CSR Calculation in Paraxial Approximation

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CSR mini Workshop

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Derivation

- Geometry
  - Use (x,y,s) coordinate
    (to make the source trajectory on grid)
  - Bending in x-plane

$$d\boldsymbol{r} = g(x,s)\boldsymbol{e}_s ds + \boldsymbol{e}_x dx + \boldsymbol{e}_y dy, \qquad g(x,s) = 1 + \frac{x}{\rho(s)}$$

- Approximation
  - Ignore d^2/ds^2 (paraxial approx.)
  - Other approx.
    - $\gamma \rightarrow$ infinite
    - a : typical aperture

$$\epsilon \equiv \sqrt{\frac{a}{\rho}} \ll 1$$

- Transverse component of Maxwell equation

$$\nabla(\nabla\cdot\boldsymbol{E}) - \nabla\times(\nabla\times\boldsymbol{E}) - \frac{\partial^2}{\partial t^2}\boldsymbol{E} = \nabla\rho_e + \frac{\partial \boldsymbol{J}}{\partial t}$$

- becomes

$$\frac{\partial}{\partial x}\left(\frac{1}{g}\frac{\partial(gE_x)}{\partial x}\right) + \frac{\partial^2 E_x}{\partial y^2} + \frac{1}{g}\frac{\partial}{\partial s}\left(\frac{1}{g}\frac{\partial E_x}{\partial s}\right) - \frac{\partial^2}{\partial t^2}E_x + \frac{2}{\rho g^2}\frac{\partial E_s}{\partial s} - \frac{E_s}{g}\frac{\partial}{\partial s}\left(\frac{1}{\rho g}\right) = \frac{\partial \rho_e}{\partial x} + \frac{\partial}{\partial t}J_x$$

$$\frac{1}{g}\frac{\partial}{\partial x}\left(g\frac{\partial E_y}{\partial x}\right) + \frac{\partial^2 E_y}{\partial y^2} + \frac{1}{g}\frac{\partial}{\partial s}\left(\frac{1}{g}\frac{\partial E_y}{\partial s}\right) - \frac{\partial^2}{\partial t^2}E_y = \frac{\partial \rho_e}{\partial y} + \frac{\partial}{\partial t}J_y$$

- Ignoring some small terms in ε and assuming

$$\boldsymbol{E} \propto e^{-ik(t-s)}$$

$$2ik\frac{\partial E_x}{\partial s} + \Delta_\perp E_x + \frac{2k^2 x}{\rho} E_x = \frac{\partial \rho_e}{\partial x} - ikZ_0 J_x - \frac{\partial^2 E_x}{\partial s^2} + x\left(\frac{\partial}{\partial s}\frac{1}{\rho}\right)\frac{\partial E_x}{\partial s} + \left(\frac{\partial}{\partial s}\frac{1}{\rho}\right) E_s - \frac{2}{\rho}\frac{\partial E_s}{\partial s}$$

$$2ik\frac{\partial E_y}{\partial s} + \Delta_\perp E_y + \frac{2k^2 x}{\rho} E_y = \frac{\partial \rho_e}{\partial y} - ikZ_0 J_y - \frac{\partial^2 E_y}{\partial s^2} - \frac{1}{g}\frac{\partial g}{\partial s}\left(ikE_y + \frac{\partial E_y}{\partial s}\right)$$

- Terms on the r.h.s. are small except

$$x\left(\frac{\partial}{\partial s}\frac{1}{\rho}\right)\frac{\partial E_x}{\partial s}$$

- This causes delta function at magnet edges
- For finite edge length, the jump of E_x is

$$\delta E_x \sim \frac{\epsilon^2}{k \cdot l_{edge}} E_x, \qquad (l_{edge} = \text{edge length})$$

- This term can be ignored if the bunch length is comparable or shorter than the edge length
- One this term is ignored, other terms are smooth and does not cause problem with hard edge magnet

- Ignoring second derivative in s (paraxial approx) we get

$$\frac{\partial}{\partial s}\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \frac{i}{2k}\left[\left(\Delta_\perp + \frac{2k^2 x}{\rho}\right)\begin{pmatrix} E_x \\ E_y \end{pmatrix} - Z_0 \begin{pmatrix} \partial\rho_e/\partial x \\ \partial\rho_e/\partial y \end{pmatrix}\right]$$

- Longitudinal field can be calculated by

$$E_s = \frac{i}{k}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}\right) - \frac{i}{k} Z_0 J_s$$

Integral in infinite exit pipe

- In the exit pipe (E_x,E_y) satisfies

$$\frac{dF}{ds} = \frac{i}{2k} M F, \qquad M = \Delta_\perp$$

- This can be solved as

$$F(s) = \exp\left(\frac{i}{2k} M (s - s_{exit})\right) F(s_{exit})$$
$$\int_{s_{exit}}^{\infty} F(s) ds = 2ikM^{-1} F(0)$$

- Actually, M^{-1} is not needed (solve linear equation)
- Integral of E_s can be calculated from this integral using the relation between (E_x,E_y) and E_s

Advantages of Paraxial Approx

- Mesh size can be independent of the wavelength 1/k
- Can be solved as an initial value problem in s
  - Oppositely traveling wave ignored → information one way
  - Only (Ex,Ey) is needed at entrance (no derivatives needed)
  - Analytic treatment also easier
- Can easily include resistive wall effect

Agreement with Theories

- Agree with Derbenev-Shiltsev formula in vacuum (simulation and analytically)
- Agree with Saldin et.al. for finite length magnet (simulation)
- Agree with Warnock's formula for infinite parallel plates (analytically)
- Agreement of troidal chamber formula has not checked yet (→ Agoh)

What has not been successful

- Method to calculate the transverse wake
  - It seems more careful treatment near the charge is needed
- Going back to the time domain and solve it by mesh