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HAMILTONIAN APPROACH TO DISTORTION FUNCTIONS

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ABSTRACT

The distortion functions as introduced by Collins are derived using the canonical Hamiltonian formalism. Beam shape distortions in the horizontal and vertical phase spaces due to skew quadrupoles, normal and skew sextupoles and normal and skew octupoles are in turn calculated in terms of these distortion functions. The lowest nonvanishing contributions to the tuneshifts introduced by the above multipoles are also computed analytically. Finally applications demonstrate the degree to which the above calculations agree with experimental data.

CONTENTS

1 Introduction

No machine is perfectly linear. There are systematic sextupole components in dipole fields from steel saturation, remanent fields, persistent currents, eddy currents, and random sextupole components due to field errors. Of course, there are also sextupoles placed around the ring on purpose to counteract the above and to modify chromaticity. Higher multipoles are also possible; for example, the octupole components from beam-beam collision. The theory therefore becomes nonlinear. This does not mean though that we lose all our prediction of the beam shape by the beta functions. For a large size storage ring, the need for sophisticated diagnosis of minor faults demands a rational beam behavior. Such rational behavior is also required for a beam pipe of small bore so that the magnet size and consequently the cost can be reduced. All these imply a machine that is as linear as possible. As a result, perturbation theory can be used away from resonances. Collins[1] has proposed a set of distortion functions for each order of the perturbation. These distortion functions are closed, i.e., periodic. They are independent of the beam amplitude and are very similar to the beta functions and alpha functions of the linear theory. Of course, the beam profile is not so simple now, because horizontal and vertical motions are coupled together. So it no longer manifests itself as an ellipse in each transverse phase space. Instead, it becomes a four dimensional hyper-egg and we can only talk about its projections onto the transverse phase planes. However, these distortion functions can give us the exact projections. They can also give us two important numbers: the transverse betatron tuneshifts $\Delta\nu_x$ and $\Delta\nu_y$.

In sections 2, 3, 4, 5 and 6 we shall derive the beam-shape distortion and the tuneshifts due to skew quadrupoles, sextupoles, skew sextupoles, octupoles and skew octupoles, using the Hamiltonian approach. In the derivation we follow exactly the same ideas used by Ng[2] for the case of sextupoles. So this note is a continuation of Ng's note[2,3]. Lastly, in section 7 some applications are discussed followed by general remarks.

We start from the Hamiltonian describing the motion of a single beam particle,

$$\begin{aligned} H_1 = {} & \frac{1}{2}[P_x^2+K_x(s)X^2] + \frac{1}{2}[P_y^2+K_y(s)Y^2] - \frac{B_x'}{B\rho}XY \\ & +\frac{B_y''}{6(B\rho)}(X^3-3XY^2) - \frac{B_x''}{6(B\rho)}(3X^2Y-Y^3) \\ & + \frac{B_y'''}{24(B\rho)}(X^4-6X^2Y^2+Y^4) - \frac{B_x'''}{6(B\rho)}(X^3Y-XY^3)\ , \end{aligned} \tag{1.1}$$

where P_x and P_y are the canonical momenta conjugate to the horizontal and vertical displacements X and Y, $K_x(s)$ and $K_y(s)$ are proportional to the restoring forces due to the ring's curvature and the field gradients of the normal quadrupoles. The term $-(B_x'/B\rho)XY$ gives only the skew quadrupole potential with $B\rho$ denoting the magnetic rigidity of the particle. The term $[B_y''/6(B\rho)](X^3-3XY^2)$ gives the normal sextupole potential, $[B_x''/6(B\rho)](3X^2Y-Y^3)$ gives the skew sextupole potential, $[B_y'''/24(B\rho)](X^4-6X^2Y^2+Y^4)$ gives the normal octupole potential and $[B_x'''/6(B\rho)](X^3Y-XY^3)$ gives the skew octupole potential.

We next perform a canonical transformation into the Floquet space using the generating function

$$G_1(x, P_x, y, P_y; s) = \sum_{u=x,y} \left[-\left(\frac{\beta_u}{\beta_0}\right)^{1/2} P_u u + \frac{\beta_u'}{4\beta_0} u^2 \right] . \tag{1.2}$$

The new Hamiltonian becomes

$$\begin{aligned} H_2 = & \frac{R}{2\beta_x}\left(\beta_0 p_x^2 + \frac{x^2}{\beta_0}\right) + \frac{R}{2\beta_y}\left(\beta_0 p_y^2 + \frac{y^2}{\beta_0}\right) - R\frac{B_x'}{B\rho}\left(\frac{\beta_x\beta_y}{\beta_0^2}\right)^{1/2} xy \\ & + \frac{RB_y''}{6(B\rho)}\left[\left(\frac{\beta_x}{\beta_0}\right)^{3/2} x^3 - 3\left(\frac{\beta_x\beta_y^2}{\beta_0^3}\right)^{1/2} xy^2\right] - \frac{RB_x''}{6(B\rho)}\left[3\left(\frac{\beta_x^2\beta_y}{\beta_0^3}\right)^{1/2} x^2 y - \left(\frac{\beta_y}{\beta_0}\right)^{3/2} y^3\right] \\ & + \frac{RB_y'''}{24(B\rho)}\left[\left(\frac{\beta_x}{\beta_0}\right)^2 x^4 - \frac{6\beta_x\beta_y}{\beta_0^2} x^2 y^2 + \left(\frac{\beta_y}{\beta_0}\right)^2 y^4\right] - \frac{RB_x'''}{6(B\rho)}\left[\left(\frac{\beta_x^3\beta_y}{\beta_0^4}\right)^{1/2} x^3 y - \left(\frac{\beta_x\beta_y^3}{\beta_0^4}\right)^{1/2} xy^3\right] . \end{aligned} \tag{1.3}$$

In the above, the independent variable s has been changed to the more convenient $\theta = s/R$, where R is the average radius of the storage ring.

This Hamiltonian is now solved exactly to zeroth order in multipole strength by canonical transformation to the action-angle variables I_x, a_x and I_y, a_y. The generating function

$$G_2(a_x, p_x, a_y, p_y; \theta) = \sum_{u=x,y} \frac{1}{2}\beta_0 p_u^2 \cot\left[Q_u(\theta) + a_u\right] \tag{1.4}$$

is used to obtain the transformation

$$u = (2I_u\beta_0)^{1/2} \cos\left[Q_u(\theta) + a_u\right] , \tag{1.5}$$

$$\beta_0 p_u = -(2I_u\beta_0)^{1/2} \sin\left[Q_u(\theta) + a_u\right] , \tag{1.6}$$

where $Q_u(\theta) = \psi_u(\theta) - \nu_u\theta$, $\beta_0 p_u = du/d\psi_u$ and is denoted by u' below. In the above, ν_u is the betatron tune and

$$\psi_u(s) = \int^s \frac{ds'}{\beta_u(s')}$$

is the Floquet phase at the location s. After the transformation, the new Hamiltonian becomes

$$H_3 = \nu_x I_x + \nu_y I_y + \text{ multipole terms } . \tag{1.7}$$

From here on, we treat each multipole term separately.

2 The Skew Quadrupole Term

2.1 Beam Shape Distortions due to Skew Quadrupoles

The skew quadrupole term in the Hamiltonian H_3 is

$$-R\frac{B_x'}{B\rho}\left(\frac{\beta_x\beta_y}{\beta_0^2}\right)^{1/2}xy \ .$$

Transformation to the action-angle variables I_u and a_u yields

$$\Delta H_3\,|_{\rm sq} = -\frac{RB_x'}{2(B\rho)}(\beta_x\beta_y)^{1/2}(2I_x)^{1/2}(2I_y)^{1/2}[\cos(Q_+ + a_+) + \cos(Q_- + a_-)] \ , \tag{2.1}$$

with $Q_\pm = Q_x \pm Q_y$, $a_\pm = a_x \pm a_y$. We note that the expressions

$$\frac{RB_x'}{2(B\rho)}(\beta_x\beta_y)^{1/2}e^{iQ_\pm}$$

are periodic functions of θ, so they can be expanded into harmonics. And so we get

$$\Delta H_3\,|_{\rm sq} = -(2I_x)^{1/2}(2I_y)^{1/2}\sum_m[A_{+m}\cos q_+ + A_{-m}\cos q_-] \ , \tag{2.2}$$

where $q_{\pm m} = \alpha_{\pm m} - m\theta + a_\pm$, and

$$A_{\pm m}e^{i\alpha_{\pm m}} = \frac{1}{4\pi}\sum_k q_k e^{i(Q_\pm + m\theta)_k} \ . \tag{2.3}$$

The summations in Eq. (2.2) are over all integers m from $-\infty$ to $+\infty$. The summations in Eq. (2.3) are over all skew quadrupoles at position θ_k along the ring. Here we treat the skew quadrupoles as elements of infinitesimal length ℓ_k, at position θ_k and with strengths

$$q_k = (\beta_x\beta_y)_k^{1/2}\frac{(B_x'\ell)_k}{B\rho} \ . \tag{2.4}$$

In Eq. (2.3), the harmonic amplitudes A_{+m}, A_{-m} and the phases α_{+m}, α_{-m} are real numbers.

For the first-order beam shape, we can solve the equations of motion obtained from the Hamiltonian H_3 to the first order. However, because we are interested in the second-order tuneshifts also, it will be advantageous for us to make another canonical transformation from (a_u, I_u) to (b_u, J_u) so that the J_u's become constants of motion up to first order in q_k. This is called a Moser transformation with generating function

$$G_3(a_x, J_x, a_y, J_y; \theta) = a_xJ_x + a_yJ_y$$
$$- (2J_x)^{1/2}(2J_y)^{1/2}\sum_m\left(\frac{A_{+m}}{m-\nu_+}\sin q_{+m} + \frac{A_{-m}}{m-\nu_-}\sin q_{-m}\right) \ , \tag{2.5}$$

where $\nu_\pm = \nu_x \pm \nu_y$. By definition the new Hamiltonian is

$$H_4 = \nu_xJ_x + \nu_yJ_y + \Delta H_4\,|_{\rm sq} \ , \tag{2.6}$$

where $\Delta H_4|_{sq}$ does not contain any zero-order or first-order terms q_k. The first-order changes in I_u and a_u are therefore given by

$$\delta I_u = I_u - J_u = \frac{\partial G_3}{\partial a_u} - J_u \ , \tag{2.7}$$

$$\delta a_u = a_u - b_u = a_u - \frac{\partial G_3}{\partial J_u} \ . \tag{2.8}$$

Explicitly they are

$$\delta I_x = -(2I_x)^{1/2}(2I_y)^{1/2} \sum_m \left(\frac{A_{+m}}{m-\nu_+} \cos q_{+m} + \frac{A_{-m}}{m-\nu_-} \cos q_{-m} \right) \ ,$$

$$\delta I_y = -(2I_x)^{1/2}(2I_y)^{1/2} \sum_m \left(\frac{A_{+m}}{m-\nu_+} \cos q_{+m} - \frac{A_{-m}}{m-\nu_-} \cos q_{-m} \right) \ ,$$

$$\delta a_x = \left(\frac{I_y}{I_x} \right)^{1/2} \sum_m \left(\frac{A_{+m}}{m-\nu_+} \sin q_{+m} + \frac{A_{-m}}{m-\nu_-} \sin q_{-m} \right) \ ,$$

$$\delta a_y = \left(\frac{I_x}{I_y} \right)^{1/2} \sum_m \left(\frac{A_{+m}}{m-\nu_+} \sin q_{+m} + \frac{A_{-m}}{m-\nu_-} \sin q_{-m} \right) \ . \tag{2.9}$$

These are related to the changes in amplitudes and phases. Recalling from Eqs. (1.5) and (1.6) that

$$u = \mathcal{A}_u \cos\left[Q_u(\theta) + a_u\right] \ ,$$
$$u' = -\mathcal{A}_u \sin\left[Q_u(\theta) + a_u\right] \ , \tag{2.10}$$

where

$$\mathcal{A}_u = (2I_u\beta_0)^{1/2} \ , \tag{2.11}$$

we have changes in amplitudes

$$\delta \mathcal{A}_u = \left(\frac{\beta_0}{2I_u} \right)^{1/2} \delta I_u \ . \tag{2.12}$$

As for the angle variable a_u, if we solve the Hamiltonian H_3, we get

$$\frac{da_u}{d\theta} = \frac{\partial H_3}{\partial I_u} = \nu_u + \text{ quadrupole terms} \ . \tag{2.13}$$

Thus, for the unperturbed part,

$$a_u(\theta) = \nu_u\theta + \text{ constant} \ . \tag{2.14}$$

Here, the constant should be chosen as $\phi_u - \psi_u$, where $\phi_u(\theta)$ is the *instantaneous* betatron phase and $\psi_u(\theta)$ is the Floquet phase designating the location at the point θ. Although both of them depend on θ, their difference is θ-independent. Such a choice of the constant is necessary, because substitution of

$$a_u = \nu_u\theta - \psi_u + \phi_u = \phi_u - Q_u \tag{2.15}$$

into Eqs. (2.10) gives

$$u = \mathcal{A}_u \cos\phi_u \ , \qquad \text{and} \qquad u' = -\mathcal{A}_u \sin\phi_u \ . \tag{2.16}$$

Therefore, the change in the angle variable a_u is just the change in the instantaneous phase, or

$$\delta\phi_u = \delta a_u \ . \tag{2.17}$$

But before we calculate the changes in amplitudes and phases, let us simplify Eqs. (2.9) by performing the summation over m. This can be accomplished easily using the formula

$$\sum_{m=-\infty}^{\infty} \frac{e^{i(m\theta+b)}}{m-\nu} = \begin{cases} -\dfrac{\pi}{\sin\pi\nu} e^{i[b+\nu(\theta-\pi)]} & 0<\theta<2\pi \\ -\pi\cot\pi\nu e^{ib} & \theta = 0 \ . \end{cases} \tag{2.18}$$

This leads us to*

$$\sum_m \frac{A_{+m} e^{iq_{+m}}}{m-\nu_+} = e^{i\phi_+}(B_+ + iA_+) \ ,$$

$$\sum_m \frac{A_{-m} e^{iq_{-m}}}{m-\nu_-} = e^{i\phi_-}(B_- + iA_-) \ , \tag{2.19}$$

where $\phi_\pm = \phi_x \pm \phi_y$, and B_+, A_+ and B_-,A_- are two sets of distortion functions defined by Collins

$$B_\pm(\psi_\pm) = -\frac{1}{2\sin\pi\nu_\pm} \sum_k \frac{q_k}{2} \cos[\psi'_{\pm k} - \psi_\pm(\theta) - \nu_\pm\pi] \ ,$$

$$A_\pm(\psi_\pm) = B'_\pm(\psi_\pm) \ , \tag{2.20}$$

where $\psi_\pm = \psi_x \pm \psi_y$,

$$\psi'_{uk} = \begin{cases} \psi_{uk} & \theta_k - \theta \le 2\pi \\ \psi_{uk} + 2\pi\nu_u & \theta_k - \theta > 2\pi \quad u = x \text{ or } y \ , \end{cases} \tag{2.21}$$

and the prime on $B_\pm$ denotes differentiation with respect to the argument. Instead of $\psi'_\pm$, we can also write the argument of the cosine in Eq. (2.20) as

$$|\psi_{xk} - \psi_x(\theta)| \pm |\psi_{yk} - \psi_y(\theta)| - \nu_\pm\pi \ . \tag{2.22}$$

The way that it was written in Refs. 2, 3, and 4 is incorrect. The distortion functions defined above are in fact, as explained in Refs. 2 and 4, lattice functions due to the presence

*For an illustration of how this can be done see Refs. 2 and 4 of skew quadrupoles, just as the β and α are lattice functions due to the presence of normal quadrupoles. They are periodic functions of the ring and closed after one revolution. The vector (B_+, A_+) rotates around the ring according to the angle equal to the phase advanced. At a skew quadrupole of strength q_k, A_+ jumps by $q_k/2$ while B_+ remains continuous but exhibits a cusp.

