Translation among SAD, MAD-X and Bmad

Demin Zhou

Acknowledgements:

Workshop SAD2019, KEK, Sep. 19, 2019
Outline

➤ Introduction
➤ Current status
  ● Actually not the latest
➤ Cases of Benchmark studies
➤ Some applications
➤ Symplectic tracking in SAD
➤ Summary and future plan
1. Introduction

➤ Motivation: To improve communications

- **SAD**: TRISTAN, KEKB, J-PARC, SuperKEKB, FCC-ee, ...
- **Bmad/PTC**: CESR, ERL, …
- **MAD/MAD-X/PTC**: PS, LEP, LHC, FCCs, …
1. Introduction

➤ Motivation: To improve collaborations on projects like SuperKEKB and FCCs

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SuperKEKB = FCC-ee demonstrator

- top up injection at high current
- $\beta_y^* = 300 \mu m$ (FCC-ee: 1 mm)
- lifetime 5 min (FCC-ee: $\geq 20$ min)
- $\varepsilon_y/\varepsilon_x = 0.25\%$ (similar to FCC-ee)
- off momentum acceptance
  - ($\pm 1.5\%$, similar to FCC-ee)
- $e^+$ production rate
  - $2.5 \times 10^{12}/s$, FCC-ee: $< 1.5 \times 10^{12}/s$ (Z cr.waist)

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Ref. F. Zimmermann et al., IPAC’14, MOXAA01.
1. Introduction

➤ References for this topic


● MAD-X user’s guide, http://mad.web.cern.ch/mad/

2. Current status

➤ Efforts for lattice translations: SAD <-> MAD/MAD-X
  ● MAD -> SAD: H. Koiso(KEK), Y. Wang(IHEP), et al.
  ● MAD-X -> SAD: A. Morita(KEK, 2008)
  ● SAD -> MAD: Y. Wang(IHEP), et al.
  ● SAD -> MAD-X: A. Morita(KEK), K. Oide(KEK), Y. Wang(IHEP), et al.
  ● ... ...

➤ Efforts for lattice translations: Others
  ● SAD <-> Bmad: D. Sagan(Cornell), E. Forest(KEK), et al.
  ● Bmad <-> UAP <-> MAD-X: D. Sagan(Cornell), et al.
  ● SAD -> AT: N. Carmignani, S.M. Liuzzo(ESRF), et al.
  ● ... ...
2. Current status

➤ Efforts for lattice translations

- SAD and PTC: developed at KEK, many shared features (transfer maps, symplectic integrator, ...)
- PTC integrated into MAD-X and Bmad

NOTE:
The widths of the arrows indicate “translatability”.

Asia

Europe

AML/UAP

Bmad

PTC

MAD

MAD-X

SAD

America
2. Current status

- Archive for examples of lattice translation
  - MAD-X svn repository: http://svnweb.cern.ch/world/wsvn/madx/branches/madX-SAD/tools/translators/ [Thanks to L. Deniau and I. Tecker]
  - Alternative: http://research.kek.jp/people/dmzhou/SAD/Translator/
  - Classified by routes of translations
  - Used programs: MAD-X/PTC, SAD, Bmad/PTC, UAP
  - Sample jobs prepared for demonstration and benchmarks

NOTE: Please inform me IF there are possible copyright violations!
2. Current status

➤ Archive for examples of lattice translation

● Sub-directories for SAD to MAD-X translation:
3. Cases of benchmark studies

➤ A benchmark of MAD-X and SAD: CLIC FFS

- Conditions:
  * Track “orbit” for a test particle
  * No soft edge fringe, no solenoid

- Hard edge fringe: SAD = PTC ≈ MAD-X

- For longitudinal transformation: SAD = PTC ≈ MAD-X (Settings for PTC: ICASE=6, TIME=true)

![Graphs showing comparison between MAD-X, PTC, and SAD results](image)
3. Cases of benchmark studies

➤ A benchmark of MAD-X and SAD: CLIC FFS

- Conditions:
  - Twiss with momentum offset
  - No soft edge fringe, no solenoid
- Hard edge fringe: Negligible
- SAD: High-order chromaticity due to DRIFT
- MAD-X/PTC: ICASE=5 (Need proper settings?)

For a DRIFT:
\[
\beta(\delta, s) = \beta_0 + \frac{s^2}{\beta_0(1+\delta)^2}
\]
3. Cases of benchmark studies

A benchmark of Bmad, PTC and SAD: SuperKEKB sler_1689

- Benchmark done in 2015
- Great agreement in closed orbit, even in the complicated IR
- “Time patching” was treated differently
- Closed orbit (or “fixed point”) is the first thing to compare
3. Cases of benchmark studies

➤ A benchmark of Bmad and SAD: SuperKEKB sler_1684

- Benchmark done in 2014
- Similar size of dynamic aperture
- Details are different, showing minor differences for nonlinear maps
3. Cases of benchmark studies

➤ A benchmark of Bmad and SAD: SuperKEKB sler_1684

- Benchmark done in 2014
- Similar footprint of tune spread
- Details are different, showing minor differences for nonlinear maps
3. Cases of benchmark studies

➤ A benchmark of Bmad and SAD: FCC-ee
- Recent study by L. van Riesen-Haupt
- Great agreement between MAD-X and SAD

Momentum Detuning: Results (MADX and SAD)

Initial Twiss

Closed Twiss
3. Cases of benchmark studies

➤ A benchmark of Bmad and SAD: FCC-ee

- Recent study by L. van Riesen-Haupt
- Discrepancy against PTC might be due to slicing (and/or integration scheme)?

