Gravity Dual of Spatially Modulated Phase

Instability of Black Holes Induced by Chern-Simons Terms

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Based on S.N.-Ooguri-Park, arXiv:0911.0679
Motivation

Chern-Simons terms in super-gravity play important roles. Especially from the viewpoint of holography,

- chiral current anomaly
- baryons
- ..........

in gauge theories are realized through the CS terms in their gravity duals.

We consider effects of the Chern-Simons terms in details in this talk.
Chern-Simons term in 5d SUGRA

An $S^5$ reduction of 10d type IIB super-gravity:

\[ 16\pi G_5 L = \sqrt{-g} \left( R + 12 - \frac{1}{4} F_{MN} F^{MN} \right) + \frac{\alpha}{3!} \epsilon^{IJKLM} A_I F_{JK} F_{LM} \]

\[ \alpha = \frac{1}{2\sqrt{3}} \]

The bosonic part of the N=2 5d minimal gauged super-gravity with a negative cosmological constant.

5-dim. Einstein ($\Lambda < 0$) + Maxwell theory with a CS term.

- Any solution to this theory can be consistently embedded into the 10d type IIB super-gravity (namely, type IIB superstring).
- This system is dual to 4d N=4 SYM theory with R-charge.
- The CS coupling $\alpha$ is at some fixed value for the closure of SUSY.
Equations of motion

\[ R_{MN} - \frac{1}{2} g_{MN} (R + 12) = \frac{1}{2} \left( F_{ML} F_{N}^{L} - \frac{1}{4} g_{MN} F^{2} \right), \]

\[ \sqrt{-g} \nabla_{M} F^{MN} + \frac{\alpha}{2} \varepsilon^{N_IJ_K_L} F_{IJ} F_{KL} = 0. \]

Let us consider only electrically charged solutions. Then the CS term does not affect the solutions.

The solution is just an ordinary AdS-Reissner-Nordström black hole.

However, if consider the fluctuations around the solution, the CS term plays an important role.
Preparation:
A flat-background example

Maxwell + CS on flat 5d (without gravity)

\[ \partial_{M} F^{MN} + \frac{\alpha}{2} \varepsilon^{NIJKL} F_{IJ} F_{KL} = 0. \]

If we have a background electric field \( F_{01} = E \), the e.o.m. under the Lorenz gauge *is

\[ \partial^2 A_N + 4\alpha E \varepsilon^{01NKL} \partial_K A_L + O(A^2) = 0. \]

For linear perturbations of \( A_N = a_N(x^1) e^{-i\omega t + ik x^2} \),

\[ \partial^2 a_3 - 4\alpha E (ik) a_4 = 0, \]

\[ \partial^2 a_4 + 4\alpha E (ik) a_3 = 0. \]

(* we can also formulate in a gauge invariant way.*)
\[
\partial^2 a_3 - 4\alpha E (ik)a_4 = 0 , \quad (1)
\]
\[
\partial^2 a_4 + 4\alpha E (ik)a_3 = 0 . \quad (2)
\]

(1) \pm i (2)

\[
\left( \omega^2 - k^2 \right)(a_3 \pm ia_4) \mp 4\alpha Ek(a_3 \pm ia_4) = 0
\]

The dispersion relation is

\[
\omega^2 - \left( k \pm 2\alpha E \right)^2 = -(2\alpha E)^2
\]

tachyonic?

for the circular-polarized modes.

This does not necessarily mean the system is unstable, but it does in the following momentum range:

\[
0 < |k| < 4\alpha E
\]
Dispersions relation

The dispersion relation is non-standard:

\[ \omega^2 = \frac{\kappa}{\kappa^2 + \alpha^2} \]

The minimum of the spectrum is located at \( k = 2\alpha E \neq 0 \).
Inhomogeneous condensation

The instability occurs at finite momenta.

The resulting condensation is inhomogeneous: the condensation is spatially modulated (and circularly polarized).

Spontaneous breaking of translational and rotational symmetry.

5d Maxwell theory + CS term + electric field is very interesting system.
What happens in the 5d SUGRA?

The background spacetime is asymptotically AdS:

A slightly “tachyonic” mode is still stable: there is Breitenlohner-Freedman bound.


We need to check whether the effect of the CS term exceeds the BF bound or not.

The U(1) gauge field couples with gravitons:

We need to diagonalize the possible fluctuations.

The background geometry is curved non-trivially.
**Strategy**

• The **electric field** in the Reissner-Nordström black hole is *maximum at the horizon* and decreases towards the boundary.

  Probably, the system is **most unstable** around the horizon since the “instability” was proportional to the electric field in 5d Maxwell.

• The **maximum electric field** is realized in the **extremal BH**.

  \(T=0\), but SUSY is broken)

  We take the **near-horizon limit** of the

  5d extremal RN black hole.

  \(\text{AdS}_2 \times \mathbb{R}^3\)

  Much **simpler** geometry than RN-BH.