We are now in a position to calculate the distortion of the beam shape projections. Substituting Eq. (2.19) into Eqs. (2.9) and using Eqs. (2.12) and (2.17) we arrive at

$$\delta\mathcal{A}_x = \mathcal{A}_y[(A_+ \sin\phi_+ - B_+ \cos\phi_+) + (A_- \sin\phi_- - B_- \cos\phi_-)] , \tag{2.23}$$

$$\delta\mathcal{A}_y = \mathcal{A}_x[(A_+ \sin\phi_+ - B_+ \cos\phi+) - (A_- \sin\phi_- - B_- \cos\phi_-)] , \tag{2.24}$$

$$\delta\phi_x = \frac{\mathcal{A}_y}{\mathcal{A}_x}[(A_+ \cos\phi_+ + B_+ \sin\phi_+) + (A_- \cos\phi_- + B_- \sin\phi_-)] , \tag{2.25}$$

$$\delta\phi_y = \frac{\mathcal{A}_x}{\mathcal{A}_y}[(A_+ \cos\phi_+ + B_+ \sin\phi_+) + (A_- \cos\phi_- + B_- \sin\phi_-)] . \tag{2.26}$$

Thus the distorted beam shape in phase space can be written as

$$x = (\mathcal{A}_x + \delta\mathcal{A}_x)\cos(\phi_x + \delta\phi_x) , \tag{2.27}$$

$$x' = -(\mathcal{A}_x + \delta\mathcal{A}_x)\sin(\phi_x + \delta\phi_x) , \tag{2.28}$$

$$y = (\mathcal{A}_y + \delta\mathcal{A}_y)\cos(\phi_y + \delta\phi_y) , \tag{2.29}$$

$$y' = -(\mathcal{A}_y + \delta\mathcal{A}_y)\sin(\phi_y + \delta\phi_y) , \tag{2.30}$$

where $\delta\mathcal{A}_x$, $\delta\mathcal{A}_y$, $\delta\phi_x$, $\delta\phi_y$ are given by Eqs. (2.23) to (2.26). These distortion formulæ are exactly those given by Collins.

Finally at this point we would like to remark that the term *distortion functions* used here is not very successful since we are dealing with a linear problem. The term *error functions* would be a more appropriate one.

2.2 Second-order Tuneshifts

The first-order tuneshift due to skew quadrupoles vanishes. This is because the first order-term in the perturbation Hamiltonian has the form xy. But since $\cos\phi_x \cos\phi_y$ has a zero average there is no shift in the tune to first order. The lowest nonvanishing contribution to the tuneshift comes from the second order. To obtain the second-order tuneshifts, we need to evaluate the second-order skew quadrupole terms in the Hamiltonian H_4. From the generating function G_3 of Eq. (2.5), we get

$$(2I_x)^{1/2} = (2J_x)^{1/2}\left[1 - \frac{(2J_y)^{1/2}}{(2J_x)^{1/2}}\sum_m \left(\frac{A_{+m}}{m-\nu_+}\cos q_+ + \frac{A_{-m}}{m-\nu_-}\cos q_-\right)\right] \tag{2.31}$$

and similar expression for $(2I_y)^{1/2}$. Then the second-order terms in the Hamiltonian is

$$\Delta H_4|_{\text{sq}} = \sum_{m'}(A_{+m'}\cos q_+ + A_{-m'}\cos q_-)\left[(2J_y)\sum_m\left(\frac{A_{+m}}{m-\nu_+}\cos q_+ + \frac{A_{-m}}{m-\nu_-}\cos q_-\right)\right.$$

$$+ (2J_x)\sum_m \left(\frac{A_{+m}}{m-\nu_+}\cos q_+ - \frac{A_{-m}}{m-\nu_-}\cos q_-\right)\Bigg] \ . \tag{2.32}$$

Since betatron tunes are defined per revolution, we average over θ. This leads to

$$\Delta H_4'|_{sq} = J_y \sum_m \left(\frac{A_{+m}^2}{m-\nu_+} + \frac{A_{-m}^2}{m-\nu_-}\right) + J_x \sum_m \left(\frac{A_{+m}^2}{m-\nu_+} - \frac{A_{-m}^2}{m-\nu_-}\right) \ . \tag{2.33}$$

Now we need to sum over the harmonics using again Eq. (2.18). Written in terms of the distortion functions, we have

$$\sum_m \frac{A_{+m}^2}{m-\nu_+} = \frac{1}{4\pi}\sum_k (qB_+)_k \ , \tag{2.34}$$

$$\sum_m \frac{A_{-m}^2}{m-\nu_-} = \frac{1}{4\pi}\sum_k (qB_-)_k \ . \tag{2.35}$$

The tuneshifts are given by

$$\Delta\nu_x = \frac{\partial \Delta H_4'}{\partial J_x} \qquad \text{and} \qquad \Delta\nu_y = \frac{\partial \Delta H_4'}{\partial J_y} \ . \tag{2.36}$$

Using Eqs. (2.33), (2.34), and (2.35) we obtain the tuneshifts

$$\Delta\nu_x = \frac{1}{4\pi}\sum_k (B_+ q - B_- q)_k \ , \tag{2.37}$$

$$\Delta\nu_y = \frac{1}{4\pi}\sum_k (B_+ q + B_- q)_k \ . \tag{2.38}$$

As expected the tuneshifts are independent of the amplitude.

3 The Normal Sextupole Term

Even though the formulæ for the beam shape distortions and the second-order tuneshifts due to normal sextupoles have been derived extensively before, in Refs. 2, 3 and 4, we shall include the derivation here, for completeness.

3.1 Beam Shape Distortions due to Normal Sextupoles

Let us start from the sextupole term in the Hamiltonian (1.3)

$$\Delta H_3|_{ns} = -\frac{RB_y''}{6(B\rho)}\left[3\left(\frac{\beta_y^2\beta_x}{\beta_0^3}\right)^{1/2} xy^2 - \left(\frac{\beta_x}{\beta_0}\right)^{3/2} x^3\right] \tag{3.1}$$

and expand it into harmonics to get

$$\Delta H_3|_{\rm ns} = (2I_x)^{3/2}\beta_0^{1/2}\sum_m (3A_{1m}\sin q_{1m} + A_{3m}\sin q_{3m})$$
$$-(2I_x)^{1/2}(2I_y)\beta_0^{1/2}\sum_m (2B_{1m}\sin p_{1m} + B_{+m}\sin p_{+m} + B_{-m}\sin p_{-m})\ , \quad (3.2)$$

with $q_{1m} = \alpha_{1m} - m\theta + a_x$, $q_{3m} = \alpha_{3m} - m\theta + 3a_x$, $p_{1m} = \beta_{1m} - m\theta + a_x$, $p_{\pm m} = \beta_{\pm m} - m\theta + a_\pm$, $a_\pm = 2a_y \pm a_x$, and

$$A_{1m}e^{i\alpha_{1m}} = \frac{i}{24\pi}\sum_k s_k e^{i(Q_x+m\theta)_k}\ , \quad (3.3)$$

$$A_{3m}e^{i\alpha_{3m}} = \frac{i}{24\pi}\sum_k s_k e^{i(3Q_x+m\theta)_k}\ , \quad (3.4)$$

$$B_{1m}e^{i\beta_{1m}} = \frac{i}{8\pi}\sum_k \bar{s}_k e^{i(Q_x+m\theta)_k}\ , \quad (3.5)$$

$$B_{\pm m}e^{i\beta_{\pm m}} = \frac{i}{8\pi}\sum_k \bar{s}_k e^{i(Q_\pm+m\theta)_k}\ , \quad (3.6)$$

where $Q_\pm = 2Q_y \pm Q_x$. The sextupoles are assumed to have infinitesimal length ℓ_k with strengths

$$s_k = \left(\frac{\beta_x^3}{\beta_0}\right)_k^{1/2}\frac{(B_y''\ell)_k}{2(B\rho)}, \qquad \bar{s}_k = \left(\frac{\beta_y^2\beta_x}{\beta_0}\right)_k^{1/2}\frac{(B_y''\ell)_k}{2(B\rho)}\ . \quad (3.7)$$

At this point, because of our interest in the second-order tuneshifts, we shall proceed by making a Moser transformation from (a_u, I_u) to (b_u, J_u), $u = x, y$, using the generating function

$$G_3(a_x, J_x, a_y, J_y; \theta) = a_xJ_x + a_yJ_y - (2J_x)^{3/2}\beta_0^{1/2}\sum_m\left(\frac{3A_{1m}}{m-\nu_x}\cos q_{1m} + \frac{A_{3m}}{m-3\nu_x}\cos q_{3m}\right)$$
$$+(2J_x)^{1/2}(2J_y)\beta_0^{1/2}\sum_m\left(\frac{2B_{1m}}{m-\nu_x}\cos p_{1m} - \frac{B_{+m}}{m-\nu_+}\cos p_{+m} + \frac{B_{-m}}{m-\nu_-}\cos p_{-m}\right)\ , \quad (3.8)$$

where $\nu_\pm = 2\nu_y \pm \nu_x$. By definition, the new Hamiltonian is

$$H_4 = \nu_xJ_x + \nu_yJ_y - \Delta H_4|_{\rm ns}\ , \quad (3.9)$$

where $\Delta H_4|_{\rm ns}$ does not contain any zeroth or first-order terms in s_k or $\bar{s}_k$. The first-order changes in I_u and a_u which are given by Eqs. (2.7) and (2.8) are

$$\delta I_x = (2I_x)^{3/2}\beta_0^{1/2}\sum_m\left(\frac{3A_{1m}}{m-\nu_x}\sin q_{1m} + \frac{3A_{3m}}{m-3\nu_x}\sin q_{3m}\right)$$

$$-(2I_x)^{1/2}(2I_y)\beta_0^{1/2}\sum_m\left(\frac{2B_{1m}}{m-\nu_x}\sin p_{1m} - \frac{B_{-m}}{m-\nu_+}\sin p_{+m} - \frac{B_{-m}}{m-\nu_-}\sin p_{-m}\right)\ ,$$

$$\delta I_y = -(2I_x)^{1/2}(2I_y)\beta_0^{1/2}\sum_m \left(\frac{2B_{+m}}{m-\nu_+}\sin p_{+m} + \frac{2B_{-m}}{m-\nu_-}\sin p_{-m}\right) ,$$

$$\delta a_x = 3(2I_x)^{1/2}\beta_0^{1/2}\sum_m \left(\frac{A_{1m}}{m-\nu_x}\cos q_{1m} + \frac{A_{3m}}{m-3\nu_x}\cos q_{3m}\right)$$

$$-(2I_x)^{-1/2}(2I_y)\beta_0^{1/2}\sum_m \left(\frac{2B_{1m}}{m-\nu_x}\cos p_{1m} + \frac{B_{+m}}{m-\nu_+}\cos p_{+m} + \frac{B_{-m}}{m-\nu_-}\cos p_{-m}\right) ,$$

$$\delta a_y = -2(2I_x)^{1/2}\beta_0^{1/2}\sum_m \left(\frac{2B_{1m}}{m-\nu_x}\cos p_{1m} + \frac{B_{+m}}{m-\nu_+}\cos p_{+m} + \frac{B_{-m}}{m-\nu_-}\cos p_{-m}\right) . \quad (3.10)$$

We shall relate these changes to changes in amplitudes and phases. But first we are going to simplify (3.10) by doing the summation over m. We use again formula (2.18) to arrive at

$$\begin{aligned}
\sum_m \frac{A_{1m}}{m-\nu_x} e^{iq_{1m}} &= \frac{1}{3} e^{i\phi_x}(-iB_1 + A_1) , \\
\sum_m \frac{A_{3m}}{m-3\nu_x} e^{iq_{3m}} &= \frac{1}{3} e^{3i\phi_x}(-iB_3 + A_3) , \\
\sum_m \frac{B_{1m}}{m-\nu_x} e^{ip_{1m}} &= e^{i\phi_x}(-i\bar{B} + \bar{A}) , \\
\sum_m \frac{B_{+m}}{m-\nu_+} e^{ip_{+m}} &= e^{i\phi_+}(-iB_s + A_s) , \\
\sum_m \frac{B_{-m}}{m-\nu_-} e^{ip_{-m}} &= e^{i\phi_-}(-iB_d + A_d) , \qquad (3.11)
\end{aligned}$$

where $\phi_\pm = 2\phi_y \pm \phi_x$ and the distortion functions are defined by

$$B_1(\psi_x) = \frac{1}{2\sin \pi\nu_x}\sum_k \frac{s_k}{4}\cos(\psi'_{xk} - \psi_x - \pi\nu_x) ,$$

$$A_1(\psi_x) = B'_1(\psi_x) ,$$

$$B_3(3\psi_x) = \frac{1}{2\sin 3\pi\nu_x}\sum_k \frac{s_k}{4}\cos 3(\psi'_{xk} - \psi_x - \pi\nu_x) ,$$

$$A_3(3\psi_x) = B'_3(3\psi_x) ,$$

$$\bar{B}(\psi_x) = \frac{1}{2\sin \pi\nu_x}\sum_k \frac{\bar{s}_k}{4}\cos(\psi'_{xk} - \psi_x - \pi\nu_x) ,$$

$$\bar{A}(\psi_x) = \bar{B}'(\psi_x) ,$$

$$B_s(\psi_+) = \frac{1}{2\sin \pi\nu_+}\sum_k \frac{\bar{s}_k}{4}\cos(\psi'_{+k} - \psi_+ - \pi\nu_-) ,$$

$$A_s(\psi_+) = B'_s(\psi_+) \ ,$$

$$B_d(\psi_-) = \frac{1}{2\sin\pi\nu_-}\sum_k \frac{\bar{s}_k}{4}\cos\left(\psi'_{-k} - \psi_- - \pi\nu_-\right) \ ,$$

$$A_d(\psi_-) = B'_d(\psi_-) \ , \tag{3.12}$$

with $\psi_\pm = 2\psi_y \pm \psi_x$ and ψ'_x and ψ'_y defined in Eq. (2.21). From Eqs. (2.12) and (2.17) we recall that the distortion of the amplitudes $\mathcal{A}_u$ and phases ϕ_u are given by

$$\delta\mathcal{A}_u = \left(\frac{\beta_0}{2I_u}\right)^{1/2} \delta I_u \tag{3.13}$$

and

$$\delta\phi_u = \delta a_u \ . \tag{3.14}$$

Using Eqs. (3.10) to (3.14) we obtain

$$\begin{aligned} \delta\mathcal{A}_x &= \mathcal{A}_x^2[(A_1\sin\phi_x - B_1\cos\phi_x) + (A_3\sin 3\phi_x - B_3\cos 3\phi_x)] \\ &- \mathcal{A}_y^2[2(\bar{A}\sin\phi_x - \bar{B}\cos\phi_x) + (A_s\sin\phi_+ - B_s\cos\phi_+) - (A_d\sin\phi_- - B_d\cos\phi_-)] \ , \quad (3.15) \\ \delta\mathcal{A}_y &= -2\mathcal{A}_x\mathcal{A}_y[(A_s\sin\phi_+ - B_s\cos\phi_+) + (A_d\sin\phi_- - B_d\cos\phi_-)] \ , \quad (3.16) \end{aligned}$$

$$\begin{aligned} \delta\phi_x = \mathcal{A}_x[3(A_1\cos\phi_x + B_1\sin\phi_x) + (A_3\cos 3\phi_x + B_3\sin 3\phi_x)] \\ -\frac{\mathcal{A}_y^2}{\mathcal{A}_x}[2(\bar{A}\cos\phi_x + \bar{B}\sin\phi_x) + (A_s\sin\phi_+ + B_s\cos\phi_+) + (A_d\cos\phi_- + B_d\sin\phi_-)] \ , \end{aligned} \tag{3.17}$$

$$\delta\phi_y = -2\mathcal{A}_x[2(\bar{A}\cos\phi_x + \bar{B}\sin\phi_x) + (A_s\cos\phi_+ + B_s\sin\phi_+) + (A_d\cos\phi_- + B_d\sin\phi_-)] \ . \tag{3.18}$$