Momentum Detuning: Results (MADX PTC)

![Graph showing momentum detuning results for different schemes.](image)
4. Some applications

➤ Old findings
  • My talk to 20th KEKB Accelerator Review Committee, Feb. 23, 2015.

2. BB+LN: Luminosity: LER

➤ Realistic lattice: lum. drops at low beam currents
➤ Crab-waist:
  • To cancel beam-beam driven resonances
  • Work well at high currents, but not well at low currents
4. Some applications

➤ Old findings


2. BB+LN: Nonlin. X-Y coupling

➤ Realistic lattice
➤ Poincare map in y direction as function of X offset
➤ Strong nonlinear X-Y coupling in LER

From Y. Zhang
4. Some applications

- Nonlinear optimization with chromatic constraints [by H. Sugimoto, SuperKEKB mini-optics meeting, Sep. 8, 2016]
  - Chromatic $\beta_{x,y}$ and $\nu_{x,y}$ correspond to RDTs of $h_{2000e}/h_{0200e}$ (X), $h_{0020e}/h_{0002e}$ (Y), and $h_{1100e}$ (X), $h_{0011e}$ (Y), respectively.
4. Some applications

Resonance driving terms (RDTs) indicate lattice nonlinearity

The effective Hamiltonian of a ring can be normalized in resonance bases [Ref. E. Forest, *Beam Dynamics – A New Attitude and Framework*, 1998].

For a ring with \( n \) elements, one can normalize the one turn map \( M_{1\to n} \) as [Ref. L. Yang *et al.*, *Phys. Rev. ST Accel. Beams* 14, 054001 (2011)]

\[
M_{1\to n} = A_1^{-1} e^{h \cdot R_{1\to n}} A_1,
\]

with \( R \): rotation, \( e^{h \cdot \cdot} \): nonlinear Lie map, \( A_1 \): normalizing map. Assume no coupling (the theory can be generalized for nonzero coupling), \( A_i \) in \( x \) plane at the \( i \)th element can be approximated in perturbation theory as

\[
A_i x = \sqrt{\beta_{x,i}} x + \eta_{x,i} \delta,
\]

\[
A_i p_x = \frac{-\alpha_{x,i} x + p_x}{\sqrt{\beta_{x,i}}} + \eta'_{x,i} \delta.
\]
4. Some applications

RDTs indicate lattice nonlinearity

In the resonance basis, using action-angle variables \((J, \phi)\) one can write

\[
h_\pm^x \equiv \sqrt{J_x} e^{\pm i\phi_x} = \frac{X \mp iP_x}{\sqrt{2}},
\]

\[
\mathcal{R}_{i \rightarrow j} h_\pm^x = \mathcal{R}_{i \rightarrow j} \sqrt{J_x} e^{\pm i\phi_x} = e^{\pm i\mu_{i \rightarrow j,x}} h_\pm^x,
\]

where \(\mu_{i \rightarrow j,x}\) is the phase advance of \(i \rightarrow j\). Consequently, the potential of a multipole magnetic field can be expanded in the resonance bases of \(h_{abcde}\) as

\[
h = \sum h_{abcde} h_x^+ a h_x^- b h_y^+ c h_y^- d \delta^e.
\]

Each \(h_{abcde}\) (a complex number in general) drives a certain resonance, and is an explicit function of magnet strengths, beta functions and dispersions.
4. Some applications

➤ RDTs indicate lattice nonlinearity

The effective Hamiltonian corresponding to chromaticity is

\[ h_c = \sum h_{1100e} h_x^{+1} h_x^{-1} \delta^e + \sum h_{0011e} h_y^{+1} h_y^{-1} \delta^e, \]

\[ h_c = J_x \sum h_{1100e} \delta^e + J_y \sum h_{0011e} \delta^e. \]

Then the tunes are calculated as

\[ \nu_x = -\frac{1}{2\pi} \frac{\partial h_c}{\partial J_x} = -\frac{1}{2\pi} \sum h_{1100e} \delta^e, \]

\[ \nu_y = -\frac{1}{2\pi} \frac{\partial h_c}{\partial J_y} = -\frac{1}{2\pi} \sum h_{0011e} \delta^e. \]

Therefore the RDTs of \( h_{1100e} \) and \( h_{0011e} \) correspond to linear and high-order chromaticity.
4. Some applications

➤ RDTs indicate lattice nonlinearity

<table>
<thead>
<tr>
<th>$h_{abcde}$</th>
<th>Driving effects</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_{11001}, h_{00111}$</td>
<td>Linear chromaticity $\zeta_x, \zeta_y$</td>
</tr>
<tr>
<td>$h_{21000}, h_{12000} \parallel h_{10110}, h_{01110}$</td>
<td>$\nu_x \left[ (J_x)^{3/2} \right] \parallel \left[ (J_x)^{1/2} (J_y) \right]$</td>
</tr>
<tr>
<td>$h_{30000}, h_{03000} \parallel h_{00300}, h_{00030}$</td>
<td>$3\nu_x \left[ (J_x)^{3/2} \right] \parallel 3\nu_y \left[ (J_y)^{3/2} \right]$</td>
</tr>
<tr>
<td>$h_{10020}, h_{01200} \parallel h_{10200}, h_{01020}$</td>
<td>$\nu_x - 2\nu_y \parallel \nu_x + 2\nu_y \left[ (J_x)^{1/2} (J_y) \right]$</td>
</tr>
<tr>
<td>$h_{20010}, h_{02100} \parallel h_{20100}, h_{02010}$</td>
<td>$2\nu_x - \nu_y \parallel 2\nu_x + \nu_y \left[ (J_x)(J_y)^{1/2} \right]$</td>
</tr>
<tr>
<td>$h_{00210}, h_{00120} \parallel h_{11100}, h_{11010}$</td>
<td>$\nu_y \left[ (J_y)^{3/2} \right] \parallel \left[ (J_x)(J_y)^{1/2} \right]$</td>
</tr>
</tbody>
</table>

