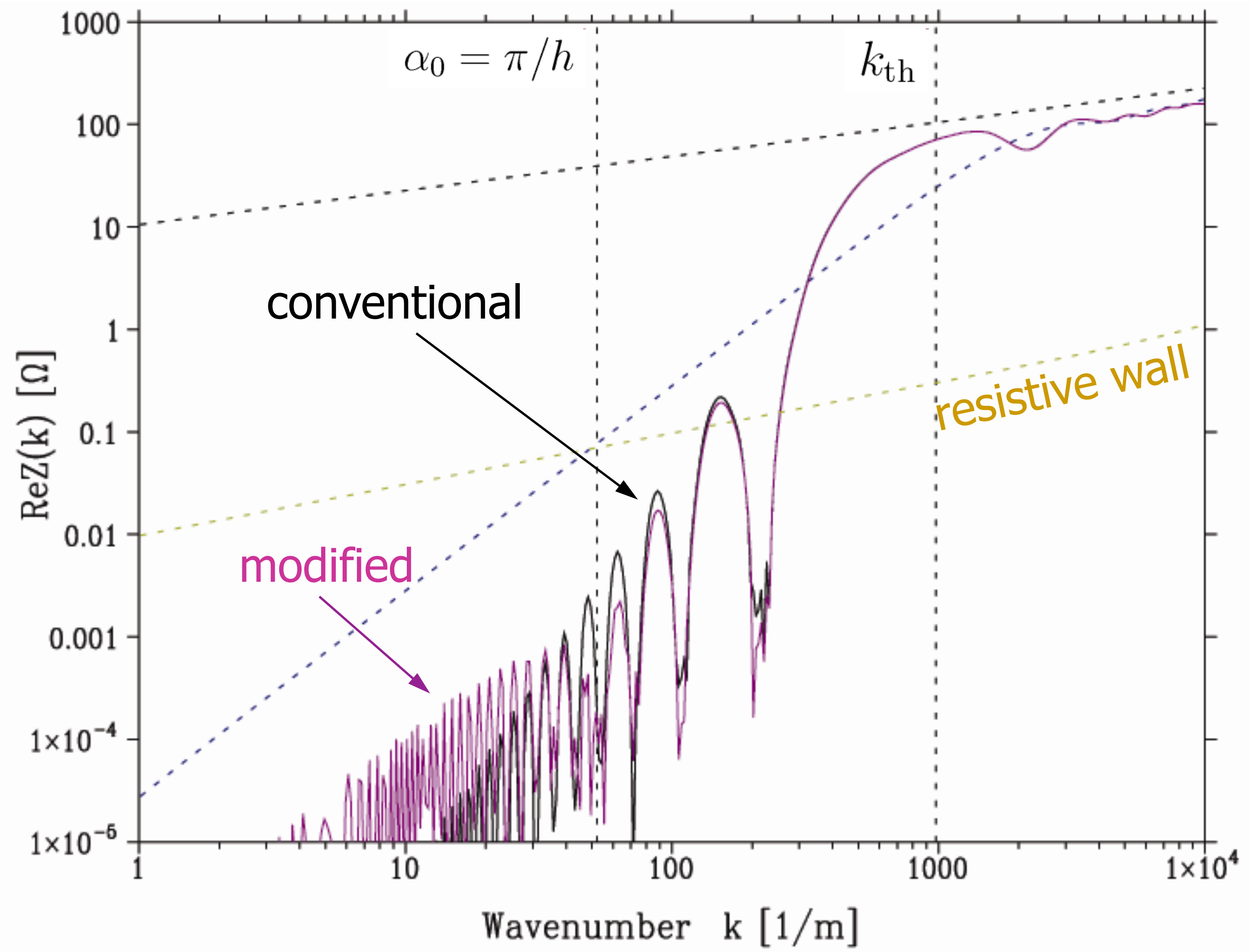
radiation condition

$$\beta g > 1 \;\Rightarrow\; \beta\frac{\rho_b}{\rho} > 1$$

Imaginary impedance in a perfectly conducting pipe

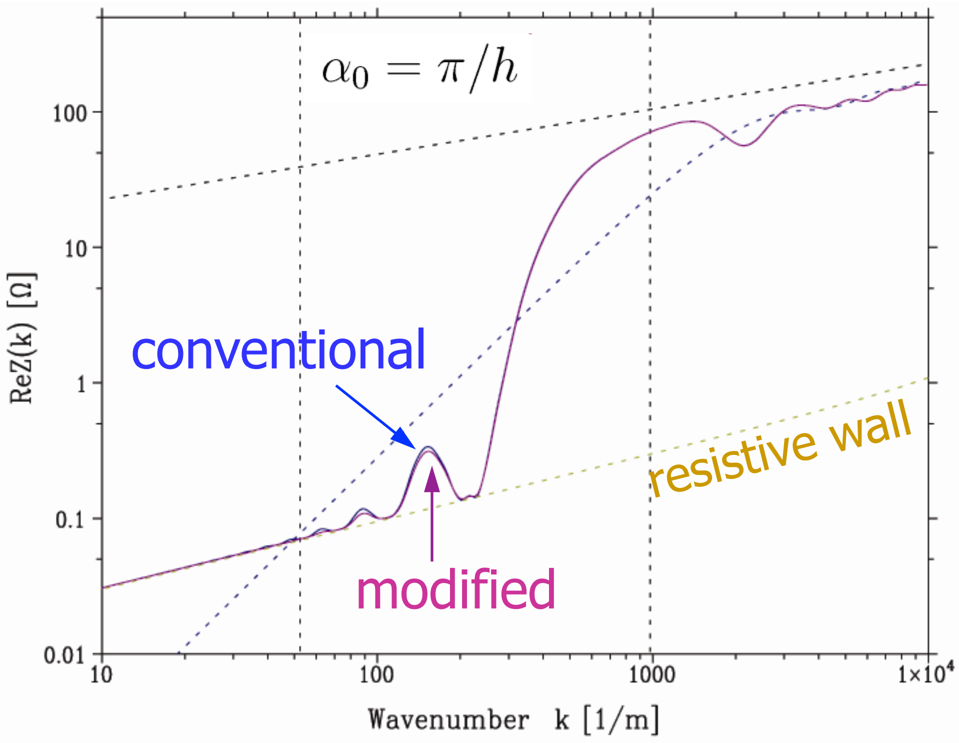


Real impedance in a perfectly conducting pipe

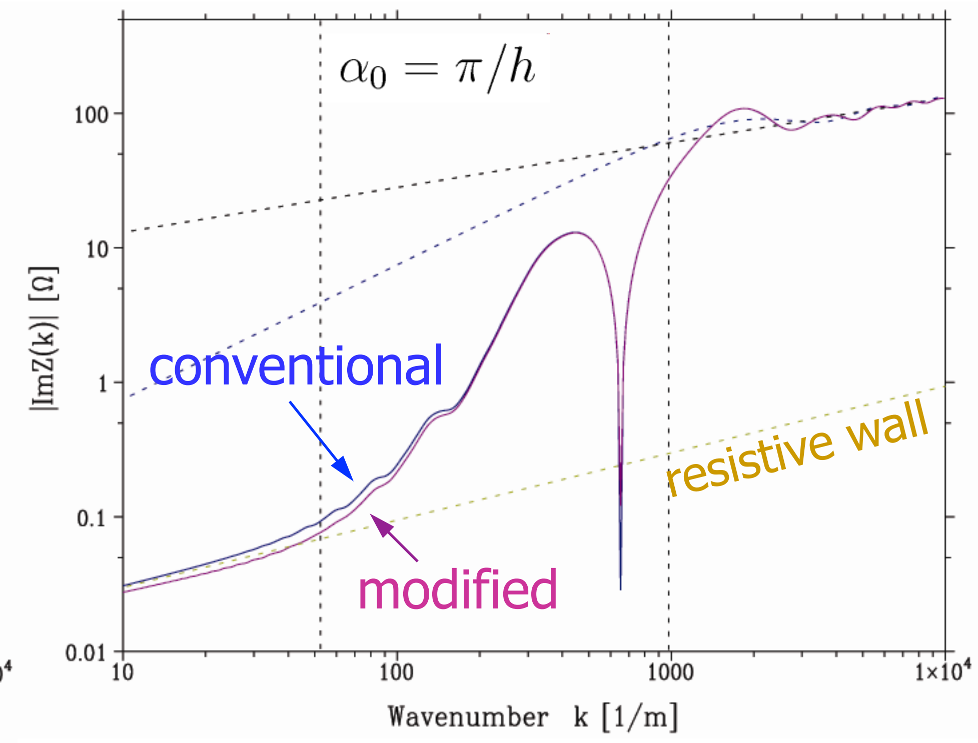


CSR in a resistive pipe

Real part



Imaginary part



The difference is almost buried in the resistive wall impedance.

somewhat different (~20%) around the cutoff wavenumber

Summary

- Modified parabolic equation

$$\left[2ik\partial_s + g^2\nabla_\perp^2 + k^2\left(\beta^2 g^2 - 1\right)\right]\boldsymbol{E}_\perp - \frac{2ik}{\rho}E_s\boldsymbol{e}_x = g^2\nabla_\perp J_0$$

$$-ikE_s = g\left(\nabla_\perp \cdot \boldsymbol{E}_\perp - J_0\right) \qquad g = 1 + \frac{x}{\rho}$$

- The imaginary impedance for this parabolic equation has better behavior than the conventional equation.
- However, the error is still large (~40%) below the cutoff wavenumber, compared to the exact equation.
- The impedance has strange structure in transient state, though the picture is not clear.