The sextupoles have an average dipole effect on a charged particle which leads to a distortion of the ideal closed orbit. This can be obtained by separating out from Eqs. (3.15) and (3.17),

$$\delta\mathcal{A}'_x = 2\mathcal{A}_x^2(A_1\sin\phi_x - B_1\cos\phi_x) - 2\mathcal{A}_y^2(\bar{A}\sin\phi_x - \bar{B}\cos\phi_x) \ , \tag{3.19}$$

$$\mathcal{A}_x\delta\phi'_x = 2\mathcal{A}_x^2(B_1\sin\phi_x + A_1\cos\phi_x) - 2\mathcal{A}_y^2(\bar{B}\sin\phi_x + \bar{A}\cos\phi_x) \ , \tag{3.20}$$

which correspond to a closed orbit distortion of

$$\delta x = 2(\mathcal{A}_y^2\bar{B} - \mathcal{A}_x^2 B_1) \ , \tag{3.21}$$

$$\delta x' = 2(\mathcal{A}_y^2\bar{A} - \mathcal{A}_x^2 A_1) \ . \tag{3.22}$$

Thus the distorted beam shape in phase space can be written as

$$\begin{aligned} x &= \delta x + (\mathcal{A}_x + \delta\mathcal{A}_x)\cos(\phi_x + \delta\phi_x) \ , & (3.23) \\ x' &= \delta x' - (\mathcal{A}_x + \delta\mathcal{A}_x)\sin(\phi_x + \delta\phi_x) \ , & (3.24) \\ y &= (\mathcal{A}_y + \delta\mathcal{A}_y)\cos(\phi_y + \delta\phi_y) \ , & (3.25) \\ y' &= -(\mathcal{A}_y + \delta\mathcal{A}_y)\sin(\phi_y + \delta\phi_y) \ , & (3.26) \end{aligned}$$

where $\delta\mathcal{A}_y$ and $\delta\phi_y$ are given by Eqs. (3.16) and (3.18), δx and $\delta x'$ by (3.21) and (3.22) and $\delta\mathcal{A}_x$ and $\delta\phi_x$ by the differences of Eqs. (3.15), (3.17) and (3.19), (3.20), or

$$\begin{aligned}\delta\mathcal{A}_x &= \mathcal{A}_x^2[-(A_1\sin\phi_x - B_1\cos\phi_x) + (A_3\sin 3\phi_x - B_3\cos 3\phi_x)] \\ &\quad -\mathcal{A}_y^2[(A_s\sin\phi_+ - B_s\cos\phi_+) - (A_d\sin\phi_- - B_d\cos\phi_-)]\ ,\end{aligned} \tag{3.27}$$

$$\begin{aligned}\delta\phi_x &= \mathcal{A}_x[(A_1\cos\phi_x + B_1\sin\phi_x) + (A_3\cos 3\phi_x + B_3\sin 3\phi_x)] \\ &\quad -\frac{\mathcal{A}_y^2}{\mathcal{A}_x}[(A_s\cos\phi_+ + B_s\sin\phi_+) + (A_d\cos\phi_- + B_d\sin\phi_-)]\ .\end{aligned} \tag{3.28}$$

3.2 Second-order Tuneshifts

The first-order perturbation produces no tuneshifts. The reason, as in the case of skew quadrupoles, is that the first-order term in the perturbation Hamiltonian is of the form $x^3 - 3xy^2$. But since $\cos^3\phi_x - 3\cos\phi_x\cos^2\phi_y$ averages to zero, there is no resultant shift in the tune to first order and we must seek higher approximations. The lowest contribution to the tuneshift comes from the second order. From the generating function G_3 of Eq. (3.8), we get

$$\begin{aligned}(2I_x)^{3/2} &= (2J_x)^{3/2} + 9(2J_x)^2\beta_0^{1/2}\sum_m\left(\frac{A_{1m}}{m-\nu_x}\sin q_{1m} + \frac{A_{3m}}{m-3\nu_x}\sin q_{3m}\right) \\ &\quad -3(2J_y)(2J_x)\beta_0^{1/2}\sum_m\left(\frac{2B_{1m}}{m-\nu_x}\sin p_{1m} + \frac{B_{+m}}{m-\nu_+}\sin p_{+m} + \frac{B_{-m}}{m-\nu_-}\sin p_{-m}\right)\end{aligned} \tag{3.29}$$

and a similar expression for $(2I_x)^{1/2}(2I_y)$. Then the second-order terms in the Hamiltonian is

$$\begin{aligned}\Delta H_4|_{\rm ns} &= \sum_{m'}(3A_{1m'}\sin q_{1m'} + A_{3m'}\sin q_{3m'})\times \\ &\quad\times\left[9\beta_0(2J_x)^2\sum_m\left(\frac{A_{1m}}{m-\nu_x}\sin q_{1m} + \frac{A_{3m}}{m-3\nu_x}\sin q_{3m}\right)\right. \\ &\quad\left.-3(2J_x)(2J_y)\beta_0\sum_m\left(\frac{2B_{1m}}{m-\nu_x}\sin p_{1m} + \frac{B_{+m}}{m-\nu_+}\sin p_{+m} + \frac{B_{-m}}{m-\nu_-}\sin p_{-m}\right)\right] \\ &\quad + \ldots\ .\end{aligned} \tag{3.30}$$

If we now consider only the θ-independent terms we obtain

$$\begin{aligned}\Delta H_4'|_{\rm ss} &= \frac{9}{2}\beta_0(2J_x)^2\sum_m\left(\frac{3A_{1m}^2}{m-\nu_x} + \frac{A_{3m}^2}{m-3\nu_x}\right) \\ &\quad +\frac{1}{2}\beta_0(2J_y)^2\sum_m\left(\frac{4B_{1m}^2}{m-\nu_x} + \frac{B_{+m}^2}{m-\nu_+} + \frac{B_{-m}^2}{m-\nu_-}\right) \\ &\quad +2\beta_0(2J_y)(2J_x)\sum_m\left[\frac{B_{+m}^2}{m-\nu_+} - \frac{B_{-m}^2}{m-\nu_-} - \frac{6A_{1m}B_{1m}}{m-\nu_x}\cos(\alpha_{1m}-\beta_{1m})\right]\ .\end{aligned} \tag{3.31}$$

Summation over the harmonics leads to the following result

$$\sum_m \frac{A_{1m}^2}{m-\nu_x} = -\frac{1}{72\pi}\sum_k (B_1 s)_k ,$$
$$\sum_m \frac{A_{3m}^2}{m-3\nu_x} = -\frac{1}{72\pi}\sum_k (B_3 s)_k ,$$
$$\sum_m \frac{B_{1m}^2}{m-\nu_x} = -\frac{1}{8\pi}\sum_k (\bar{B}\bar{s})_k ,$$
$$\sum_m \frac{B_{+m}^2}{m-\nu_+} = -\frac{1}{8\pi}\sum_k (B_s\bar{s})_k ,$$
$$\sum_m \frac{B_{-m}^2}{m-\nu_-} = \frac{1}{8\pi}\sum_k (B_d\bar{s})_k ,$$

$$\sum_m \frac{A_{1m}B_{1m}}{m-\nu_x}\cos(\alpha_{1m}-\beta_{1m}) = -\frac{1}{24\pi}\sum_k (B_1\bar{s})_k . \tag{3.32}$$

So the tuneshifts given by (2.37) are

$$\Delta\nu_x = -\frac{1}{4\pi}\mathcal{A}_x^2\sum_k (3B_1 s + B_3 s)_k - \frac{1}{2\pi}\mathcal{A}_y^2\sum_k (B_s\bar{s} + B_d\bar{s} - 2B_1\bar{s})_k , \tag{3.33}$$

$$\Delta\nu_y = -\frac{1}{4\pi}\mathcal{A}_y^2\sum_k (4\bar{B}\bar{s} + B_s\bar{s} - B_d\bar{s})_k - \frac{1}{2\pi}\mathcal{A}_x^2\sum_k (B_s\bar{s} + B_d\bar{s} - 2B_1\bar{s})_k . \tag{3.34}$$

4 The Skew Sextupole Term

4.1 Beam Shape Distortions due to Skew Sextupoles

We start from the skew sextupole term in the Hamiltonian (1.3)

$$\Delta H_3|_{ss} = -\frac{RB_x''}{6(B\rho)}\left[3\left(\frac{\beta_x^2\beta_y}{\beta_0^3}\right)^{1/2} x^2 y - \left(\frac{\beta_y}{\beta_0}\right)^{3/2} y^3\right] , \tag{4.1}$$

and expand it into harmonics to get

$$\begin{aligned}\Delta H_3|_{ss} = {} & (2I_y)^{3/2}\beta_0^{1/2}\sum_m (3A_{1m}'\cos p_{1m} + A_{3m}'\cos p_{3m}) \\ & -(2I_y)^{1/2}(2I_x)\beta_0^{1/2}\sum_m (2B_{1m}'\cos q_{1m} + B_{+m}'\cos q_{+m} + B_{-m}'\cos q_{-m}) ,\end{aligned} \tag{4.2}$$

with $p_{1m} = \alpha_{1m}' - m\theta + a_y$, $p_{3m} = \alpha_{3m}' - m\theta + 3a_y$, $q_{1m} = \beta_{1m}' - m\theta + a_y$, $q_{\pm m} = \beta_{\pm m}' - m\theta + a_\pm$, $a_\pm = 2a_x \pm a_y$, and

$$A_{1m}' e^{i\alpha_{1m}'} = \frac{1}{24\pi}\sum_k s_k' e^{i(Q_y + m\theta)_k} , \tag{4.3}$$

$$A'_{3m}e^{i\alpha'_{3m}} = \frac{1}{24\pi}\sum_k s'_k e^{i(3Q_y+m\theta)_k} , \tag{4.4}$$

$$B'_{1m}e^{i\beta'_{1m}} = \frac{1}{8\pi}\sum_k \bar{s}'_k e^{i(Q_y+m\theta)_k} , \tag{4.5}$$

$$B'_{\pm m}e^{i\beta'_{\pm m}} = \frac{1}{8\pi}\sum_k \bar{s}'_k e^{i(Q_\pm+m\theta)_k} , \tag{4.6}$$

where $Q_\pm = 2Q_x \pm Q_y$. The skew sextupoles are assumed to have infinitesimal length ℓ_k with strengths

$$s'_k = \left(\frac{\beta_y^3}{\beta_0}\right)_k^{1/2}\frac{(B''_x\ell)_k}{2(B\rho)} , \qquad \bar{s}'_k = \left(\frac{\beta_x^2\beta_y}{\beta_0}\right)_k^{1/2}\frac{(B''_x\ell)_k}{2(B\rho)} . \tag{4.7}$$

Again here, because we are also interested in the second-order tuneshifts, we shall proceed by making a Moser transformation from (a_u, I_u) to (b_u, J_u), $u = x, y$, using the generating function

$$G_3(a_x, J_x, a_y, J_y; \theta) = a_xJ_x + a_yJ_y + (2J_y)^{3/2}\beta_0^{1/2}\sum_m\left(\frac{3A'_{1m}}{m-\nu_y}\sin p_{1m} + \frac{A'_{3m}}{m-3\nu_y}\sin p_{3m}\right)$$

$$-(2J_y)^{1/2}(2J_x)\beta_0^{1/2}\sum\left(\frac{2B'_{1m}}{m-\nu_y}\sin q_{1m} + \frac{B'_{+m}}{m-\nu_+}\sin q_{+m} + \frac{B'_{-m}}{m-\nu_-}\sin q_{-m}\right) , \tag{4.8}$$

where $\nu_\pm = 2\nu_x \pm \nu_y$. By definition, the new Hamiltonian is

$$H_4 = \nu_xJ_x + \nu_yJ_y + \Delta H_4|_{ss} , \tag{4.9}$$

where $\Delta H_4|_{ss}$ does not contain any zeroth or first-order terms in s'_k or $\bar{s}'_k$. The first-order changes in I_u and a_u are given by Eqs. (2.7) and (2.8). Explicitly they are

$$\delta I_x = -(2I_y)^{1/2}(2I_x)\beta_0^{1/2}\sum_m\left(\frac{2B'_{+m}}{m-\nu_+}\cos q_{+m} + \frac{2B'_{-m}}{m-\nu_-}\cos q_{-m}\right) ,$$

$$\delta I_y = (2I_y)^{3/2}\beta_0^{1/2}\sum_m\left(\frac{3A'_{1m}}{m-\nu_y}\cos p_{1m} + \frac{3A'_{3m}}{m-3\nu_y}\cos p_{3m}\right)$$

$$-(2I_y)^{1/2}(2I_x)\beta_0^{1/2}\sum_m\left(\frac{2B'_{1m}}{m-\nu_y}\cos q_{1m} + \frac{B'_{+m}}{m-\nu_+}\cos q_{+m} - \frac{B'_{-m}}{m-\nu_-}\cos q_{-m}\right) ,$$

$$\delta a_x = 2(2I_y)^{1/2}\beta_0^{1/2}\sum_m\left(\frac{2B'_{1m}}{m-\nu_y}\sin q_{1m} + \frac{B'_{+m}}{m-\nu_+}\sin q_{+m} + \frac{B'_{-m}}{m-\nu_-}\sin q_{-m}\right) ,$$

$$\delta a_y = -3(2I_y)^{1/2}\beta_0^{1/2}\sum_m\left(\frac{3A'_{1m}}{m-\nu_y}\sin p_{1m} + \frac{A'_{3m}}{m-3\nu_y}\sin p_{3m}\right)$$

$$+(2I_y)^{-1/2}(2I_x)\beta_0^{1/2}\sum_m\left(\frac{2B'_{1m}}{m-\nu_y}\sin q_{1m} + \frac{B'_{+m}}{m-\nu_+}\sin q_{+m} - \frac{B'_{-m}}{m-\nu_-}\sin q_{-m}\right) . \tag{4.10}$$