| $h_{22000}, h_{00220}, h_{11110}$ | $d\nu_x / dJ_x, d\nu_y / dJ_y, d\nu_{x,y} / dJ_{y,x}$ |
| $h_{40000}, h_{04000} \parallel h_{00400}, h_{00040}$ | $4\nu_x \left[ (J_x)^2 \right] \parallel 4\nu_y \left[ (J_y)^2 \right]$ |
| $h_{31000}, h_{13000} \parallel h_{20110}, h_{02110}$ | $2\nu_x \left[ (J_x)^2 \right] \parallel \left[ (J_x)(J_y) \right]$ |
| $h_{00310}, h_{00130} \parallel h_{11200}, h_{11020}$ | $2\nu_y \left[ (J_y)^2 \right] \parallel \left[ (J_x)(J_y) \right]$ |
| $h_{20020}, h_{02200} \parallel h_{20200}, h_{02020}$ | $2\nu_x - 2\nu_y \parallel 2\nu_x + 2\nu_y \left[ (J_x)(J_y) \right]$ |
| $h_{30010}, h_{03100} \parallel h_{30100}, h_{03010}$ | $3\nu_x - \nu_y \parallel 3\nu_x + \nu_y \left[ (J_x)^{3/2} (J_y)^{1/2} \right]$ |
| $h_{10030}, h_{01300} \parallel h_{10300}, h_{01030}$ | $\nu_x - 3\nu_y \parallel \nu_x + 3\nu_y \left[ (J_x)^{1/2} (J_y)^{3/2} \right]$ |

Table: Low-order driving terms.
4. Some applications

➤ RDTs indicate lattice nonlinearity: Analytic theories according to J. Bengtsson, SLS Note 9/97

● Linear chromaticity:

\[
h_{11001} = \frac{1}{4} \sum_{i=1}^{N} \left[ (b_2 L)_i - 2(b_3 L)_i \eta^{(1)}_{xi} \right] \beta_{xi} + O \left( \delta^2 \right),
\]

\[
h_{00111} = -\frac{1}{4} \sum_{i=1}^{N} \left[ (b_2 L)_i - 2(b_3 L)_i \eta^{(1)}_{xi} \right] \beta_{yi} + O \left( \delta^2 \right)
\]

● Chromatic beta functions:

\[
h_{20001} = h^*_{02001} = \frac{1}{8} \sum_{i=1}^{N} \left[ (b_2 L)_i - 2(b_3 L)_i \eta^{(1)}_{xi} \right] \beta_{xi} e^{i2\mu_{xi}} + O \left( \delta^2 \right),
\]

\[
h_{00201} = h^*_{00021} = -\frac{1}{8} \sum_{i=1}^{N} \left[ (b_2 L)_i - 2(b_3 L)_i \eta^{(1)}_{xi} \right] \beta_{yi} e^{i2\mu_{yi}} + O \left( \delta^2 \right)
\]

● Chromatic dispersion:

\[
h_{10002} = h^*_{01002} = \frac{1}{2} \sum_{i=1}^{N} \left[ (b_2 L)_i - (b_3 L)_i \eta^{(1)}_{xi} \right] \eta^{(1)}_{xi} \sqrt{\beta_{xi}} e^{i\mu_{xi}} + O \left( \delta^3 \right)
\]
4. Some applications

RDTs indicate lattice nonlinearity: Analytic theories according to J. Bengtsson, SLS Note 9/97

- First order geometric terms (amplitude-dependent):

\[
\begin{align*}
    h_{21000} &= h^*_{12000} = -\frac{1}{8} \sum_{i=1}^{N} (b_{3i} L) \beta_{x_i}^{3/2} e^{i\mu x_i}, \\
    h_{30000} &= h^*_{03000} = -\frac{1}{24} \sum_{i=1}^{N} (b_{3i} L) \beta_{x_i}^{3/2} e^{i3\mu x_i}, \\
    h_{10110} &= h^*_{01110} = \frac{1}{4} \sum_{i=1}^{N} (b_{3i} L) \beta_{x_i}^{1/2} \beta_{y_i} e^{i\mu x_i}, \\
    h_{10020} &= h^*_{01200} = \frac{1}{8} \sum_{i=1}^{N} (b_{3i} L) \beta_{x_i}^{1/2} \beta_{y_i} e^{i(\mu x_i - 2\mu y_i)}, \\
    h_{10200} &= h^*_{01020} = \frac{1}{8} \sum_{i=1}^{N} (b_{3i} L) \beta_{x_i}^{1/2} \beta_{y_i} e^{i(\mu x_i + 2\mu y_i)}
\end{align*}
\]
4. Some applications

