A Calculation of CSR

K. Oide

Nov 8, 2010
Mini CSR Workshop @ KEK
**MAXWELL’S EQUATIONS**

\[ \begin{align*}
\frac{1}{r} \frac{\partial r E_{\phi}}{\partial r} - \frac{1}{r} \frac{\partial E_r}{\partial \phi} &= -\frac{\partial B_y}{\partial t} \\
\frac{1}{r} \frac{\partial E_r}{\partial \phi} - \frac{\partial E_{\phi}}{\partial y} &= -\frac{\partial B_y}{\partial t} \\
\frac{\partial E_r}{\partial y} - \frac{\partial E_{\phi}}{\partial r} &= -\frac{\partial B_y}{\partial t} \\
\frac{1}{r} \frac{\partial r B_{\phi}}{\partial r} - \frac{1}{r} \frac{\partial B_r}{\partial \phi} &= \mu_0 j_y + \frac{1}{c^2} \frac{\partial E_y}{\partial t} \\
\frac{1}{r} \frac{\partial B_y}{\partial \phi} - \frac{\partial B_{\phi}}{\partial y} &= \mu_0 j_r + \frac{1}{c^2} \frac{\partial E_r}{\partial t} \\
\frac{\partial B_r}{\partial y} - \frac{\partial B_y}{\partial r} &= \mu_0 j_\phi + \frac{1}{c^2} \frac{\partial E_\phi}{\partial t} \\
\frac{1}{r} \frac{\partial r E_r}{\partial r} + \frac{1}{r} \frac{\partial E_{\phi}}{\partial \phi} + \frac{\partial E_y}{\partial y} &= \frac{\rho}{\varepsilon_0} \\
\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial r E_r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 E_r}{\partial \phi^2} + \frac{\partial^2 E_r}{\partial y^2} - \frac{1}{c^2} \frac{\partial^2 E_r}{\partial t^2} - \frac{2}{r^2} \frac{\partial E_\phi}{\partial \phi} &= \frac{1}{\varepsilon_0} \frac{\partial \rho}{\partial r} \\
\frac{\partial}{\partial \phi} \left( \frac{1}{r} \frac{\partial r E_{\phi}}{\partial \phi} \right) + \frac{1}{r^2} \frac{\partial^2 E_{\phi}}{\partial \phi^2} + \frac{\partial^2 E_\phi}{\partial y^2} - \frac{1}{c^2} \frac{\partial^2 E_\phi}{\partial t^2} + \frac{2}{r^2} \frac{\partial E_r}{\partial \phi} &= \frac{1}{\varepsilon_0} \left( \frac{1}{r} \frac{\partial \rho}{\partial \phi} + \frac{1}{c} \frac{\partial \rho}{\partial t} \right)
\end{align*} \]

\[ j_r = j_y = 0, \quad j_\phi = \rho c \]
MAXWELL’S EQUATIONS

\[
\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial r E_r}{\partial r} + \frac{1}{r^2} \frac{\partial^2 E_r}{\partial \phi^2} + \frac{\partial^2 E_r}{\partial y^2} - \frac{1}{c^2} \frac{\partial^2 E_r}{\partial t^2} - \frac{2}{r^2} \frac{\partial E_\phi}{\partial \phi} = \frac{1}{\varepsilon_0} \frac{\partial \rho}{\partial r}
\]

\[
\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial r E_\phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 E_\phi}{\partial \phi^2} + \frac{\partial^2 E_\phi}{\partial y^2} - \frac{1}{c^2} \frac{\partial^2 E_\phi}{\partial t^2} + \frac{2}{r^2} \frac{\partial E_r}{\partial \phi} = \frac{1}{\varepsilon_0} \left( \frac{1}{r} \frac{\partial \rho}{\partial \phi} + \frac{1}{c} \frac{\partial \rho}{\partial t} \right)
\]

\[
\rho \propto \delta(r - R) \delta(y) \exp \left( ik(R\phi - ct) \right)
\]

\[
E_{r,\phi} = (i\overline{E}_r(\phi), \overline{E}_\phi(\phi)) \exp \left( ik(R\phi - ct) \right)
\]

\[
\overline{E}_r = \overline{E}_r + \overline{E}_{r0},
\]

\[
\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial r \overline{E}_{r0}}{\partial r} + \frac{\partial^2 \overline{E}_{r0}}{\partial y^2} = \frac{1}{\varepsilon_0} \frac{\partial \rho}{\partial r}
\]

★ Ignore \( \frac{\partial^2 E}{\partial \phi^2} \) terms (Stupakov-Agoh-Yokoya)
MAXWELL’S EQUATIONS

Then we obtain the first order differential equations for \( \overline{E}_{r,\phi} \).