We shall relate these changes to changes in amplitudes and phases. But first we need to simplify (4.10) by doing the summation over m. We use again formula (2.18) to arrive at

$$\begin{aligned}
\sum_m \frac{A'_{1m}}{m-\nu_y} e^{ip_{1m}} &= -\frac{1}{3} e^{i\phi_y}(B_1 + iA_1) \,, \\
\sum_m \frac{A'_{3m}}{m-3\nu_y} e^{ip_{3m}} &= -\frac{1}{3} e^{3i\phi_y}(B_3 + iA_3) \,, \\
\sum_m \frac{B'_{1m}}{m-\nu_y} e^{iq_{1m}} &= -e^{i\phi_y}(\bar{B} + i\bar{A}) \,, \\
\sum_m \frac{B'_{+m}}{m-\nu_+} e^{iq_{+m}} &= -e^{i\phi_+}(B_s + iA_s) \,, \\
\sum_m \frac{B'_{-m}}{m-\nu_-} e^{iq_{-m}} &= -e^{i\phi_-}(B_d + iA_d) \,.
\end{aligned} \tag{4.11}$$

where $\phi_\pm = 2\phi_x \pm \phi_y$ and the distortion functions are defined by

$$B_1(\psi_y) = \frac{1}{2\sin\pi\nu_y} \sum_k \frac{s'_k}{4} \cos\left(\psi'_{yk} - \psi_y - \pi\nu_y\right) \,,$$

$$A_1(\psi_y) = B'_1(\psi_y) \,,$$

$$B_3(3\psi_y) = \frac{1}{2\sin 3\pi\nu_y} \sum_k \frac{s'_k}{4} \cos 3(\psi'_{yk} - \psi_y - \pi\nu_y) \,,$$

$$A_3(3\psi_y) = B'_3(3\psi_y) \,,$$

$$\bar{B}(\psi_y) = \frac{1}{2\sin\pi\nu_y} \sum_k \frac{\bar{s}'_k}{4} \cos\left(\psi'_{yk} - \psi_y - \pi\nu_y\right) \,,$$

$$\bar{A}(\psi_y) = \bar{B}'(\psi_y) \,,$$

$$B_s(\psi_+) = \frac{1}{2\sin\pi\nu_+} \sum_k \frac{\bar{s}'_k}{4} \cos\left(\psi'_{+k} - \psi_+ - \pi\nu_+\right) \,,$$

$$A_s(\psi_+) = B'_s(\psi_+) \,,$$

$$B_d(\psi_-) = \frac{1}{2\sin\pi\nu_-} \sum_k \frac{\bar{s}'_k}{4} \cos\left(\psi'_{-k} - \psi_- - \pi\nu_-\right) \,,$$

$$A_d(\psi_-) = B'_d(\psi_-) \,, \tag{4.12}$$

where $\psi_{\pm} = 2\psi_x \pm \psi_y$ and ψ_x' and ψ_y' are defined in Eq. (2.21). From Eqs. (2.12) and (2.17), we recall that the distortion of the amplitudes $\mathcal{A}_u$ and phases ϕ_u are given by

$$\delta\mathcal{A}_u = \left(\frac{\beta_0}{2I_u}\right)^{1/2} \delta I_u \tag{4.13}$$

and

$$\delta\phi_u = \delta a_u \; . \tag{4.14}$$

Using Eqs. (4.10) to (4.14), we obtain

$$\delta\mathcal{A}_x = -2\mathcal{A}_x\mathcal{A}_y[(A_s \sin\phi_+ - B_s\cos\phi_+) + (A_d\sin\phi_- - B_d\cos\phi_-)] \; , \tag{4.15}$$

$$\begin{aligned} \delta\mathcal{A}_y = {} & \mathcal{A}_y^2[(A_1\sin\phi_y - B_1\cos\phi_y) + (A_3\sin 3\phi_y - B_3\cos 3\phi_y)] \\ & - \mathcal{A}_x^2[2(\bar{A}\sin\phi_y - \bar{B}\cos\phi_y) + (A_s\sin\phi_+ - B_s\cos\phi_+) - (A_d\sin\phi_- - B_d\cos\phi_-)] \; , \end{aligned} \tag{4.16}$$

$$\delta\phi_x = -2\mathcal{A}_y[2(\bar{A}\cos\phi_y + \bar{B}\sin\phi_y) + (A_s\cos\phi_+ + B_s\sin\phi_+) + (A_d\cos\phi_- + B_d\sin\phi_-)] \; , \tag{4.17}$$

$$\begin{aligned} \delta\phi_y = {} & \mathcal{A}_y[3(A_1\cos\phi_y + B_1\sin\phi_y) + (A_3\cos 3\phi_y + B_3\sin 3\phi_y)] \\ & - \frac{\mathcal{A}_x^2}{\mathcal{A}_y}[2(\bar{A}\cos\phi_y + \bar{B}\sin\phi_y) + (A_s\cos\phi_+ + B_s\sin\phi_+) + (A_d\cos\phi_- + B_d\sin\phi_-)] \; . \end{aligned} \tag{4.18}$$

The skew sextupoles have an average dipole effect on a charged particle which leads to a distortion of the ideal closed orbit. This can be obtained by separating out from Eqs. (4.16) and (4.18),

$$\delta\mathcal{A}'_y = 2\mathcal{A}_y^2(A_1\sin\phi_y - B_1\cos\phi_y) - 2\mathcal{A}_x^2(\bar{A}\sin\phi_y - \bar{B}\cos\phi_y) \; , \tag{4.19}$$

$$\mathcal{A}_y\delta\phi'_y = 2\mathcal{A}_y^2(B_1\sin\phi_y + A_1\cos\phi_y) - 2\mathcal{A}_x^2(\bar{B}\sin\phi_y + \bar{A}\cos\phi_y) \; , \tag{4.20}$$

which correspond to a closed orbit distortion of

$$\delta y = 2(\mathcal{A}_x^2\bar{B} - \mathcal{A}_y^2 B_1) \; , \tag{4.21}$$

$$\delta y' = 2(\mathcal{A}_x^2\bar{A} - \mathcal{A}_y^2 A_1) \; . \tag{4.22}$$

Thus the distorted beam shape in phase space can be written as

$$\begin{aligned} x &= (\mathcal{A}_x + \delta\mathcal{A}_x)\cos(\phi_x + \delta\phi_x) \; , & (4.23) \\ x' &= -(\mathcal{A}_x + \delta\mathcal{A}_x)\sin(\phi_x + \delta\phi_x) \; , & (4.24) \\ y &= \delta y + (\mathcal{A}_y + \delta\mathcal{A}_y)\cos(\phi_y + \delta\phi_y) \; , & (4.25) \\ y' &= \delta y' - (\mathcal{A}_y + \delta\mathcal{A}_y)\sin(\phi_y + \delta\phi_y) \; , & (4.26) \end{aligned}$$

where $\delta\mathcal{A}_x$ and $\delta\phi_x$ are given by Eqs. (4.15) and (4.17), δy and $\delta y'$ by Eqs. (4.21) and (4.22) and $\delta\mathcal{A}_y$ and $\delta\phi_y$ by the differences of Eqs. (4.16), (4.18) and (4.19), (4.20), or

$$\begin{aligned} \delta\mathcal{A}_y = {} & \mathcal{A}_y^2[-(A_1\sin\phi_y - B_1\cos\phi_y) + (A_3\sin 3\phi_y - B_3\cos 3\phi_y)] \\ & - \mathcal{A}_x^2[(A_s\sin\phi_+ - B_s\cos\phi_+) - (A_d\sin\phi_- - B_d\cos\phi_-)] \; , \end{aligned} \tag{4.27}$$

$$\delta\phi_y = \mathcal{A}_y[(A_1\cos\phi_y + B_1\sin\phi_y) + (A_3\cos 3\phi_y + B_3\sin 3\phi_y)]$$
$$-\frac{\mathcal{A}_x^2}{\mathcal{A}_y}[(A_s\cos\phi_+ + B_s\sin\phi_+) + (A_d\cos\phi_- + B_d\sin\phi_-)] \,. \quad (4.28)$$

4.2 Second-order Tuneshifts

As expected from the symmetry between the equations for the normal sextupoles and the ones for the skew sextupoles (if one interchanges x and y in the equations for normal sextupoles, one gets the equations for skew sextupoles and vice versa), the first-order tuneshift due to skew sextupoles also vanishes. To obtain the second-order tuneshifts, we need to evaluate the second-order sextupole terms in the Hamiltonian H_4. From the generating function G_3 of Eq. (4.8), we get

$$(2I_y)^{3/2} = (2J_y)^{3/2} + 9(2J_y)^2\beta_0^{1/2}\sum_m\left(\frac{A'_{1m}}{m-\nu_y}\cos p_{1m} + \frac{A'_{3m}}{m-3\nu_y}\cos p_{3m}\right)$$
$$-3(2J_x)(2J_y)\beta_0^{1/2}\sum_m\left(\frac{2B'_{1m}}{m-\nu_y}\cos q_{1m} + \frac{B'_{+m}}{m-\nu_+}\cos q_{+m} - \frac{B'_{-m}}{m-\nu_-}\cos q_{-m}\right) \quad (4.29)$$

and similar expressions for $(2I_y)^{1/2}(2I_x)$. Then the second-order terms in the Hamiltonian is

$$\Delta H_4|_{ss} = \sum_{m'}(3A'_{1m'}\cos p_{1m'} + A'_{3m'}\cos p_{3m'})\times$$
$$\times\left[9\beta_0(2J_y)^2\sum_m\left(\frac{A'_{1m}}{m-\nu_y}\cos p_{1m} + \frac{A'_{3m}}{m-3\nu_y}\cos p_{3m}\right)\right.$$
$$\left.-3(2J_y)(2J_x)\beta_0\sum_m\left(\frac{2B'_{1m}}{m-\nu_y}\cos q_{1m} + \frac{B'_{+m}}{m-\nu_+}\cos q_{+m} - \frac{B'_{-m}}{m-\nu_-}\cos q_{-m}\right)\right]$$
$$+\dots \,. \quad (4.30)$$

If we now consider only the θ-independent terms we obtain

$$\Delta H'_4|_{ss} = \frac{9}{2}\beta_0(2J_y)^2\sum_m\left(\frac{3A'^2_{1m}}{m-\nu_y} + \frac{A'^2_{3m}}{m-3\nu_y}\right)$$
$$+\frac{1}{2}\beta_0(2J_x)^2\sum_m\left(\frac{4B'^2_{1m}}{m-\nu_y} + \frac{B'^2_{+m}}{m-\nu_+} - \frac{B'^2_{-m}}{m-\nu_-}\right)$$
$$+2\beta_0(2J_x)(2J_y)\sum_m\left[\frac{B'^2_{+m}}{m-\nu_+} + \frac{B'^2_{-m}}{m-\nu_-} - \frac{6A'_{1m}B'_{1m}}{m-\nu_y}\cos(\alpha'_{1m} - \beta'_{1m})\right] \,. \quad (4.31)$$

Summation over the harmonics leads to the following result

$$\sum_m\frac{A'^2_{1m}}{m-\nu_y} = -\frac{1}{72\pi}\sum_k(B_1s')_k \,,$$
$$\sum_m\frac{A'^2_{3m}}{m-3\nu_y} = -\frac{1}{72\pi}\sum_k(B_3s')_k \,,$$

$$\sum_m \frac{B'^2_{1m}}{m-\nu_y} = -\frac{1}{8\pi}\sum_k (\bar{B}\bar{s}')_k \ ,$$

$$\sum_m \frac{B'^2_{+m}}{m-\nu_+} = -\frac{1}{8\pi}\sum_k (B_s\bar{s}')_k \ ,$$

$$\sum_m \frac{B'^2_{-m}}{m-\nu_-} = \frac{1}{8\pi}\sum_k (B_d\bar{s}')_k \ ,$$

$$\sum_m \frac{A'_{1m}B'_{1m}}{m-\nu_y}\cos(\alpha'_{1m}-\beta'_{1m}) = -\frac{1}{24\pi}\sum_k (B_1\bar{s}')_k \ . \qquad (4.32)$$

So the tuneshifts given by Eq. (2.37) are

$$\Delta\nu_x = -\frac{1}{4\pi}\mathcal{A}_x^2\sum_k (4\bar{B}\bar{s}'+B_s\bar{s}'-B_d\bar{s}')_k - \frac{1}{2\pi}\mathcal{A}_y^2\sum_k (B_s\bar{s}'+B_d\bar{s}'-2B_1\bar{s})_k \ , \qquad (4.33)$$

and

$$\Delta\nu_y = -\frac{1}{4\pi}\mathcal{A}_y^2\sum_k (3B_1s'+B_3s')_k - \frac{1}{2\pi}\mathcal{A}_x^2\sum_k (B_s\bar{s}'+B_d\bar{s}'-2B_1\bar{s}')_k \ . \qquad (4.34)$$

5 The Octupole Term

5.1 Beam Shape Distortions due to Normal Octupoles

The normal octupole term in the Hamiltonian (1.3) is

$$\Delta H_3|_{\rm oct} = \frac{RB'''_y}{24(B\rho)}\left[\left(\frac{\beta_x}{\beta_0}\right)^2 x^4 - \frac{6\beta_x\beta_y}{\beta_0^2}x^2y^2 + \left(\frac{\beta_y}{\beta_0}\right)^2 y^4\right] \ . \qquad (5.1)$$

If we expand this into harmonics, we get

$$\begin{aligned}\Delta H_3|_{\rm oct} = {} & (2I_x)^2\beta_0\sum_m (3\underline{A}^0_m\cos q_{0m} + 4\underline{A}_{2m}\cos q_{2m} + \underline{A}_{4m}\cos q_{4m}) \\ & -(2I_x)(2I_y)\beta_0\sum_m (2B^0_m\cos p_{0m} - 2B_{xm}\cos p_{xm} \\ & +2B_{ym}\cos p_{ym} + B_{+m}\cos p_{+m} - B_{-m}\cos p_{-m}) \\ & +(2I_y)^2\beta_0\sum_m (3\bar{A}^0_m\cos r_m + 4\bar{A}_{2m}\cos r_{2m} + \bar{A}_{4m}\cos r_{4m}) \ ,\end{aligned} \qquad (5.2)$$

where $q_{0m} = \underline{\alpha}^0_m - m\theta$, $q_{2m} = \underline{\alpha}_{2m} - m\theta + 2a_x$, $q_{4m} = \underline{\alpha}_{4m} - m\theta + 4a_x$, $p_{0m} = \beta^0_m - m\theta$, $p_{xm} = \beta_{xm} - m\theta + 2a_x$, $p_{ym} = \beta_{ym} - m\theta + 2a_y$, $p_{+m} = \beta_{+m} - m\theta + 2a_+$, $p_{-m} = \beta_{-m} - m\theta + 2a_-$, $r_{0m} = \bar{\alpha}^0_m - m\theta$, $r_{2m} = \bar{\alpha}_{2m} - m\theta + 2a_y$, $r_{4m} = \bar{\alpha}_{4m} - m\theta + 4a_y$, $a_\pm = a_x \pm a_y$, and

$$\underline{A}^0_m e^{i\underline{\alpha}_m} = \frac{1}{64\pi}\sum_k \mathrm{m}_k e^{im\theta_k} \ ,$$

$$\underline{A}_{2m} e^{i\underline{\alpha}_{2m}} = \frac{1}{64\pi}\sum_k \mathrm{m}_k e^{i(2Q_x+m\theta)_k} \ ,$$

$$\underline{A}_{4m}e^{i\underline{\alpha}_{4m}} = \frac{1}{64\pi}\sum_k \underline{m}_k e^{i(4Q_x+m\theta)_k} ,$$
$$B^0_m e^{i\beta^0_m} = \frac{3}{32\pi}\sum_k m_k e^{im\theta_k} ,$$
$$B_{xm}e^{i\beta_{xm}} = \frac{3}{32\pi}\sum_k m_k e^{i(2Q_x+m\theta)_k} ,$$
$$B_{ym}e^{i\beta_{ym}} = \frac{3}{32\pi}\sum_k m_k e^{i(2Q_y+m\theta)_k} ,$$
$$B_{\pm m}e^{i\beta_{\pm m}} = \frac{3}{32\pi}\sum_k m_k e^{i(2Q_\pm+m\theta)_k} ,$$
$$\bar{A}^0_m e^{i\bar{\alpha}^0_m} = \frac{1}{64\pi}\sum_k \bar{m}_k e^{im\theta_k} ,$$
$$\bar{A}_{2m}e^{i\bar{\alpha}_{2m}} = \frac{1}{64\pi}\sum_k \bar{m}_k e^{i(2Q_y+m\theta)_k} ,$$
$$\bar{A}_{4m}e^{i\bar{\alpha}_{4m}} = \frac{1}{64\pi}\sum_k \bar{m}_k e^{i(4Q_y+m\theta)_k} , \tag{5.3}$$

with $Q_\pm = Q_x \pm Q_y$. The normal octupoles are assumed to have infinitesimal length ℓ_k with strengths

$$\underline{m}_k = \left(\frac{\beta_x^2}{\beta_0}\right)\frac{(B_y'''\ell)_k}{6(B\rho)} , \quad m_k = \left(\frac{\beta_x\beta_y}{\beta_0}\right)\frac{(B_y'''\ell)_k}{6(B\rho)} , \quad \bar{m}_k = \left(\frac{\beta_y^2}{\beta_0}\right)\frac{(B_y'''\ell)_k}{6(B\rho)} . \tag{5.4}$$

For the first-order beam shape, we shall solve the equations of motion obtained from the Hamiltonian H_3 to first order, instead of performing a Moser transformation. This is because, unlike the previously analyzed multipoles, normal octupole has a first-order tuneshift which can and will be calculated directly from the Hamiltonian H_3.