- Integration of RDTs along the whole ring
  - Almost perfect cancellation of 3rd order RDTs in the arc sections

\[ v_x - 2v_y \left[ (J_x)^{1/2} (J_y) \right] \]

\[ 3v_x \left[ (J_x)^{3/2} \right] \]

\[ v_x \left[ (J_x)^{1/2} (J_y) \right] \]

\[ v_x \left[ (J_x)^{3/2} \right] \]
4. Some applications

Integration of RDTs along the whole ring

- FFS contributes most of residual RDTs

\[ v_x = 2v_y \left[ (J_x)^{1/2} (J_y) \right] \]

\[ 3v_x \left[ (J_x)^{3/2} \right] \]

\[ v_x \left[ (J_x)^{1/2} (J_y) \right] \]

\[ v_x \left[ (J_x)^{3/2} \right] \]
4. Some applications

Integration of RDTs along the whole ring

- Almost perfect cancellation of 3rd order RDTs in the arc sections

\[ v_y \left[ (J_y)^{1/2}(J_x) \right] \]

\[ 3v_y \left[ (J_y)^{3/2} \right] \]

\[ 2v_x - v_y \left[ (J_x)(J_y)^{1/2} \right] \]

\[ v_y \left[ (J_y)^{3/2} \right] \]
4. Some applications

Integration of RDTs along the whole ring

- FFS contributes most of residual RDTs

\[ v_y \left( (J_y)^{1/2} (J_x) \right) \]

\[ 3v_y \left( (J_y)^{3/2} \right) \]

\[ 2v_x - v_y \left( (J_x) (J_y)^{1/2} \right) \]

\[ v_y \left( (J_y)^{3/2} \right) \]
4. Some applications

- Detuning along the whole ring
  - w/ constraints: chromatic correction

\[
d\beta_x/d\delta
\]

\[
d\nu_x/d\delta
\]

\[
d\beta_y/d\delta
\]

\[
d\nu_y/d\delta
\]
4. Some applications

➤ Detuning along the whole ring
  • w/ constraints: chromatic correction

\[ \frac{d\beta_x}{d\delta}, \frac{d\beta_y}{d\delta}, \frac{d\nu_x}{d\delta}, \frac{d\nu_y}{d\delta} \]
4. Some applications

➤ Detuning along the whole ring - second order
  - w/ constraints: chromatic correction

\[ \frac{d^2 \beta_x}{d \delta^2} \]
\[ \frac{d^2 \beta_y}{d \delta^2} \]
\[ \frac{d^2 v_x}{d \delta^2} \]
\[ \frac{d^2 v_y}{d \delta^2} \]
4. Some applications

- Detuning along the whole ring - second order
  - w/ constraints: chromatic correction

\[ \frac{d^2\beta_x}{d\delta^2} \]

\[ \frac{d^2\beta_y}{d\delta^2} \]

\[ \frac{d^2v_x}{d\delta^2} \]

\[ \frac{d^2v_y}{d\delta^2} \]
4. Some applications

Dispersion along the whole ring

- w/ constraints: No special control on chromatic dispersions?

\[
\frac{d\eta_x}{d\delta}, \quad \frac{d\eta_y}{d\delta}, \quad \frac{d^2\eta_x}{d\delta^2}, \quad \frac{d^2\eta_y}{d\delta^2}
\]
4. Some applications

➢ Dispersion along the whole ring
  ● w/ constraints: No special control on chromatic dispersions?

\[
\frac{d\eta_x}{d\delta} \quad \frac{d\eta_y}{d\delta} \\
\frac{d^2\eta_x}{d\delta^2} \quad \frac{d^2\eta_y}{d\delta^2}
\]
4. Some applications

➤ Chromatic coupling along the whole ring
  ● with constraints: Chromatic coupling controlled

\[
\frac{dR}{d\delta} \quad \frac{d^2R}{d\delta^2} \quad \frac{d^3R}{d\delta^3}
\]
4. Some applications

➤ Chromatic coupling along the whole ring
  ○ w/ constraints: Chromatic coupling controlled

\[
\begin{align*}
\frac{dR}{d\delta} & \quad \text{sler\_1689} \quad \text{sler\_1689\_w\_const001} \\
\frac{d^2R}{d\delta^2} & \quad \text{sler\_1689} \quad \text{sler\_1689\_w\_const001} \\
\frac{d^3R}{d\delta^3} & \quad \text{sler\_1689} \quad \text{sler\_1689\_w\_const001}
\end{align*}
\]
4. Some applications

\( p_x^2 p_y \) term: Compare SAD

- Hard-edge fringe fields of final focus quads are important sources
4. Some applications

- $p_x^2 p_y$ term
  - How quad. hard-edge fringes contribute?

Vert. offsets in FF quads

Table 2: Shift Amount of Magnet Axis

<table>
<thead>
<tr>
<th>Magnet</th>
<th>$\Delta Y$</th>
<th>Magnet</th>
<th>$\Delta X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>QC1RP</td>
<td>-1.0 mm</td>
<td>QC1RE</td>
<td>-0.7 mm</td>
</tr>
<tr>
<td>QC2RP</td>
<td>-1.0 mm</td>
<td>QC2RE</td>
<td>-0.7 mm</td>
</tr>
<tr>
<td>QC1LP</td>
<td>-1.5 mm</td>
<td>QC1LE</td>
<td>+0.7 mm</td>
</tr>
<tr>
<td>QC2LP</td>
<td>-1.5 mm</td>
<td>QC2LE</td>
<td>+0.7 mm</td>
</tr>
</tbody>
</table>

N. Ohuchi et al., IPAC’13
4. Some applications

- $p_x^2 p_y$ term
  - How quad. hard-edge fringes contribute?