\[
\begin{align*}
\frac{\partial \overline{E}_r}{\partial \phi} &= \frac{i}{2(k^2 R^2 - 1)} \left[ kR \left( (k^2(r^2 - R^2) + 1) (\overline{E}_r + \overline{E}_{r0}) + r \frac{\partial}{\partial r} (\overline{E}_r + \overline{E}_{r0}) + r^2 \left( \frac{\partial^2 \overline{E}_r}{\partial r^2} + \frac{\partial^2 \overline{E}_r}{\partial y^2} \right) \right) \\
&\quad + (k^2(r^2 + R^2) - 1) \overline{E}_\phi + r \frac{\partial \overline{E}_\phi}{\partial r} + r^2 \left( \frac{\partial^2 \overline{E}_\phi}{\partial r^2} + \frac{\partial^2 \overline{E}_\phi}{\partial y^2} \right) \right]\end{align*}
\]

\[
\begin{align*}
\frac{\partial \overline{E}_\phi}{\partial \phi} &= \frac{i}{2(k^2 R^2 - 1)} \left[ kR \left( (k^2(r^2 - R^2) + 1) \overline{E}_\phi + r \frac{\partial \overline{E}_\phi}{\partial r} + r^2 \left( \frac{\partial^2 \overline{E}_\phi}{\partial r^2} + \frac{\partial^2 \overline{E}_\phi}{\partial y^2} \right) \right) \\
&\quad + (k^2(r^2 + R^2) - 1) (\overline{E}_r + \overline{E}_{r0}) + r \frac{\partial}{\partial r} (\overline{E}_r + \overline{E}_{r0}) + r^2 \left( \frac{\partial^2 \overline{E}_r}{\partial r^2} + \frac{\partial^2 \overline{E}_r}{\partial y^2} \right) \right]\end{align*}
\]
SOLVER

\[ \frac{d f}{d \phi} = Af + b , \quad f = (E_r, E_\phi) , \]

\[ f(\phi) = f_0 \exp(A\phi) + b \int_{0}^{\phi} \exp (A(\phi' - \phi)) \, d\phi' \]

\( A \): Spatial differentiation matrix with boundary conditions

\( b \): Driving source term by \( E_{r0} \).

The exponent is evaluated by the eigen system of \( A \).

The cross section of the beam pipe must be uniform along \( \phi \).

The mesh size for \( A \) is varied with \( k \) under the condition:

\[ (\Delta x, \Delta y) = \left( \frac{R/k^2}{{M_x, M_y}} \right)^{1/3} , \quad M_{x,y} \gtrsim (4, 1) \]
Implementation of the boundary condition

\[ f_{\text{boundary}} = 0 : \quad f_{i+1} = -f_{i-1}, \quad f_i'' = \frac{f_{i-1} - 3f_i}{2} \]

\[ f'_{\text{boundary}} = 0 : \quad f_{i+1} = f_i, \quad f_i'' = \frac{f_{i-1} - f_i}{2} \]

• Are these right choice?
Results for KEKB antechamber

Pipe height = 90 mm, Pipe width = 184 mm,
TiN thickness = .2 μm, TiN Cond. = 1.4 (μΩm)^{-1},
Maximum k = 3.5/σ_z, # of k = 32, Mesh Ratio = {4, 1}, σ_z = .3 mm

ρ = 16.3 m, L = 0.89 m, ∞ drift, 11/7/2010
Some eigen modes

\[ \rho = 16.3 \text{ m}, \ k = 10 \text{ /mm}, 11/7/2010 \]

\[ w = 187 \text{ /mm} \]

\[ w = 521 \text{ /mm} \]

\[ w = 595 \text{ /mm} \]

\[ w = 940 \text{ /\mu m} \]
Results with an asymmetric pipe

- $\Delta s$ and $\Delta k$ agree with the path difference (next page) between the reflection.

- Also the modulation ratio of $Z$ roughly agrees with the ratio of interference lengths:

$$L/(2L_0 - L) = 2.55/(82 \times 4 - 2.55) = 0.47$$

$\rho = 16.3$ m, $L = 4$ m, $\infty$ drift, 11/7/2010
\[ \rho = 16.3 \text{ m}, w = 100 \text{ mm} : \]
\[ \downarrow \]
\[ L = 2.55 \text{ m}, \]
\[ \Delta s = 5.2 \text{ mm}, \]
\[ \Delta k = \frac{2\pi}{\Delta s} = 1210 \text{ m}^{-1} \]

\[ \theta \approx \tan^{-1} \sqrt{\frac{w}{\rho}} \]
\[ L = 2\rho \theta \approx 2\sqrt{\rho w} \]
\[ \Delta s = 2\rho (\tan \theta - \theta) \approx 2\rho \frac{\theta^3}{3} \approx \frac{2w^{3/2}}{3\rho^{1/2}} \]
Unphysical results with a round pipe

- Converges to an unphysical result, even for $M \geq 128$.
- The reason has not been identified.

$\rho = 16.3 \text{ m}, L = 4 \text{ m}, \infty \text{ drift, 11/7/2010}$
Discussions

• The eigen mode method may have some merits:
  • Capability to handle arbitral shape of the beam pipe.
  • Saving computation for a repetitive arrangement.

• But it has demerits:
  • Heavy computation, if finer mesh is necessary.
  • Not suitable for varying cross section.