The equations of motion are given by

$$\frac{dI_x}{d\theta} = -\frac{\partial H_3}{\partial a_x} = (2I_x)^2\beta_0\sum_m(8\underline{A}_{2m}\sin q_{2m}+4\underline{A}_{4m}\sin q_{4m})$$
$$-(2I_x)(2I_y)\beta_0\sum_m(4B_{xm}\sin p_{xm}+2B_{+m}\sin p_{+m}-2B_{-m}\sin p_{-m}) , \tag{5.5}$$

$$\frac{dI_y}{d\theta} = -\frac{\partial H_3}{\partial a_y} = -(2I_x)(2I_y)\beta_0\sum_m(4B_{ym}\sin p_{ym}-2B_{-m}\sin p_{+m}$$
$$-2B_{-m}\sin p_{-m}) + (2I_y)^2\beta_0\sum_m(8\bar{A}_{2m}\sin r_{2m}-4\bar{A}_{4m}\sin r_{4m}) , \tag{5.6}$$

$$\frac{da_x}{d\theta} = \frac{\partial H_3}{\partial I_x} = \nu_x + 4(2I_x)\beta_0\sum_m(3\underline{A}^0_m\cos q_{0m}-4\underline{A}_{2m}\cos q_{2m}+\underline{A}_{4m}\cos q_{4m})$$
$$-2(2I_y)\beta_0\sum_m(2B^0_m\cos p_{0m}+2B_{xm}\cos p_{xm}-2B_{ym}\cos p_{ym}+B_{+m}\cos p_{+m}+B_{-m}\cos p_{-m}) , \tag{5.7}$$

$$\frac{da_y}{d\theta} = \frac{\partial H_3}{\partial I_y} = \nu_y - 2(2I_x)\beta_0\sum_m(2B^0_m\cos p_{0m}-2B_{xm}\cos p_{xm}+2B_{ym}\cos p_{ym}+B_{+m}\cos p_{+m}$$

$$+B_{-m}\cos p_{-m}) + 4(2I_y)\beta_0 \sum_m (3\bar{A}^0_m \cos q_{0m} + 4\bar{A}_{2m}\cos r_{2m} + \bar{A}_{4m}\cos r_{4m}) \ . \tag{5.8}$$

The solution of Eqs. (5.7) and (5.8) gives, in the absence of octupoles, $a_u = \nu_u\theta +$ constant. Again, we choose

$$a_u = \nu_u\theta - \psi_u + \phi_u \ , \tag{5.9}$$

where ψ_u is the Floquet phase at position θ and ϕ_u is the instantaneous phase of the betatron oscillation. Since we are interested in solutions accurate up to lowest order in $\underline{m}_k$, m_k and $\bar{m}_k$ only, on the right hand side of Eqs. (5.5) to (5.8), I_x and I_y can be considered as θ-independent and Eq. (5.9) can be substituted for a_u. Then we can integrate all four differential equations easily. Denoting by δ the deviation from the situation where the octupoles are absent, we obtain

$$\begin{aligned}
\delta I_x &= (2I_x)^2\beta_0 \sum_m \left(\frac{8\underline{A}_{2m}}{m-2\nu_x}\cos q_{2m} + \frac{4\underline{A}_{4m}}{m-4\nu_x}\cos q_{4m}\right) \\
&\quad -(2I_x)(2I_y)\beta_0 \sum_m \left(\frac{4B_{xm}}{m-2\nu_x}\cos p_{xm} + \frac{2B_{+m}}{m-2\nu_+}\cos p_{+m} + \frac{2B_{-m}}{m-2\nu_-}\cos p_{-m}\right) , \qquad (5.10)\\
\delta I_y &= -(2I_x)(2I_y)\beta_0 \sum_m \left(\frac{4B_{ym}}{m-2\nu_y}\cos p_{ym} + \frac{2B_{+m}}{m-2\nu_+}\cos p_{+m} - \frac{2B_{-m}}{m-2\nu_-}\cos p_{-m}\right) \\
&\quad +(2I_y)^2\beta_0 \sum_m \left(\frac{8\bar{A}_{2m}}{m-2\nu_y}\cos r_{2m} + \frac{4\bar{A}_{4m}}{m-4\nu_y}\cos r_{4m}\right) , \qquad (5.11)\\
\delta a_x &= -4(2I_x)\beta_0 \sum_m \left(\frac{3\underline{A}^0_m}{m}\sin q_{0m} + \frac{4\underline{A}_{2m}}{m-2\nu_x}\sin q_{2m} + \frac{\underline{A}_{4m}}{m-4\nu_x}\sin q_{4m}\right) \\
&\quad +2(2I_y)\beta_0 \sum_m \left(\frac{2B^0_m}{m}\sin p_{0m} + \frac{2B_{xm}}{m-2\nu_x}\sin p_{xm}\right. \\
&\quad \left. + \frac{2B_{ym}}{m-2\nu_y}\sin p_{ym} + \frac{B_{+m}}{m-2\nu_+}\sin p_{+m} + \frac{B_{-m}}{m-2\nu_-}\sin p_{-m}\right) , \qquad (5.12)\\
\delta a_y &= 2(2I_x)\beta_0 \sum_m \left(\frac{2B^0_m}{m}\sin p_{0m} + \frac{2B_{xm}}{m-2\nu_x}\sin p_{xm}\right. \\
&\quad \left. + \frac{2B_{ym}}{m-2\nu_y}\sin p_{ym} + \frac{B_{+m}}{m-2\nu_+}\sin p_{+m} + \frac{B_{-m}}{m-2\nu_-}\sin p_{-m}\right) \\
&\quad -4(2I_y)\beta_0 \sum_m \left(\frac{3\bar{A}^0_m}{m}\sin r_{0m} + \frac{4\bar{A}_{2m}}{m-2\nu_y}\sin r_{2m} + \frac{\bar{A}_{4m}}{m-4\nu_y}\sin r_{4m}\right) , \qquad (5.13)
\end{aligned}$$

where $\nu_\pm = \nu_x \pm \nu_y$. Finally we are going to perform the summation over m, using formula (2.18). This yields the following results

$$\begin{aligned}
\sum_m \frac{\underline{A}_{2m}}{m-2\nu_x} e^{iq_{2m}} &= -\frac{e^{2i\phi_x}}{4}(B_2 + iA_2) \ , \\
\sum_m \frac{\underline{A}_{4m}}{m-4\nu_x} e^{iq_{4m}} &= -\frac{e^{4i\phi_x}}{4}(B_1 + iA_1) \ ,
\end{aligned}$$

$$\sum_m \frac{B_{xm}}{m-2\nu_x} e^{ip_{xm}} = -\frac{3e^{2i\phi_x}}{2}(B_5 + iA_5) ,$$

$$\sum_m \frac{B_{+m}}{m-2\nu_+} e^{ip_{+m}} = -\frac{3e^{2i\phi_+}}{2}(B_3 + iA_3) ,$$

$$\sum_m \frac{B_{-m}}{m-2\nu_-} e^{ip_{-m}} = -\frac{3e^{2i\phi_-}}{2}(B_4 + iA_4) ,$$

$$\sum_m \frac{B_{ym}}{m-2\nu_y} e^{ip_{ym}} = -\frac{3e^{2i\phi_y}}{2}(B_6 + iA_6) ,$$

$$\sum_m \frac{\bar{A}_{2m}}{m-2\nu_y} e^{ir_{2m}} = -\frac{e^{2i\phi_y}}{4}(B_8 + iA_8) ,$$

$$\sum_m \frac{\bar{A}_{4m}}{m-4\nu_y} e^{ir_{4m}} = -\frac{e^{4i\phi_y}}{4}(B_7 + iA_7) , \tag{5.14}$$

where $\phi_\pm = \phi_x \pm \phi_y$ and the various sets of distortion functions are defined as follows

$$B_1(4\psi_x) = \frac{1}{2\sin\pi 4\nu_x} \sum_k \frac{\mathrm{m}_k}{8} \cos 4(\psi'_{xk} - \psi_x - \pi\nu_x) ,$$

$$A_1(4\psi_x) = B'_1(4\psi_x) ,$$

$$B_2(2\psi_x) = \frac{1}{2\sin\pi 2\nu_x} \sum_k \frac{\mathrm{m}_k}{8} \cos 2(\psi'_{xk} - \psi_x - \pi\nu_x) ,$$

$$A_2(2\psi_x) = B'_2(2\psi_x) ,$$

$$B_3(2\psi_+) = \frac{1}{2\sin\pi 2\nu_+} \sum_k \frac{m_k}{8} \cos 2(\psi'_{+k} - \psi_+ - \pi\nu_+) ,$$

$$A_3(2\psi_+) = B'_3(2\psi_+) ,$$

$$B_4(2\psi_-) = \frac{1}{2\sin\pi 2\nu_-} \sum_k \frac{m_k}{8} \cos 2(\psi'_{-k} - \psi_- - \pi\nu_-) ,$$

$$A_4(2\psi_-) = B'_4(2\psi_-) ,$$

$$B_5(2\psi_x) = \frac{1}{2\sin\pi 2\nu_x} \sum_k \frac{m_k}{8} \cos 2(\psi'_{xk} - \psi_x - \pi\nu_x) ,$$

$$A_5(2\psi_x) = B'_5(2\psi_x) ,$$

$$B_6(2\psi_y) = \frac{1}{2\sin\pi 2\nu_y} \sum_k \frac{m_k}{8} \cos 2(\psi'_{yk} - \psi_y - \pi\nu_y) ,$$

$$A_6(2\psi_y) = B'_6(2\psi_y) ,$$

$$B_7(4\psi_y) = \frac{1}{2\sin\pi 4\nu_y} \sum_k \frac{\bar{m}_k}{8} \cos 4(\psi'_{yk} - \psi_y - \pi\nu_y) ,$$

$$A_7(4\psi_y) = B'_7(4\psi_y) ,$$

$$B_8(2\psi_y) = \frac{1}{2\sin\pi 2\nu_y}\sum_k \frac{\bar{m}_k}{8}\cos 2(\psi'_{yk} - \psi_y - \pi\nu_y) ,$$

$$A_8(2\psi_y) = B'_8(2\psi_y) . \tag{5.15}$$

Here, again ψ'_x and ψ'_y are defined by Eq. (2.21).

We recall that the distortion of the amplitudes $\mathcal{A}_u$ and phases ϕ_u are given by $\delta\mathcal{A}_u = \delta(2I_u\beta_0)^2$ and $\delta\phi_u = \delta a_u$. Then using Eqs. (5.10) to (5.13) and (5.14) we arrive at

$$\begin{aligned}\delta\mathcal{A}_x = {} & \mathcal{A}_x^3[(A_1\sin 4\phi_x - B_1\cos 4\phi_x) + 2(A_2\sin 2\phi_x - B_2\cos 2\phi_x)] \\ & -3\mathcal{A}_x\mathcal{A}_y^2[2(A_5\sin 2\phi_x - B_5\cos 2\phi_x) + (A_3\sin 2\phi_+ - B_3\cos 2\phi_+) + (A_4\sin 2\phi_- - B_4\cos 2\phi_-)],\end{aligned} \tag{5.16}$$

$$\begin{aligned}\delta\mathcal{A}_y = {} & -3\mathcal{A}_x^2\mathcal{A}_y[2(A_6\sin 2\phi_y - B_6\cos 2\phi_y) + (A_3\sin 2\phi_+ - B_3\cos 2\phi_+) \\ & -(A_4\sin 2\phi_- - B_4\cos 2\phi_-)] + \mathcal{A}_y^3[2(A_8\sin 2\phi_y - B_8\cos 2\phi_y) + (A_7\sin 4\phi_y - B_7\cos 4\phi_y)] ,\end{aligned} \tag{5.17}$$

$$\begin{aligned}\delta\phi_x = {} & \mathcal{A}_x^2[(B_1\sin 4\phi_x + A_1\cos 4\phi_x) + 4(B_2\sin 2\phi_x + A_2\cos 2\phi_x)] \\ & -3\mathcal{A}_y^2[2(B_5\sin 2\phi_x + A_5\cos 2\phi_x) + 2(B_6\sin 2\phi_y + A_6\cos 2\phi_y) \\ & +(B_3\sin 2\phi_+ + A_3\cos 2\phi_+) + (B_4\sin 2\phi_- + A_4\cos 2\phi_-)] ,\end{aligned} \tag{5.18}$$

$$\begin{aligned}\delta\phi_y = {} & -3\mathcal{A}_x^2[2(B_5\sin 2\phi_x + A_5\cos 2\phi_x) + 2(B_6\sin 2\phi_y + A_6\cos 2\phi_y) \\ & +(B_3\sin 2\phi_+ + A_3\cos 2\phi_+) + (B_4\sin 2\phi_- + A_4\cos 2\phi_-)] \\ & +\mathcal{A}_y^2[4(A_8\cos 2\phi_y + B_8\sin 2\phi_y) + (A_7\cos 4\phi_y + B_7\sin 4\phi_y)] .\end{aligned} \tag{5.19}$$

Here we would like to comment on those terms of δa_x and δa_y in Eqs. (5.12) and (5.13), whose denominator is m. Even though they seem to diverge when $m = 0$, this should not be the case because they really come from the Hamiltonian (5.1) which is a finite quantity. In fact, there is a missing term in Eqs. (5.12) and (5.13) corresponding to the lower limit, θ_0, of integration over the angle θ. So in reality these "divergent-like" terms are of the form $\sum_m(e^{im\theta} - e^{im\theta_0})/m$ which does not diverge for $m = 0$. This lower limit of integration θ_0, determines the position around the ring where the initial conditions are considered. So, in general, there will be one more term contributing to the expression for $\delta\phi_x$ and $\delta\phi_y$ which will be some complicated function of θ.

In the particular case of integration of the equations of motion over exactly *one turn* around the ring, one obtains an interesting, though expected, result. Recall that the origin of the "divergent-like" terms is the part of the Hamiltonian which is independent of the angle variable, a. If one writes the equations of motion for a_x and a_y for this part of the Hamiltonian and integrates them over *one turn*, one gets

$$\delta a'_x = \frac{3}{8}\mathcal{A}_x^2\sum_k \underline{m}_k - \frac{3}{4}\mathcal{A}_y^2\sum_k m_k , \tag{5.20}$$

and

$$\delta a'_y = -\frac{3}{4}\mathcal{A}_x^2\sum_k m_k + \frac{3}{8}\mathcal{A}_y^2\sum_k \bar{m}_k . \tag{5.21}$$

As we shall see in the next section these expressions are just the first-order tuneshifts (in units of 2π). This particular case represents the experimental reality more closely. Indeed, one usually chooses a point around the ring as the observation point and then one follows the behavior of the beam at this same point for every turn. In this case the expressions for the phase distortions are simply given by the sum of Eqs. (5.18) and (5.20) for the horizontal plane and the sum of Eqs. (5.19) and (5.21) for the vertical plane.