![Graph showing COD and DX versus s]
4. Some applications

➤ $p_x^2p_y$ term

- How quad. hard-edge fringes contribute?
  + Magnet offsets + COD => 3rd geometric terms

In[1]:\[
(* f1 = K1 / (12 (1+\delta) L) *)
\]

$$HQfr = f1 \times \left( (x^3 + 3 x \times y^2) \times px - (y^3 + 3 x^2 y) \times py \right);$$

\[
D[HQfr, x] \times \Delta X
\]

\[
D[HQfr, px] \times \Delta PX
\]

\[
D[HQfr, y] \times \Delta Y
\]

\[
D[HQfr, py] \times \Delta PY
\]

Out[2]:\[
f1 \left( -6 py \times x \times y + px \times (3 x^2 + 3 y^2) \right) \times \Delta X
\]

Out[3]:\[
f1 \left( x^3 + 3 x \times y^2 \right) \times \Delta PX
\]

Out[4]:\[
f1 \left( 6 px \times x \times y - py \times (3 x^2 + 3 y^2) \right) \times \Delta Y
\]

Out[5]:\[
f1 \left( -3 x^2 \times y - y^3 \right) \times \Delta PY
\]
4. Some applications

➤ Luminosity calculations

- ~1/3 caused by $p_x^2 p_y$ term (from FFS, strength calculated by PTC)
- ~1/2 caused by chromatic effects (including interplay with geometric nonlinearities?)
- ~1/6 minor contribution from other nonlinearities
4. Some applications

➢ Luminosity calculations

● Important chromatic nonlinear terms (specific to sler_1689.sad):
  \( p_y^2 \delta, \gamma p_y \delta, p_x y \delta, x p_y \delta \)

● Basically SuperKEKB is sensitive to Y-motion coupled to X- and Z-directions
5. Symplectic tracking in SAD

- **Hamiltonian**
  - Hamiltonian for a relativistic particle in an electromagnetic field in Cartesian coordinate system:
    \[
    H = \sqrt{(\vec{p} - q\vec{A})^2 c^2 + m_0^2 c^4 + q\phi}
    \]
  - Hamiltonian used in SAD:
    \[
    H(x, p_x, y, p_y, z, \delta) = \frac{E}{P_0^\gamma_0} - (1 + \frac{x}{\rho_x} + \frac{y}{\rho_y})\sqrt{(1 + \delta)^2 - (p_x - \hat{A}_x)^2 - (p_y - \hat{A}_y)^2 - (1 + \frac{x}{\rho_x} + \frac{y}{\rho_y})\hat{A}_s}
    \]

- **Reference synchronous particle:**
  \[P_0 = \gamma_0 m_0 v_0\]

- **P**
  \[P = \frac{P}{P_0} = 1 + \delta\]

- **p_x**
  \[p_x = \frac{P_x}{P_0}, \quad p_y = \frac{P_y}{P_0}\]

- **\hat{A}_x**
  \[\hat{A}_x = \frac{q A_x}{P_0} = \frac{A_x}{B\rho}, \quad \hat{A}_y = \frac{q A_y}{P_0} = \frac{A_y}{B\rho}, \quad \hat{A}_s = \frac{q A_s}{P_0} = \frac{A_s}{B\rho}\]

- **Quadrupole:**
  \[\vec{A} \equiv (A_x, A_y, A_s) = (0, 0, \frac{1}{2} B_1 (y^2 - x^2))\]

- **Solenoid:**
  \[\vec{B} = (0, 0, B_s)\]
  \[\vec{A} \equiv (A_x, A_y, A_s) = (\frac{1}{2} B_s y, \frac{1}{2} B_s x, 0)\]

- **B_1**
  \[B_1 = \frac{\partial B_y}{\partial x}\]

- **K_1**
  \[K_1 = \frac{B_1}{B_0 \rho} = e B_1 P_0\]
5. Symplectic tracking in SAD

➤ A DRIFT (L) is nonlinear ...

- Hamiltonian for a DRIFT:

$$H(x, p_x, y, p_y, z, \delta) = \frac{1}{v_0} \sqrt{p^2 c^2 + \left(\frac{m_0 c^2}{P_0}\right)^2} - \sqrt{p^2 - p_x^2 - p_y^2}$$

- Symplectic transformation (exact solution):

$$x_2 = x_1 + \frac{p_x}{\sqrt{p^2 - p_x^2 - p_y^2}} L, \quad p_{x2} = p_{x1},$$
$$y_2 = y_1 + \frac{p_y}{\sqrt{p^2 - p_x^2 - p_y^2}} L, \quad p_{y2} = p_{y1},$$
$$z_2 = z_1 - \left(\frac{p}{\sqrt{p^2 - p_x^2 - p_y^2}} - \frac{v}{v_0}\right) L = z_1 + \left(1 - \frac{p}{\sqrt{p^2 - p_x^2 - p_y^2}}\right) L - \frac{v_0 - v}{v_0} L$$
5. Symplectic tracking in SAD

➤ A DRIFT with solenoid field (L, BZ)