5.2 First-order Tuneshifts

In the case of a normal octupole, there exist first-order terms in the Hamiltonian $\Delta H_3|_{\rm oct}$ which are θ-independent and yield a first-order tuneshift. The θ-independent part of $\Delta H_3|_{\rm oct}$ is

$$\Delta H_3'|_{\rm oct} = 3(2I_x)^2\beta_0 \underline{A}_0^0 \cos\underline{\alpha}_0^0 - 2(2I_x)(2I_y)\beta_0 B_0^0 \cos\beta_0^0 + 3(2I_y)^2\beta_0 \bar{A}_0^0 \cos\bar{\alpha}_0^0 \ . \tag{5.22}$$

If we recall Eqs. (5.3) we find that

$$\begin{aligned} \underline{A}_0^0 \cos\underline{\alpha}_0^0 &= \frac{1}{64\pi}\sum_k \underline{m}_k \ , \\ \bar{A}_0^0 \cos\bar{\alpha}_0^0 &= \frac{1}{64\pi}\sum_k \bar{m}_k \ , \\ B_0^0 \cos\beta_0^0 &= \frac{3}{32\pi}\sum_k m_k \ . \end{aligned} \tag{5.23}$$

So $\Delta H_3'|_{\rm oct}$ becomes

$$\Delta H_3'|_{\rm oct} = \frac{3}{64\pi}(2I_x)^2\beta_0\sum_k \underline{m}_k - \frac{3}{16\pi}(2I_x)(2I_y)\beta_0\sum_k m_k + \beta_0\sum_k \bar{m}_k \ . \tag{5.24}$$

And the tuneshifts are

$$\begin{aligned} \Delta\nu_x &= \frac{3}{16\pi}(2I_x\beta_0)\sum_k \underline{m}_k - \frac{3}{8\pi}(2I_y\beta_0)\sum_k m_k \\ \Delta\nu_y &= -\frac{3}{8\pi}(2I_x\beta_0)\sum_k m_k + \frac{3}{16\pi}(2I_y\beta_0)\sum_k \bar{m}_k \ . \end{aligned} \tag{5.25}$$

Recalling that the amplitudes $\mathcal{A}_x$ and $\mathcal{A}_y$ are $\mathcal{A}_x = (2I_x\beta_0)^{1/2}$ and $\mathcal{A}_y = (2I_y\beta_0)^{1/2}$, we arrive at

$$2\pi\Delta\nu_x = \frac{3}{8}\mathcal{A}_x^2\sum_k \underline{m}_k - \frac{3}{4}\mathcal{A}_y^2\sum_k m_k \ , \tag{5.26}$$

and

$$2\pi\Delta\nu_y = -\frac{3}{4}\mathcal{A}_x^2\sum_k m_k + \frac{3}{8}\mathcal{A}_y^2\sum_k \bar{m}_k \ . \tag{5.27}$$

6 The Skew Octupole Term

6.1 Beam Shape Distortions due to Skew Octupoles

We recall that the skew octupole term in the Hamiltonian (1.3) is

$$\Delta H_3\,|_{so}= -\frac{RB_x'''}{6(B\rho)}\left[\left(\frac{\beta_x^3\beta_y}{\beta_0^4}\right)^{1/2} x^3y - \left(\frac{\beta_x\beta_y^3}{\beta_0^4}\right)^{1/2} xy^3\right] . \tag{6.1}$$

If we expand this into harmonics we get

$$\begin{aligned}\Delta H_3\,|_{so}= &-(2I_x)^{3/2}(2I_y)^{1/2}\beta_0\sum_m(3A_{1+m}\cos q_{1+m} + 3A_{1-m}\cos q_{1-m}\\ &+A_{3+m}\cos q_{3+m} + A_{3-m}\cos q_{3-m})\\ &+(2I_y)^{3/2}(2I_x)^{1/2}\beta_0\sum_m(3B_{1+m}\cos p_{1+m} + 3B_{1-m}\cos p_{1-m}\\ &+B_{3+m}\cos p_{3+m} + B_{3-m}\cos p_{3-m}) ,\end{aligned} \tag{6.2}$$

where $q_{1+m} = \alpha_{1+m} - m\theta + a_+$, $q_{1-m} = \alpha_{1-m} - m\theta + a_-$, $q_{3+m} = \alpha_{3+m} - m\theta + 3a_x + a_y$, $q_{3-m} = \alpha_{3-m} - m\theta + 3a_x - a_y$, $p_{1+m} = \beta_{1+m} - m\theta + a_+$, $p_{1-m} = \beta_{1-m} - m\theta + a_-$, $p_{3+m} = \beta_{3+m} - m\theta + a_x + 3a_y$, $p_{3-m} = \beta_{3-m} - m\theta + a_x - 3a_y$, $a_\pm = a_x \pm a_y$, and

$$\begin{aligned}A_{1+m}e^{i\alpha_{1+m}} &= \frac{1}{16\pi}\sum_k m_k e^{i(Q_+ + m\theta)_k} ,\\ A_{1-m}e^{i\alpha_{1-m}} &= \frac{1}{16\pi}\sum_k m_k e^{i(Q_- + m\theta)_k} ,\\ A_{3+m}e^{i\alpha_{3+m}} &= \frac{1}{16\pi}\sum_k m_k e^{i(3Q_x + Q_y + m\theta)_k} ,\\ A_{3-m}e^{i\alpha_{3-m}} &= \frac{1}{16\pi}\sum_k m_k e^{i(3Q_x - Q_y + m\theta)_k} ,\\ B_{1+m}e^{i\beta_{1+m}} &= \frac{1}{16\pi}\sum_k \bar{m}_k e^{i(Q_+ + m\theta)_k} ,\\ B_{1-m}e^{i\beta_{1-m}} &= \frac{1}{16\pi}\sum_k \bar{m}_k e^{i(Q_- + m\theta)_k} ,\\ B_{3+m}e^{i\beta_{3+m}} &= \frac{1}{16\pi}\sum_k \bar{m}_k e^{i(Q_x + 3Q_y + m\theta)_k} ,\\ B_{3-m}e^{i\beta_{3-m}} &= \frac{1}{16\pi}\sum_k \bar{m}_k e^{i(Q_x - 3Q_y + m\theta)_k} .\end{aligned} \tag{6.3}$$

where $Q_\pm = Q_x \pm Q_y$. Again here we assume skew octupoles with infinitesimal length ℓ_k and strengths

$$m_k = \left(\frac{\beta_x^3\beta_y}{\beta_0^2}\right)^{1/2}\frac{(B_x'''\ell)_k}{6(B\rho)} , \qquad \bar{m}_k = \left(\frac{\beta_x\beta_y^3}{\beta_0^2}\right)^{1/2}\frac{(B_x'''\ell)_k}{6(B\rho)} . \tag{6.4}$$

Skew octupoles induce a second-order tuneshift, which we would like eventually to calculate. So instead of solving the equations of motion obtained from H_3, we proceed by making a Moser transformation from (a_u, I_u) to (b_u, J_u), $u = x, y$, using the generating function

$$\begin{aligned} G_3 = a_x J_x + a_y J_y - \beta_0 (2J_x)^{3/2}(2J_y)^{1/2} \sum_m \bigg(&\frac{3A_{1+m}}{m-\nu_+} \sin q_{1+m} \\ + \frac{3A_{1-m}}{m-\nu_-} \sin q_{1-m} + \frac{A_{3+m}}{m-(3\nu_x+\nu_y)} \sin q_{3+m} &+ \frac{A_{3-m}}{m-(3\nu_x-\nu_y)} \sin q_{3-m} \bigg) \\ + \beta_0 (2J_y)^{3/2}(2J_x)^{1/2} \sum_m \bigg(\frac{3B_{1+m}}{m-\nu_+} \sin p_{1+m} &+ \frac{3B_{1-m}}{m-\nu_-} \sin p_{1-m} \\ + \frac{B_{3+m}}{m-(\nu_x+3\nu_y)} \sin p_{3+m} &+ \frac{B_{3-m}}{m-(\nu_x-3\nu_y)} \sin p_{3-m} \bigg) , \end{aligned} \tag{6.5}$$

where $\nu_\pm = \nu_x \pm \nu_y$. So the new Hamiltonian H_4 is

$$H_4 = \nu_x J_x + \nu_y J_y + \Delta H_4 |_{\text{so}} , \tag{6.6}$$

where $\Delta H_4 |_{\text{so}}$ does not contain any zeroth or first-order terms in m_k and $\bar{m}_k$. The first-order changes in I_u and a_u which are given by Eqs. (2.7) and (2.8) are explicitly the following

$$\begin{aligned} \delta I_x = -(2I_x)^{3/2}(2I_y)^{1/2}\beta_0 \sum_m \bigg[&\frac{3A_{1+m}}{m-\nu_+} \cos q_{1+m} + \frac{3A_{1-m}}{m-\nu_-} \cos q_{1-m} + \frac{3A_{3+m} \cos q_{3+m}}{m-(3\nu_x+\nu_y)} \\ + \frac{3A_{3-m}}{m-(3\nu_x-\nu_y)} \cos q_{3-m} \bigg] &+ (2I_y)^{3/2}(2I_x)^{1/2}\beta_0 \sum_m \bigg[\frac{3B_{1+m}}{m-\nu_+} \cos p_{1+m} \\ + \frac{3B_{1-m}}{m-\nu_-} \cos p_{1-m} + \frac{B_{3+m}}{m-(\nu_x+3\nu_y)} &\cos p_{3+m} + \frac{B_{3-m}}{m-(\nu_x-3\nu_y)} \cos p_{3-m} \bigg] , \end{aligned} \tag{6.7}$$

$$\begin{aligned} \delta I_y = (2I_x)^{3/2}(2I_y)^{1/2}\beta_0 \sum_m \bigg[&-\frac{3A_{1+m}}{m-\nu_+} \cos q_{1+m} + \frac{3A_{1-m}}{m-\nu_-} \cos q_{1-m} - \frac{A_{3+m} \cos q_{3+m}}{m-(3\nu_x+\nu_y)} \\ + \frac{A_{3-m}}{m-(3\nu_x-\nu_y)} \cos q_{3-m} \bigg] &- (2I_y)^{3/2}(2I_x)^{1/2}\beta_0 \sum_m \bigg[-\frac{3B_{1+m}}{m-\nu_+} \cos p_{1+m} \\ + \frac{3B_{1-m}}{m-\nu_-} \cos p_{1-m} - \frac{3B_{3+m}}{m-(\nu_x+3\nu_y)} &\cos p_{3+m} + \frac{3B_{3-m}}{m-(\nu_x-3\nu_y)} \cos p_{3+m} \bigg] , \end{aligned} \tag{6.8}$$

$$\begin{aligned} \delta a_x = 3(2I_x)^{1/2}(2I_y)^{1/2}\beta_0 \sum_m \bigg[&\frac{3A_{1+m}}{m-\nu_+} \sin q_{1+m} + \frac{3A_{1-m}}{m-\nu_-} \sin q_{1-m} + \frac{A_{3+m} \sin q_{3+m}}{m-(3\nu_x+\nu_y)} \\ + \frac{A_{3-m}}{m-(3\nu_x-\nu_y)} \sin q_{3-m} \bigg] &- (2I_x)^{-1/2}(2I_y)^{3/2}\beta_0 \sum_m \bigg[\frac{3B_{1+m}}{m-\nu_+} \sin p_{1+m} \\ + \frac{3B_{1-m}}{m-\nu_-} \sin p_{1-m} + \frac{B_{3+m}}{m-(\nu_x+3\nu_y)} &\sin p_{3+m} + \frac{B_{3-m}}{m-(\nu_x-3\nu_y)} \sin p_{3-m} \bigg] , \end{aligned} \tag{6.9}$$

$$\begin{aligned} \delta a_y = (2I_y)^{-1/2}(2I_x)^{3/2}\beta_0 \sum_m \bigg[&\frac{3A_{1+m}}{m-\nu_+} \sin q_{1+m} + \frac{3A_{1-m}}{m-\nu_-} \sin q_{1-m} + \frac{A_{3+m} \sin q_{3+m}}{m-(3\nu_x+\nu_y)} \\ + \frac{A_{3-m}}{m-(3\nu_x-\nu_y)} \sin q_{3-m} \bigg] &- 3(2I_x)^{1/2}(2I_y)^{1/2}\beta_0 \sum_m \bigg[\frac{3B_{1+m}}{m-\nu_+} \sin p_{1+m} \end{aligned}$$

$$+\frac{3B_{1-m}}{m-\nu_-}\sin p_{1-m}+\frac{B_{3+m}}{m-(\nu_x+3\nu_y)}\sin p_{3+m}+\frac{B_{3-m}}{m-(\nu_x-3\nu_y)}\sin p_{3-m}\Bigg] \ . \quad (6.10)$$

These changes will yield the changes in amplitudes and phases. But let us first simplify Eqs. (6.7) to (6.10) by performing the summation over the harmonics with the aid of formula (2.18). The result of the summation is

$$\begin{aligned}
\sum_m \frac{A_{1+m}}{m-\nu_+}e^{iq_{1+m}} &= e^{i\phi_+}(B_3+iA_3)\ ,\\
\sum_m \frac{A_{1-m}}{m-\nu_-}e^{iq_{1-m}} &= e^{i\phi_-}(B_4+iA_4)\ ,\\
\sum_m \frac{A_{3+m}}{m-(3\nu_x+\nu_y)}e^{iq_{3+m}} &= e^{i(3\phi_x+\phi_y)}(B_1+iA_1)\ ,\\
\sum_m \frac{A_{3-m}}{m-(3\nu_x-\nu_y)}e^{iq_{3-m}} &= e^{i(3\phi_x-\phi_y)}(B_2+iA_2)\ ,\\
\sum_m \frac{B_{1+m}}{m-\nu_+}e^{ip_{1+m}} &= e^{i\phi_+}(B_5+iA_5)\ ,\\
\sum_m \frac{B_{1-m}}{m-\nu_-}e^{ip_{1-m}} &= e^{i\phi_-}(B_6+iA_6)\ ,\\
\sum_m \frac{B_{3+m}}{m-(\nu_x+3\nu_y)}e^{ip_{3+m}} &= e^{i(\phi_x+3\phi_y)}(B_7+iA_7)\ ,\\
\sum_m \frac{B_{3-m}}{m-(\nu_x-3\nu_y)}e^{ip_{3-m}} &= e^{i(\phi_x-3\phi_y)}(B_8+iA_8)\ . \qquad (6.11)
\end{aligned}$$

where $\phi_\pm = \phi_x \pm \phi_y$. The various sets of distortion functions are

$$B_{1,2}(3\psi_x\pm\psi_y) = -\frac{1}{2\sin\pi(3\nu_x\pm\nu_y)}\sum_k\frac{m_k}{8}\cos[(3\psi'_{xk}\pm\psi'_{yk})-(3\psi_x\pm\psi_y)-\pi(3\nu_x\pm\nu_y)]\ ,$$

$$A_{1,2}(3\psi_x\pm\psi_y) = B'_{1,2}(3\psi_x\pm\psi_y)\ ,$$

$$B_3(\psi_+) = -\frac{1}{2\sin\pi\nu_+}\sum_k\frac{m_k}{8}\cos(\psi'_{+k}-\psi_+-\pi\nu_+)\ ,$$

$$A_3(\psi_+) = B'_3(\psi_+)\ ,$$

$$B_4(\psi_-) = -\frac{1}{2\sin\pi\nu_-}\sum_k\frac{m_k}{8}\cos(\psi'_{-k}-\psi_--\pi\nu_-)\ ,$$

$$A_4(\psi_-) = B'_4(\psi_-)\ ,$$

$$B_{5,6}(\psi_\pm) = -\frac{1}{2\sin\pi\nu_\pm}\sum_k\frac{\bar{m}_k}{8}\cos\psi'_{\pm k}-\psi_\pm-\pi\nu_\pm)\ ,$$

$$A_{5,6}(\psi_\pm) = B'_{5,6}(\psi_\pm)\ ,$$

$$B_{7,8}(\psi_x\pm3\psi_y) = -\frac{1}{2\sin\pi(\nu_x\pm3\nu_y)}\sum_k\frac{\bar{m}_k}{8}\cos[(\psi'_{xk}\pm3\psi'_{yk})-(\psi_x\pm3\psi_y)-\pi(\nu_x\pm3\nu_y)]\ ,$$

$$A_{7,8}(\psi_x \pm 3\psi_y) = B'_{7,8}(\psi_x \pm 3\psi_y) \ , \tag{6.12}$$

where $\psi_\pm = \psi_x \pm \psi_y$ and $\psi'x$ and ψ'_y are defined in Eq. (2.21).