- Hamiltonian for a DRIFT + solenoid field:

\[ H(x, p_x, y, p_y, z, \delta) = \frac{E}{P_0 v_0} - \sqrt{p^2 - (p_x + \frac{1}{2} b_z y)^2 - (p_y - \frac{1}{2} b_z x)^2} \quad b_z = \frac{e B_z}{P_0} \]

- Symplectic transformation (exact solution):

\[
\begin{align*}
x_2 &= x_1 + \frac{(1+\delta) \sin \phi}{b_z} p_{xi} + \frac{(1+\delta)(1-\cos \phi)}{b_z} p_{yi}, \\
y_2 &= y_1 - \frac{(1+\delta)(1-\cos \phi)}{b_z} p_{xi} + \frac{(1+\delta) \sin \phi}{b_z} p_{yi}, \\
p_{x2} &= p_{xi} \cos \phi + p_{yi} \sin \phi - \frac{b_z}{2(1+\delta)} y_2, \\
p_{y2} &= -p_{xi} \sin \phi + p_{yi} \cos \phi + \frac{b_z}{2(1+\delta)} x_2, \\
z_2 &= z_1 + \left[ \frac{\sqrt{1-p_{xi}^2-p_{yi}^2} - 1}{\sqrt{1-p_{xi}^2-p_{yi}^2}} - \Delta \nu \right] L \\
\end{align*}
\]

- The SOL element in SAD is special: NO attribute of L
- The next case: L≠0, BZ≠0, K0≠0, SK0≠0 [Solvable]
5. Symplectic tracking in SAD

➤ Fringe fields: Bend soft edge fringe (From Bmad manual)

*Bmad* defines the bend soft edge map in terms of the field integral $F_{H_1}$ for the entrance end and $F_{H_2}$ for the exit end given by (see Eq. (3.5))

$$F_{H_1} = F_{int} H_{gap} = \int_{pole} ds \frac{B_y(s) (B_{y0} - B_y(s))}{2 B_{y0}^2}$$

(19.46)

With a similar equation for $F_{H_2}$. The soft edge map is then

$$x_2 = x_1 + c_1 p_z$$

$$p_{y2} = p_{y1} + c_2 y_1 - c_3 y_1^3$$

$$z_2 = z_1 + \frac{1}{1 + p_{z1}} \left( c_1 p_{z1} + \frac{1}{2} c_2 y_1^2 - \frac{1}{4} c_3 y_1^4 \right)$$

(19.47)

For the entrance face:

$$c_1 = \frac{g_{tot} F_{H_1}^2}{2 (1 + p_z)}, \quad c_2 = \frac{2 g_{tot} F_{H_1}}{1 + p_z}, \quad c_3 = 0$$

(19.48)

with $g_{tot}$ is the total bending strength

$$g_{tot} = g + g_{err}$$

(19.49)

g being the reference bend strength and $g_{err}$ being bend the difference between the actual and reference bend strengths (§3.5).

For the exit face, the substitution is made

$$F_{H_1} \rightarrow F_{H_2}$$

$$g_{tot} \rightarrow -g_{tot}$$

(19.50)

When the SAD bend soft edge map is used (§4.20), the map is the same except that the value of $c_3$ is

$$c_3 = \frac{8 g_{tot}^2}{F_{H_1} (1 + p_z)}$$

(19.51)
5. Symplectic tracking in SAD

➤ Fringe fields: Quad. soft edge fringe (From Bmad manual)

Only the quadrupole soft edge fringe is modeled in Bmad. The model is adapted from SAD[SAD]. The fringe map is:

\[
\begin{align*}
    x_2 &= x_1 e^{g_1} + g_2 p_{x_1} \\
    p_{x_2} &= p_{x_1} e^{-g_1} \\
    y_2 &= y_1 e^{-g_1} - g_2 p_{y_1} \\
    p_{y_2} &= p_{y_1} e^{g_1} \\
    z_2 &= z_1 - \left[ g_1 x_1 p_{x_1} + g_2 \left(1 + \frac{g_1}{2}\right) e^{-g_1} p_{x_1}^2 \right] + \left[ g_1 y_1 p_{y_1} + g_2 \left(1 - \frac{g_1}{2}\right) e^{g_1} p_{y_1}^2 \right]
\end{align*}
\]  

(19.53)

where

\[
\begin{align*}
    g_1 &= K_1 f_{q1} \\
    g_2 &= K_1 f_{q2}
\end{align*}
\]  

(19.54)

\(K_1\) is the quadrupole strength, and \(f_{q1}\) and \(f_{q2}\) are the fringe quadrupole parameters. These parameters are related to the field integral \(I_n\) via

\[
\begin{align*}
    f_{q1} &= I_1 - \frac{1}{2} I_0^2 \\
    f_{q2} &= I_2 - \frac{1}{3} I_0^3
\end{align*}
\]  

(19.55)

where \(I_n\) is defined by

\[
I_n = \frac{1}{K_1} \int_{-\infty}^{\infty} (K_1(s) - H(s - s_0) K_1) (s - s_0)^n ds
\]  

(19.56)

and \(H(s)\) is the step function

\[
H(s) = \begin{cases} 
1 & s > 0 \\
0 & s < 0
\end{cases}
\]  

(19.57)

and it is assumed that the quadrupole edge is at \(s_0\) and the interior is in the region \(s > s_0\).
5. Symplectic tracking in SAD

➤ Fringe fields: Hard edge fringe (From Bmad manual)