Recalling that the distortion of the amplitudes and phases are given by Eqs. (2.12) and (2.17) and using Eqs. (6.7) to (6.11), we arrive at

$$\begin{aligned}
\delta\mathcal{A}_x =\ & 3\mathcal{A}_x^2\mathcal{A}_y\{(A_3 \sin\phi_+ - B_3\cos\phi_+) + (A_4\sin\phi_- - B_4\cos\phi_-) \\
& -[A_1\sin(3\phi_x+\phi_y) - B_1\cos(3\phi_x+\phi_y)] + [A_2\sin(3\phi_x-\phi_y) - B_2\cos(3\phi_x-\phi_y)]\} \\
& -\mathcal{A}_y^3\{3(A_5\sin\phi_+ - B_5\cos\phi_+) + 3(A_6\sin\phi_- - B_6\cos\phi_-) \\
& +[A_7\sin(\phi_x+3\phi_y) - B_7\cos(\phi_x+3\phi_y)] + [A_8\sin(\phi_x-3\phi_y) - B_8\cos(\phi_x-3\phi_y)]\} \ , \\
\delta\mathcal{A}_y =\ & \mathcal{A}_x^3\{3(A_3\sin\phi_+ - B_3\cos\phi_+) - 3(A_4\sin\phi_- - B_4\cos\phi_-) \\
& +[A_1\sin(3\phi_x+\phi_y) - B_1\cos(3\phi_x+\phi_y)] - [A_2\sin(3\phi_x-\phi_y) - B_2\cos(3\phi_x-\phi_y)]\} \\
& -3\mathcal{A}_x\mathcal{A}_y^2\{(A_5\sin\phi_+ - B_5\cos\phi_+) - (A_6\sin\phi_- - B_6\cos\phi_-) \\
& +[A_7\sin(\phi_x+3\phi_y) - B_7\cos(\phi_x+3\phi_y)] - [A_8\sin(\phi_x-3\phi_y) - B_8\cos(\phi_x-3\phi_y)]\} \ , \\
\delta\phi_x =\ & 3\mathcal{A}_x\mathcal{A}_y\{3(A_3\cos\phi_+ + B_3\sin\phi_+) + 3(A_4\cos\phi_- + B_4\sin\phi_-) \\
& +[A_1\cos(3\phi_x+\phi_y) + B_1\sin(3\phi_x+\phi_y)] + [A_2\cos(3\phi_x-\phi_y) + B_2\sin(3\phi_x-\phi_y)]\} \\
& -\frac{\mathcal{A}_y^3}{\mathcal{A}_x}\{3(A_5\cos\phi_+ + B_5\sin\phi_+) + 3(A_6\cos\phi_- + B_6\sin\phi_-) \\
& -[A_7\cos(\phi_x+3\phi_y) + B_7\sin(\phi_x+3\phi_y)] + [A_8\cos(\phi_x-3\phi_y) + B_8\sin(\phi_x-3\phi_y)]\} \ , \\
\delta\phi_y =\ & \frac{\mathcal{A}_x^3}{\mathcal{A}_y}\{3(A_3\cos\phi_+ + B_3\sin\phi_+) + 3(A_4\cos\phi_- + B_4\sin\phi_-) \\
& +[A_1\cos(3\phi_x+\phi_y) + B_1\sin(3\phi_x+\phi_y)] + [A_2\cos(3\phi_x-\phi_y) + B_2\sin(3\phi_x-\phi_y)]\} \\
& -3\mathcal{A}_x\mathcal{A}_y\{3(A_5\cos\phi_+ + B_5\sin\phi_+) + 3(A_6\cos\phi_- + B_6\sin\phi_-) \\
& -[A_7\cos(\phi_x+3\phi_y) + B_7\sin(\phi_x+3\phi_y)][A_8\cos(\phi_x-3\phi_y) + B_8\sin(\phi_x-3\phi_y)]\} \ .
\end{aligned} \tag{6.13}$$

6.2 Second-order Tuneshifts

Skew octupoles do not produce any first-order tuneshifts. The first-order perturbation Hamiltonian which has the form $x^3y - xy^3$ or $\cos^3\phi_x\cos\phi_y - \cos\phi_x\cos^3\phi_y$ averages to zero, therefore the first-order tuneshift vanishes. In order to obtain the second-order tuneshifts we need to evaluate the second-order terms in the Hamiltonian H_4. From the generating function G_3 of Eq. (6.5), we get

$$\begin{aligned}
(2I_x)^{3/2}(2I_y)^{1/2} = (2J_x)^{3/2}(2J_y)^{1/2}[1 - 3\beta_0(2J_x)^{1/2}(2J_y)^{1/2} \times \\
\times \sum_m \left(\frac{3A_{1+m}}{m-\nu_+}\cos q_{1+m} + \frac{3A_{1-m}}{m-\nu_-}\cos q_{1-m} + \dots\right.
\end{aligned} \tag{6.14}$$

and similarly for $(2I_y)^{3/2}(2I_x)^{1/2}$. Then we can write the second-order terms in the Hamiltonian and from them keep only the θ-independent terms which are

$$\Delta H_4'|_{so} = (2J_x)^2(2J_y)\beta_0^2\sum_m\left\{\frac{3}{2}\left[\frac{9A_{1+m}^2}{m-\nu_+} + \frac{9A_{1-m}^2}{m-\nu_-} + \frac{3A_{3+m}^2}{m-(3\nu_x+\nu_y)} + \frac{3A_{3-m}^2}{m-(3\nu_x-\nu_y)}\right]\right.$$

$$
\begin{aligned}
&-\frac{4}{2}\left[\frac{9A_{1+m}B_{1+m}}{m-\nu_+}\cos\left(\beta_{1+m}-\alpha_{1+m}\right)-\frac{9A_{1-m}B_{1-m}}{m-\nu_-}\cos\left(\beta_{1-m}-\alpha_{1-m}\right)\right] \\
&+(2J_x)(2J_y)^2\beta_0^2\sum_m\left\{\frac{3}{2}\left[\frac{9B_{1+m}^2}{m-\nu_+}-\frac{9B_{1-m}^2}{m-\nu_-}+\frac{3B_{3+m}^2}{m-(\nu_x+3\nu_y)}-\frac{3B_{3-m}^2}{m-(\nu_x-3\nu_y)}\right]\right. \\
&\left.-\frac{4}{2}\left[\frac{9A_{1+m}B_{1+m}}{m-\nu_+}\cos\left(\beta_{1+m}-\alpha_{1+m}\right)+\frac{9A_{1-m}B_{1-m}}{m-\nu_-}\cos\left(\beta_{1-m}-\alpha_{1-m}\right)\right]\right\} \\
&+(2J_y)^3\beta_0^2\sum_m\left\{\frac{1}{2}\left[\frac{9B_{1+m}^2}{m-\nu_+}+\frac{9B_{1-m}^2}{m-\nu_-}+\frac{B_{3+m}^2}{m-(\nu_x+3\nu_y)}+\frac{B_{3-m}^2}{m-(\nu_x-3\nu_y)}\right]\right\} \\
&+(2J_x)^3\beta_0^2\sum_m\left\{\frac{1}{2}\left[\frac{9A_{1+m}^2}{m-\nu_+}-\frac{9A_{1-m}^2}{m-\nu_-}+\frac{A_{3+m}^2}{m-(3\nu_x+\nu_y)}-\frac{A_{3-m}^2}{m-(3\nu_x-\nu_y)}\right]\right\} .
\end{aligned}
\tag{6.15}
$$

If we sum over the harmonics we obtain

$$
\begin{aligned}
\sum_m \frac{A_{1+m}^2}{m-\nu_+} &= \frac{1}{16\pi}\sum_k (B_3m)_k , \\
\sum_m \frac{A_{1-m}^2}{m-\nu_-} &= \frac{1}{16\pi}\sum_k (B_4m)_k , \\
\sum_m \frac{A_{3+m}^2}{m-(3\nu_x+\nu_y)} &= \frac{1}{16\pi}\sum_k (B_1m)_k , \\
\sum_m \frac{A_{3-m}^2}{m-(3\nu_x-\nu_y)} &= \frac{1}{16\pi}\sum_k (B_2m)_k , \\
\sum_m \frac{B_{1+m}^2}{m-\nu_+} &= \frac{1}{16\pi}\sum_k (B_5\bar{m})_k , \\
\sum_m \frac{B_{1-m}^2}{m-\nu_-} &= \frac{1}{16\pi}\sum_k (B_6\bar{m})_k , \\
\sum_m \frac{B_{3+m}^2}{m-(\nu_x+3\nu_y)} &= \frac{1}{16\pi}\sum_k (B_7\bar{m})_k , \\
\sum_m \frac{B_{3-m}^2}{m-(\nu_x-3\nu_y)} &= \frac{1}{16\pi}\sum_k (B_8\bar{m})_k ,
\end{aligned}
$$

$$
\begin{aligned}
\sum_m \frac{A_{1+m}B_{1+m}}{m-\nu_+}\cos\left(\beta_{1+m}-\alpha_{1+m}\right) &= \frac{1}{16\pi}\sum_k (B_3\bar{m})_k , \\
\sum_m \frac{A_{1-m}B_{1-m}}{m-\nu_-}\cos\left(\beta_{1-m}-\alpha_{1-m}\right) &= \frac{1}{16\pi}\sum_k (B_4\bar{m})_k .
\end{aligned}
\tag{6.16}
$$

Now we are in a position to calculate the tuneshifts

$$
\begin{aligned}
\Delta\nu_x = &\frac{1}{8\pi}\mathcal{A}_x^2\mathcal{A}_y^2\sum_k[3(9B_3m+9B_4m+3B_1m-3B_2m)-4(9B_3\bar{m}-9B_4\bar{m})]_k \\
&+\frac{1}{16\pi}\mathcal{A}_y^4\sum_k[3(9B_5\bar{m}-9B_6\bar{m}-3B_7\bar{m}-3B_8\bar{m})-4(9B_3\bar{m}+9B_4\bar{m})]_k
\end{aligned}
$$

$$+\frac{3}{16\pi}\mathcal{A}_x^4\sum_k(9B_3m-9B_4m+B_1m-B_2m)_k\ , \tag{6.17}$$

and

$$\Delta\nu_y = \frac{1}{16\pi}\mathcal{A}_x^4\sum_k[3(9B_3m+9B_4m+3B_1m+3B_2m) - 4(9B_3\bar{m}-9B_4\bar{m})]_k$$
$$+\frac{1}{8\pi}\mathcal{A}_x^2\mathcal{A}_y^2\sum_k[3(9B_5\bar{m}-9B_6\bar{m}+3B_7\bar{m}-3B_8\bar{m}) - 4(9B_3\bar{m}+9B_4\bar{m})]_k$$
$$+\frac{3}{16\pi}\mathcal{A}_y^4\sum_k(9B_5\bar{m}+9B_6\bar{m}+B_7\bar{m}+B_8\bar{m})_k\ . \tag{6.18}$$

7 Applications

Here, we present some examples in order to, first illustrate how these formulæ can be used and second to show the degree to which canonical Hamiltonian formalism provides a faithful description of the nonlinear effects on the motion of the beam in the machine. In the first example, the above formulation will be used to display the beam shape distortions due to sextupoles which will be compared with experimental data. In the second example, we show how one can have control over the amplitude dependence of tune using a set of octupoles.

7.1 Beam Shape Distortions

In 1985 some studies of the perturbation of the motion by nonlinearities were made in the Fermilab Tevatron.[5,6,7] In particular, 8 normal sextupoles at stations 32, 34, 36, 38 in C and F sectors were powered in pairs so as to excite the resonance at the betatron oscillation tune of 19 1/3. The Tevatron injection kicker produced a horizontal betatron oscillation with an initial amplitude such that a particle at the centroid would perform a stable motion close to the separatrix. Figure 1 is the phase space plot of the motion described above with small-amplitude tune 19.34 in dots. The horizontal axis is displacement x from the central orbit in mm. The vertical axis is x' normalized to mm.

There was no oscillation induced in the vertical plane, hence the vertical motion was much smaller than the horizontal. Also, there was practically no linear coupling in the machine. Within these bounds (the absence of both vertical motion and coupling), the problem can be simplified to the study of only one plane instead of 4-dimensional phase space. Even though it may seem as a digression, we will proceed with the justification of our assumption on the absence of linear coupling.

In a machine, where the only nonlinearities are introduced by normal sextupoles, and where the vertical motion is negligible compared to the horizontal, linear coupling is introduced through skew quadrupoles only. Indeed, let us start from the Hamiltonian

$$H_1 = \frac{1}{2}[P_x^2+K_x(s)X^2] + \frac{1}{2}[P_y^2+K_y(s)Y^2] - \frac{B_x'}{B\rho}XY + \frac{B_y''}{6(B\rho)}(X^3-3XY^2)\ . \tag{7.1}$$

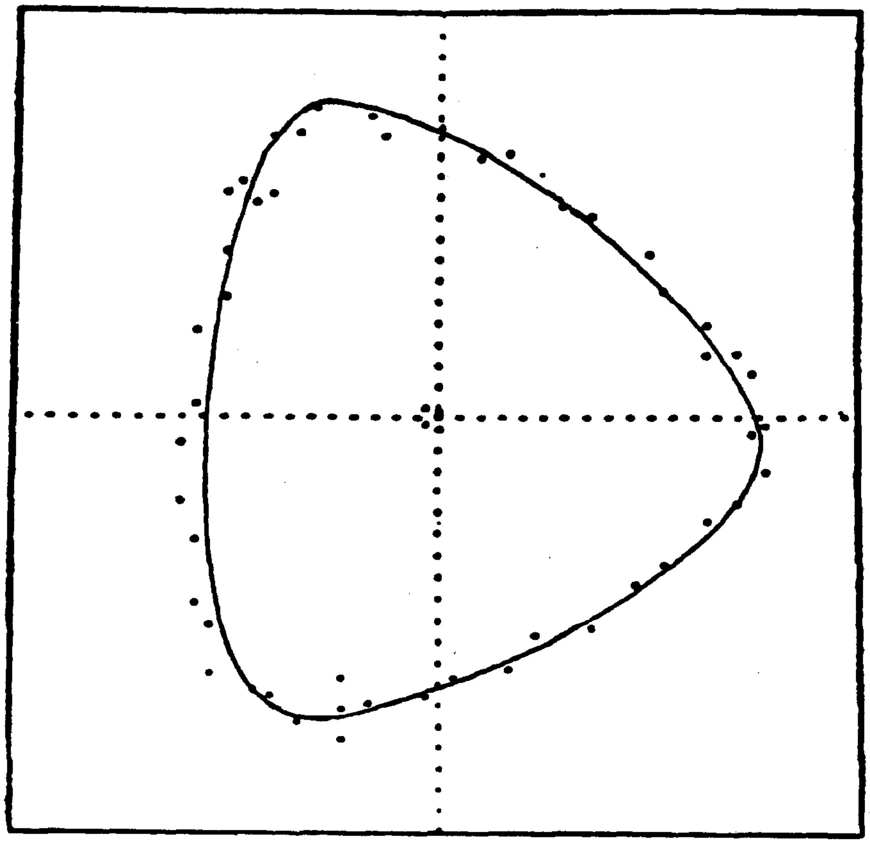

Figure 1: Phase space plot from experimental data (dots) and the predictions of perturbation theory (solid), in the presence of sextupoles.

The equations of motion for X and Y are

$$\ddot{X} + K_x(s)X = \frac{B'_x}{B\rho}Y - \frac{B''_y}{2(B\rho)}(X^2 - Y^2) , \tag{7.2}$$

$$\ddot{Y} + K_y(s)Y = \frac{B'_x}{B\rho}X + \frac{B''_y}{B\rho}XY . \tag{7.3}$$

From Eq. (7.3) we see that even though we may start from $Y = 0$, a Y-motion can develop due to its X-dependence through the skew quadrupole term $(B'_x/B\rho)X$. The traditional way to minimize the coupling is to adjust the skew quadrupole strength so as to minimize the separation of the two observed tune lines (ν_1, ν_2) and hence the coupling:[5] A measure of the linear skew field in the Tevatron is the parameter

$$|k| \equiv \frac{1}{4\pi}\left|\int ds \frac{B'_x}{B\rho}\sqrt{\beta_x\beta_y}\right| , \tag{7.4}$$

which can be measured easily from the relation

$$|\nu_1 - \nu_2|_{\min} = 2|k| . \tag{7.5}$$

In this experiment the optimum value of the current through the skew quadrupoles was found to be -6.53 A at 400 GeV. The corresponding value of the linear coupling parameter was

$$|k| \simeq .014 \tag{7.6}$$

In the presence of a small but not vanishing $|k|$, resonances of the form $\nu_x \pm \nu_y = m$, m being an integer, could be excited (see the skew quadrupole term analysis we did in Sec. 2). In order to reduce such a possibility, the tunes were split as far apart as possible so that their separation was

$$\Delta\nu = 0.1014 \tag{7.7}$$

So the Hamiltonian which describes the above experimental situation where both vertical motion and x-y coupling are absent is

$$H = \frac{1}{2}[P_x^2 + K_x(s)X^2] + \frac{B''_y}{6(B\rho)}X^3 . \tag{7.8}$$

This expression comes from the general Hamiltonian (1.1) where the only nonlinear contribution comes from the normal sextupole term.