The magnetic multipole hard edge fringe field is modeled using the method shown in Forest[Forest98]. For the $m^{th}$ order multipole the Lee transform is (Forest Eq. (13.29)):

$$f_\pm = \mp \Re \left[ \frac{(b_m + i a_m) (x + iy)^{m+1}}{4 (m+2) (1 + \delta)} \left\{ x p_x + y p_y + i \frac{m+3}{m+1} (x p_x - y p_y) \right\} \right]$$

$$\equiv \frac{p_x f^x + p_y f^y}{1 + \delta} \tag{19.58}$$

The multipole strengths $a_m$ and $b_m$ are given by (14.9) and the second equation defines $f^x$ and $f^y$. On the right hand side of the first equation, the minus sign is appropriate for particles entering the magnet and the plus sign is for particle leaving the magnet. Notice that here the multipole order $m$ is equivalent to $n - 1$ in Forest’s notation.

With this, the implicit multipole map is (Forest Eq. (13.31))

$$x^f = x - \frac{f^x}{1 + \delta}$$

$$p_x = p_x^f - \frac{p_x^f \partial_x f^x + p_y^f \partial_x f^y}{1 + \delta}$$

$$y^f = y - \frac{f_y}{1 + \delta}$$

$$p_y = p_y^f - \frac{p_x^f \partial_y f^x + p_y^f \partial_y f^y}{1 + \delta}$$

$$\delta^f = \delta$$

$$z^f = \frac{p_x^f f^x + p_y^f f^y}{(1 + \delta)^2} \tag{19.59}$$

Note:
This equation is general, applying for BEND, QUAD, SEXT, ... to arbitrary order. But BEND is special!
5. Symplectic tracking in SAD

➤ Fringe fields: Hard edge fringe for BEND

* Two models found for hard-edge fringe
  * E. Forest: “Parallel-plate” shape (popular theory)
  * Y. Cai: Round shape (SLAC-PUB-11181, apply for SC magnets?)

[Diagrams: Usual case (From SuperKEKB TDR), SC magnet (From S. Russenschuck’s textbook, 2010)]
5. Symplectic tracking in SAD

➤ Fringe fields: Hard edge fringe for BEND

- Maxwellian solution for hard-edge dipole field
  * G. Lee-Whiting et al. => E. Forest et al.
  * S. Caspi et al. => M. Bassetti et al. => Y. Cai et al.

The model for harmonics expansion. The field is confined inside a circle with $r<r_0$
(From S. Russenschuch’s textbook, 2010)

The model for wide magnet. The field is confined at region of $-b<y<b$ and $-\infty<x<\infty$
5. Symplectic tracking in SAD

➤ Fringe fields: Hard edge fringe for BEND

● Maxwellian solution for hard-edge dipole field

* G. Lee-Whiting et al. => E. Forest et al.

* S. Caspi et al. => M. Bassetti et al. => Y. Cai et al.

\[
A_s = -xB(s) = -xB_0\theta(s).
\]

\[
\vec{A} = (A_x, 0, A_s)
\]

\[
\nabla \times \nabla \times \vec{A} = 0
\]

\[
A_x = B_0 \sum_{n=1}^{\infty} \frac{(-1)^n \theta(2n-1)(s)}{(2n)!} y^{2n}
\]

\[
A_y = 0
\]

\[
A_x = \frac{1}{2}(x^2 - y^2) \sum_{p=0}^{\infty} \frac{1}{2+p} G_{1,2p+1}(s)(x^2 + y^2)^p,
\]

\[
A_y = xy \sum_{p=0}^{\infty} \frac{1}{2+p} G_{1,2p+1}(s)(x^2 + y^2)^p,
\]

\[
A_s = -x \sum_{p=0}^{\infty} G_{1,2p}(s)(x^2 + y^2)^p.
\]

\[
G_{n,2p}(s) = (-1)^p \frac{n!}{4p(n+p)!p!} \frac{d^{2p}G_{n,0}(s)}{ds^{2p}},
\]

\[
G_{n,2p+1}(s) = \frac{dG_{n,2p}(s)}{ds},
\]

\[
A_y \neq 0
\]
5. Symplectic tracking in SAD

➢ Fringe fields: Hard edge fringe for BEND

● Maxwellian solution for hard-edge dipole field
  * G. Lee-Whiting et al. => E. Forest et al.
  * S. Caspi et al. => M. Bassetti et al. => Y. Cai et al.

Field distribution with hard-edge:

\[
B_x = 0,
\]

\[
B_y(y, s) = B_0 \sum_{n=1}^{\infty} \frac{(-1)^n y^{2n} \theta(2n)(s)}{(2n)!},
\]

\[
B_s(y, s) = -2B_0 \sum_{n=1}^{\infty} \frac{(-1)^n n y^{2n-1} \theta(2n-1)(s)}{(2n)!}.
\]

Field distribution with hard-edge:

\[
B_x(x, y, s) = -\frac{1}{2} B_0 x y \sum_{p=0}^{\infty} \frac{(-1)^p (x^2 + y^2)^p \theta(2p+2)(s)}{4^n p!(p+2)!},
\]

\[
B_y(x, y, s) = B_0 \theta(s) + B_0 \sum_{p=0}^{\infty} \frac{(-1)^{p+1} (x^2 + y^2)^p \theta(2p+2)(s)}{4^{p+1} (p+1)! (p+2)!} [x^2 + (2p+3)y^2],
\]

\[
B_s(x, y, s) = B_0 y \sum_{n=1}^{\infty} \frac{(-1)^p (x^2 + y^2)^p \theta(2p+1)(s)}{4^n p!(p+1)!}.
\]
5. Symplectic tracking in SAD

➤ Fringe fields: Hard edge fringe for BEND

- Maxwellian solution for hard-edge dipole field
  * G. Lee-Whiting et al. => E. Forest et al.
  * S. Caspi et al. => M. Bassetti et al. => Y. Cai et al.

\[
f = -V_1 = -\frac{1}{2\rho(1+\delta)} p_x y^2
\]

Implemented in SAD:

\[
x_2 = x_1 - \frac{1}{\rho(1+\delta)} y_1^2,
\]

\[
p_{y2} = p_{y1} + \frac{1}{\rho(1+\delta)} y p_{x1},
\]

\[
z_2 = z_1 + \frac{y_1^2}{2\rho(1+\delta)^2} p_{x2}.
\]

Apply for LHC and FCCs?:

\[
f = \frac{1}{8\rho(1+\delta)} (-p_x x^2 + 2p_y xy - 3p_y y^2)
\]

\[
x_2 = x_1 - \frac{1}{8\rho(1+\delta)} (x_1^2 + 3y_1^2),
\]

\[
y_2 = y_1 + \frac{1}{4\rho(1+\delta)} x_1 y_1,
\]

\[
p_{x2} = \frac{1}{d} \left[ p_{x1} - \frac{1}{4\rho(1+\delta)} (y_1 p_{y1} - x_1 p_{x1}) \right],
\]

\[
p_{y2} = \frac{1}{d} \left[ p_{y1} - \frac{1}{4\rho(1+\delta)} (x_1 p_{y1} - 3y_1 p_{x1}) \right],
\]

\[
z_2 = z_1 + \frac{x_1^2 + 3y_1^2}{8\rho(1+\delta)^2} p_{x2} - \frac{x_1 y_1}{4\rho(1+\delta)^2} p_{y2},
\]

\[
d = 1 + \frac{3y_1^2 - x_1^2}{16\rho^2(1+\delta)^2}.
\]
5. Symplectic tracking in SAD

➢ Solenoid region

● The most complicated part in SAD
● SAD uses GEO and BOUND to define a solenoid region
● Acceptable elements inside solenoid region: DRIFT, BEND(ANGLE=0), QUAD and MULT

● To simplify the transformation: In a SOL region, the coordinate is shifted on the axis of the solenoid, no matter how the design orbit bends there.
5. Symplectic tracking in SAD

➤ Solenoid region

- DRIFT with \( BZ \neq 0 \)
- BEND with \( BZ \neq 0: L \neq 0, K0 \neq 0, SK0 \neq 0, \text{ANGLE}=0 \) [Solvable]
- QUAD with \( BZ \neq 0: L \neq 0, K1 \neq 0, SK1 \neq 0 \) [Solvable?]
- The general case: MULT with \( BZ \neq 0 \) [Need multi-step integration]
  * Step 1: Solenoid fringe at the entrance
  * Step 2: Rotation of coordinate to cancel SK1
  * Step 3: Calculate the number slices for tracking
  * Step 4: Nonlinear Maxwellian fringe map at the entrance
  * Step 5: Linear soft edge fringe at the entrance
  * Step 6: Body part using “drift-kick-drift” integration
  * Step 7-11: Maps at exit
5. Symplectic tracking in SAD

➤ Tilted solenoid: FCC-ee as an example

- **SAD: Orbit patching**

```
SOL  ES3L  = (BZ = 0  DX = -0.03000219149072553  DY = 3.394659937371677e-14  
       DZ = 0.0002250210956802542  BOUND = 1  
       CHI1 = 0.1499991193234515  CHI2 = -1.69694146889447e-14  
       CHI3 = -2.545603126440053e-16  F1 = .3 )
ES2L  = (BZ = 2  F1 = .1 )
ES1L  = (BZ = -2  DPX = -.015  BOUND = 1  CHI1 = -.015  GEO = 1 )
ES1R  = (BZ = 2  DPX = -.015  BOUND = 1  CHI1 = .015  CHI2 = -1.3883951931889808e-28  
       CHI3 = -2.070759876205156e-30  GEO = 1 )
ES2R  = (BZ = -2  F1 = .1 )
ES3R  = (BZ = 0  DX = 0.03000219149072553  DY = 3.394659937371677e-14  
       DZ = 0.0002250210956802542  BOUND = 1  
       CHI1 = -0.01499991193234515  CHI2 = -1.6971323927087947e-14  
       CHI3 = 1.4946246979225722e-21  F1 = .3 )
```

- **Beam line: (-ES3L -LX2 -ES2L -LX1 -ES1L -IP -IP -ES1R LX1 ES2R LX2 ES3R)**

![Solenoid axis](image)

![Graphs](image)
6. Summary

➤ Lattice translation
- Translators collected
- Examples uploaded to MAD-X svn repository and my webpage
- Benchmark of SAD, Bmad and MAD-X/PTC for several projects

➤ Applications
- Synchrotron radiation simulation using Bmad (Synrad3D)
- RDT calculations using PTC
- Analysis of lattice nonlinearity in SuperKEKB and simulations of beam-beam with nonlinearity

➤ Future plan
- Translators to be improved (joint efforts)
- Accelerator design/simulations: Applications