The equations describing this 'distorted' one dimensional motion are

$$x = \delta x + (\mathcal{A}_x + \delta\mathcal{A}_x)\cos(\phi_x + \delta\phi_x) , \tag{7.9}$$

$$x' = \delta x' - (\mathcal{A}_x + \delta\mathcal{A}_x)\sin(\phi_x + \delta\phi_x) , \tag{7.10}$$

where $\delta\mathcal{A}_x$ and $\delta\phi_x$ are the distortions of the amplitude $\mathcal{A}_x$ and phase ϕ_x due to normal sextupoles and they are given by the one degree of freedom version of Eqs. (3.27) and (3.28) correspondingly, namely,

$$\delta\mathcal{A}_x = \mathcal{A}_x^2[(A_3\sin 3\phi_x - B_3\cos 3\phi_x) - (A_1\sin\phi_x - B_1\cos\phi_x)] , \tag{7.11}$$

$$\delta\phi_x = \mathcal{A}_x[(A_3\cos 3\phi_x + B_3\sin 3\phi_x) + (A_1\cos\phi_x + B_1\sin\phi_x)] . \tag{7.12}$$

The closed orbit distortions δx and $\delta x'$ are given by[4]

$$\delta x = -2\mathcal{A}_x^2 B_1 , \tag{7.13}$$

$$\delta x' = -2\mathcal{A}_x^2 A_1 . \tag{7.14}$$

The experimental situation under consideration, with 8 normal sextupoles of 15 amperes excitation, corresponds to $\mathcal{A}_x B_1 = 7.09 \times 10^{-4}$, $\mathcal{A}_x B_3 = -.1064$, $\mathcal{A}_x A_1 = -1.88 \times 10^{-3}$, $\mathcal{A}_x A_3 = -.1137$ at the location of the horizontal beam position monitor HE24. If we use these numbers to plot Eqs. (7.9) and (7.10), we obtain the solid curve in Fig. 1 which clearly follows the real motion very closely.

7.2 Tuneshifts

Let us consider the problem of devising a set of octupoles to control the amplitude dependence of tune in both degrees of freedom in Tevatron, without driving any resonances.[8]

Octupoles give rise to amplitude dependent tuneshifts which, according to Eqs. (5.26) and (5.27), are given by

$$2\pi\Delta\nu_x = \frac{3}{8}\mathcal{A}_x^2\sum_k\left(\frac{\beta_x^2}{\beta_0}\right)\frac{(B_y'''\ell)_k}{6(B\rho)} - \frac{3}{4}\mathcal{A}_y^2\sum_k\left(\frac{\beta_x\beta_y}{\beta_0}\right)\frac{(B_y'''\ell)_k}{6(B\rho)} , \tag{7.15}$$

and

$$2\pi\Delta\nu_y = \frac{3}{8}\mathcal{A}_y^2\sum_k\left(\frac{\beta_y^2}{\beta_0}\right)\frac{(B_y'''\ell)_k}{6(B\rho)} - \frac{3}{4}\mathcal{A}_x^2\sum_k\left(\frac{\beta_x\beta_y}{\beta_0}\right)\frac{(B_y'''\ell)_k}{6(B\rho)} . \tag{7.16}$$

Let us call

$$\frac{1}{\beta_0}\frac{(B_y'''\ell)_k}{6(B\rho)} = I_k , \tag{7.17}$$

since it is a measure of the current through the k-th octupole. So the expressions for the tuneshifts take the form

$$2\pi\Delta\nu_x = \frac{3}{8}\mathcal{A}_x^2\sum_k(\beta_x^2 I)_k - \frac{3}{4}\mathcal{A}_y^2\sum_k(\beta_x\beta_y I)_k , \tag{7.18}$$

and

$$2\pi\Delta\nu_y = \frac{3}{8}\mathcal{A}_y^2\sum_k(\beta_y^2 I)_k - \frac{3}{4}\mathcal{A}_x^2\sum_k(\beta_x\beta_y I)_k . \tag{7.19}$$

In one degree of freedom, these expressions are reduced to

$$2\pi\Delta\nu_x = \frac{3}{8}\mathcal{A}_x^2\sum_k(\beta_x^2 I)_k \tag{7.20}$$

and so, it is clear that the amplitude dependence of the tune can be controlled by adjusting the current I_k through the octupoles. The next question is to find the octupole configurations which do not excite any resonances. Normal octupoles can excite half-integer resonances,

$2\nu_u = m$, $u = x, y$, or $2(\nu_x \pm \nu_y) = m$, where m is an integer, and quarter-integer resonances, $4\nu_u = m$, $u = x$, y or $4(\nu_x \pm \nu_y) = m$ [this can be seen easily from Eqs. (5.14)]. However, if one chooses the location of the octupoles carefully the resonant driving terms, more or less, cancel each other, so the octupole configuration as a whole does not excite any resonances. In this particular case, if we power 4 octupoles O_1, O_2, O_3, O_4 of the same polarity in series, the phase advance between 2 adjacent octupoles being 136°, the contribution to the above mentioned resonances is small. A clear way to see this, is by representing graphically the contributing terms to the summations of (5.3),

$$\sum_k \underline{\mathrm{m}}_k e^{i(2Q_x+m\theta)_k} , \tag{7.21}$$

and

$$\sum_k \underline{\mathrm{m}}_k e^{i(4Q_x+m\theta)_k} . \tag{7.22}$$

If we substitute $\psi_x(\theta) - \nu_x\theta$ for Q_x, the above sums become

$$\sum_k \underline{\mathrm{m}}_k e^{i[2\psi_x+(m-2\nu_x)\theta]_k} , \tag{7.23}$$

and

$$\sum_k \underline{\mathrm{m}}_k e^{i[4\psi_x+(m-4\nu_x)\theta]_k} . \tag{7.24}$$

The summation is over the 4 octupoles. Near the resonances, $m - 2\nu_x \approx 0$ and $m - 4\nu_x \approx 0$, so the two sums become really

$$\sum_k \underline{\mathrm{m}}_k e^{i(2\psi_x)_k} , \tag{7.25}$$

and

$$\sum_k \underline{\mathrm{m}}_k e^{i(4\psi_x)_k} , \tag{7.26}$$

which turn out to be small quantities as one can see from Fig. 2. Fig. 2(a) represents the contribution to the first sum from each octupole separately as well as the resultant contribution, which is indeed small. The horizontal tune is $37/2 = 19.5$. Fig. 2(b) is the corresponding vector diagram for the quarter-integer resonance. Here the tune is $77/4 = 19.25$.

In two degrees of freedom, we would like to control the following 4 quantities

$$\frac{\delta(\Delta\nu_x)}{\delta(\mathcal{A}_x^2)} , \quad \frac{\delta(\Delta\nu_x)}{\delta(\mathcal{A}_y^2)} , \quad \frac{\delta(\Delta\nu_y)}{\delta(\mathcal{A}_x^2)} , \quad \frac{\delta(\Delta\nu_y)}{\delta(\mathcal{A}_y^2)} , \tag{7.27}$$

which are given by the expressions

$$\frac{\delta(2\pi\Delta\nu_x)}{\delta(\mathcal{A}_x^2)} = \frac{3}{8}\sum_k(\beta_x^2 I)_k , \tag{7.28}$$

$$\frac{\delta(2\pi\Delta\nu_x)}{\delta(\mathcal{A}_y^2)} = -\frac{3}{4}\sum_k(\beta_x\beta_y I)_k , \tag{7.29}$$

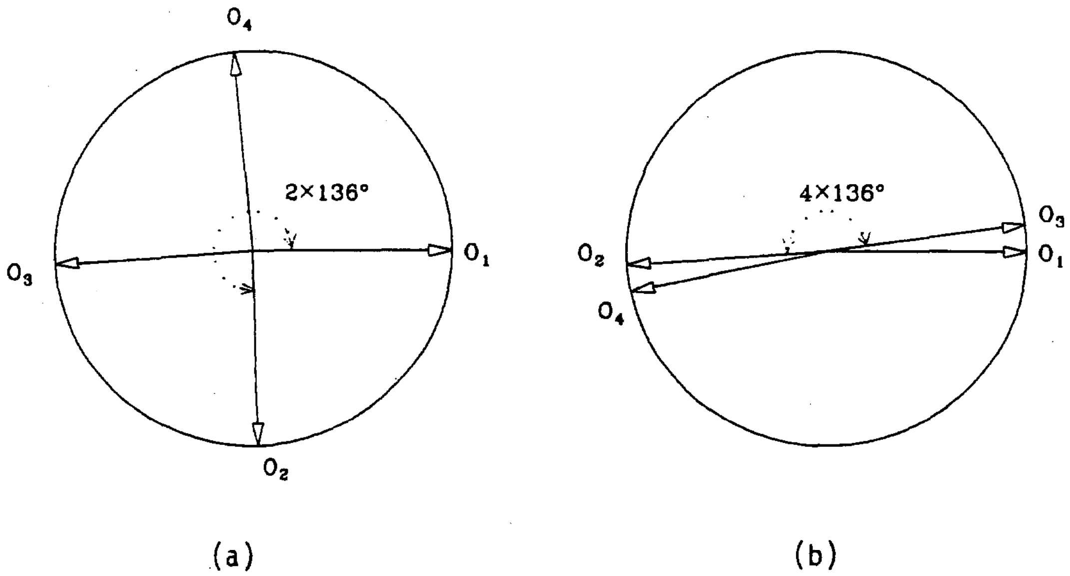

Figure 2: Vector diagrams showing the driving terms for the resonances (a) $2\nu_x = 37$ and (b) $4\nu_x = 77$. (Recall that the integer part of the Tevatron tune is 19).

$$\frac{\delta(2\pi\Delta\nu_y)}{\delta(\mathcal{A}_x^2)} = -\frac{3}{4}\sum_k(\beta_x\beta_y I)_k \ , \tag{7.30}$$

$$\frac{\delta(2\pi\Delta\nu_y)}{\delta(\mathcal{A}_y^2)} = \frac{3}{8}\sum_k(\beta_y^2 I)_k \ . \tag{7.31}$$

We see immediately that only 3 out of the 4 quantities above are independent and so it is clear that we need to power the octupoles in 3 circuits with currents, say, I_1, I_2, I_3. Each circuit will consist of 4 octupoles, according to the one degree of freedom analysis, so that no resonances will be excited. This results in the multiplicative factor 4 in the above expressions. Keeping this in mind, the problem takes the following matrix form

$$\begin{pmatrix} \frac{3}{2}(\beta_x^2)_1 & \frac{3}{2}(\beta_x^2)_2 & \frac{3}{2}(\beta_x^2)_3 \\ -3(\beta_x\beta_y)_1 & -3(\beta_x\beta_y)_2 & -3(\beta_x\beta_y)_3 \\ \frac{3}{2}(\beta_y^2)_1 & \frac{3}{2}(\beta_y^2)_2 & \frac{3}{2}(\beta_y^2)_3 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} \frac{\delta(2\pi\Delta\nu_x)}{\delta(\mathcal{A}_x^2)} \\ \frac{\delta(2\pi\Delta\nu_x)}{\delta(\mathcal{A}_y^2)} \\ \frac{\delta(2\pi\Delta\nu_y)}{\delta(\mathcal{A}_y^2)} \end{pmatrix} , \tag{7.32}$$

or

$$\mathcal{M}I = \mathcal{D} \ . \tag{7.33}$$

That is, once we know the matrix $\mathcal{M}$, and we are given $\mathcal{D}$, we can solve for the appropriate octupole strengths, denoted by the column vector I above.

So the question now is what the matrix $\mathcal{M}$ is, or equivalently, what the positions of the 3 circuits should be around the ring.

From Eqs. (7.33), we see that if we want to be able to solve for I, $\mathcal{M}$ has to be invertible, hence $\det\mathcal{M} \neq 0$. This imposes a constraint on the beta functions at the positions of the 3 circuits. Specifically

$$\det\mathcal{M} \neq 0 \quad \Leftrightarrow$$

$$\frac{27}{4}\Big\{(\beta_x)_1(\beta_y)_1[(\beta_x^2)_2(\beta_y^2)_3 - (\beta_x^2)_3(\beta_y^2)_2] + (\beta_x)_2(\beta_y)_2[(\beta_y^2)_1(\beta_x^2)_3 - (\beta_x^2)_1(\beta_y^2)_3] + (\beta_x)_3(\beta_y)_3[(\beta_x^2)_1(\beta_y^2)_2 - (\beta_x^2)_2(\beta_y^2)_1]\Big\} \neq 0 \ . \tag{7.34}$$

The following is a set of sufficient conditions for the above relation to be true,

$$(\beta_x)_2(\beta_y)_3 \neq (\beta_x)_3(\beta_y)_2 \ , \tag{7.35}$$

or

$$(\beta_x)_3(\beta_y)_1 \neq (\beta_x)_1(\beta_y)_3 \ , \tag{7.36}$$

or

$$(\beta_x)_1(\beta_y)_2 \neq (\beta_x)_2(\beta_y)_1 \ . \tag{7.37}$$

This means the ratios of the horizontal to the vertical beta function in any two circuits should be different from each other. The solutions given above are in consistency with the situation at the Tevatron where 2 of the octupole circuits are placed at $(\beta_x)_1 = 100$m, $(\beta_y)_1 = 28$m and $(\beta_x)_2 = 28$m, $(\beta_y)_2 = 100$m.

Finally we would like to add one more comment. From the matrix form Eq. (7.32), we observe that the optimum situation is when each element of $\mathcal{D}$ is controlled fully by just one of the 3 circuits and not by a linear combination of all 3 of them. This translates into $\mathcal{M}$ being a diagonal matrix. Since it is impossible for $\mathcal{M}$ to be exactly diagonal, we can search for solutions where the diagonal elements of $\mathcal{M}$ are considerably larger than the off-diagonal ones. In fact, all we need is that the (1,1) element of $\mathcal{M}$ be much larger than the (1,2) and (1,3) elements. Similarly the (2,2) element be much larger than the (2,1) and (2,3) and the (3,3) element be much larger than the (3,1) and (3,2). Unfortunately a quick inspection shows that such a solution is impossible. So, one can not have an independent control over the tune variations with amplitude in the two degree of freedom case.

However, one can think of a situation where the (1,1) element is larger than the (1,2) and (1,3), so it controls most effectively the first element of $\mathcal{D}$, while (1,2) and (1,3) provide the "fine tuning." Hence in the Tevatron, for example, one could place the first octupole circuit at a $(\beta_x)_1$ of approximately 100m (in the neighborhood of focusing quadrupoles) and $(\beta_y)_1 \approx 30$m, while the second circuit could be at $(\beta_x)_2 \approx$70m, $(\beta_y)_2 \approx$70m, and the third at $(\beta_x)_3 \approx$30m (in the neighborhood of defocusing quadrupoles) and $(\beta_y)_3 \approx$100m.

Remarks

1. Usually the strength of a normal multipole is much bigger than that of the corresponding skew multipole. Therefore, the tuneshifts due to skew multipoles are much smaller and can be neglected in most cases. Furthermore, the tuneshifts due to normal octupole are of first order while those due to a skew octupole are of second order. Thus the latter can be neglected.

2. There is no closed orbit distortion due to skew quadrupoles, octupoles and skew octupoles. In all these cases the jumps that x' and y' undergo when the particle crosses the multipole, are such that they average to zero.

3. A derivation of the generating function G_3 is given in the appendix of Ref. 4.

4. A more straightforward derivation of the distortion functions is given implicitly in Ref. 9. Starting from the Hamiltonian expressed in terms of action-angle variables, one integrates the equations of motion directly to get the distortion of the amplitudes and phases. From these expressions the distortion functions can be read out readily. Even though it may seem that the derivation we followed throughout this paper is unnecessarily elaborate, there is an advantage to the method: By expanding the multipole terms into harmonics and then summing them up, one can understand and treat the resonance cases much easier. All is needed is to keep the resonance term and neglect the rest of the harmonics.

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