  The **near-horizon analysis** is at least a **good starting point**.
Breitenlohner-Freedman bound

In $d+1$-dimensional AdS spacetime,

$$ds^2 = L^2 \eta_{\mu\nu}dx^\mu dx^\nu + dz^2$$

a scalar fluctuation is stable if its mass $m$ satisfies:

$$m^2 \geq -\frac{d^2}{4L^2}$$

Breitenlohner-Freedman bound

Stable means

the amplitude of the field fluctuation does not grow along the time evolution.
The near-horizon geometry

RN black hole:

\[ ds^2 = \frac{1}{z^2} \left( -f(z)dt^2 + d\bar{x}^2 + f(z)^{-1}dz^2 \right), \]

\[ f(z) = 1 - M z^4 + \frac{q^2}{12} z^6, \quad F_{z0} = q z, \quad M = \frac{1}{z_H^4} + \frac{q^2}{12} z_H^2, \quad T_H = \frac{1}{\pi z_H^2} \left( 1 - \frac{1}{24} q^2 z_H^6 \right) \]

**Extremal limit** choose the combination of M and q so that T=0.

\[ f(z) = \left( 1 - \frac{1}{z_H^2} \right)^2 \left( 1 + 2 \left( \frac{1}{z_H^2} \right)^2 \right), \quad M = \frac{3}{z_H^4}, \quad \frac{q^2}{12} = 2 z_H^6, \quad T_H = 0. \]

**Near-horizon limit** \( y = 1 - \frac{z}{z_H}, \quad y \approx 0 \ll 1, \quad (X^\mu = x^\mu \frac{1}{z_H}) \)

\[ ds^2 = \frac{1}{12\xi^2} \left( -dt^2 + d\bar{x}^2 + d\xi^2 \right), \quad \xi = \frac{1}{12y} \]

\[ F_{\xi0} = 2\sqrt{6} \frac{1}{12\xi^2}. \]

**AdS$_2 \times \mathbb{R}^3$** with elec. field of const. $\times$ vol. form.
The **Maxell + CS theory** at the near-horizon

\[ ds^2 = \frac{1}{12\xi^2} (-dt^2 + d\vec{x}^2 + d\xi^2), \quad F_{\xi_0} = 2\sqrt{6} \frac{1}{12\xi^2}. \]

**BF bound:** \( m^2 \geq -\frac{1^2}{4(\frac{1}{12})^2} = -3 \)

\[ \omega^2 - (k \pm 2\alpha E)^2 = -(2\alpha E)^2 \]
\[- (2\alpha E)^2 = -96 \alpha^2 < -3, \]

condition for instability

\[ |\alpha| = \frac{1}{2\sqrt{3}} > \frac{1}{4\sqrt{2}} \]

for SUGRA

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The contribution of the CS term **exceeds** the **BF bound**, if we **do not** include the gravitational fluctuations.
What happens when the graviton switched on?

The coupling between the gauge field and the graviton comes from the kinetic term:

\[-\frac{1}{4} \sqrt{-g} F_{MN} F^{MN} = -\sqrt{-g} F^\mu_\nu h^i_\mu F_{vi} + \ldots\]

\[g_{\mu i} = \bar{g}_{\mu i} + h_{\mu i}, \quad f_{vi} = \partial_v A_i - \partial_i A_v, \quad (\mu, \nu) \in (0,1)\]

Our modes couple to the off-diagonal part of the metric.

<table>
<thead>
<tr>
<th></th>
<th>$\text{AdS}_2 \times \mathbb{R}^3 (x^{\mu=0,1}, x^2, x^{i=3,4})$</th>
<th>$\text{AdS}_2 (x^{\mu=0,1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_i$</td>
<td>Maxwell</td>
<td>scalar</td>
</tr>
<tr>
<td>$h_{\mu i}$</td>
<td>off-diagonal graviton</td>
<td>KK gauge field</td>
</tr>
<tr>
<td>$h_{2i}$</td>
<td>off-diagonal graviton</td>
<td>Stückelberg field</td>
</tr>
</tbody>
</table>
graviton

Maxwell+CS coupling

The equations of motion reduce to

\[
\left(\nabla_{\text{AdS}_2}^2 + \partial_j \partial^j\right)f_i - 4\alpha E \varepsilon_{ijk} \partial_j f_k + E \varepsilon_{ijk} \partial_j K_k = 0, \\
E \nabla_{\text{AdS}_2}^2 f_i \left(\nabla_{\text{AdS}_2}^2 + \partial_j \partial^j\right) \varepsilon_{ijk} \partial_j K_k = 0.
\]

\[f_i = \frac{1}{2} \varepsilon_{ijk} F_{jk}, \quad K_i = 12 \xi^2 K_{01}^{(i)}\]

The mass eigenvalue should satisfy:

\[
\det\begin{pmatrix}
    m^2 - k^2 - 4\alpha E k & E \\
    E m^2 & m^2 - k^2
\end{pmatrix} = 0,
\]

\[E = 2\sqrt{6}, \quad \alpha = \frac{1}{2\sqrt{3}}\]

\[m_{\text{min}}^2 = \frac{E^2}{2(4\alpha^2 + 1)^2} \left(-64\alpha^6 - 24\alpha^4 + 6\alpha^2 - (16\alpha^4 + 4\alpha^2 + 1)^{3/2} + 1\right) = -2.96804\ldots\]

Oops! slightly stable!
Einstein + Maxwell + CS theory near the horizon of the RN-BH becomes unstable if the CS coupling $\alpha$ is $\alpha > \alpha_c = 0.2896$. ... while the 5d SUGRA has $\alpha = 3^{-1/2}/2 = 0.2887$... which is barely outside the unstable region.

(The SUSY is broken even at the extremal case.)

<table>
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<tr>
<th>$A_i$</th>
<th>scalar</th>
<th>$(\text{mass})^2 = k^2$</th>
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<td>$h_{\mu i}$</td>
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<td>$h_{2i}$</td>
<td>Stückelberg field</td>
<td>decouples</td>
</tr>
</tbody>
</table>

The CS makes the "scalars" tachyonic but they couples with the "massive KK gauge fields."
Full-geometry analysis

The near-horizon analysis provides a sufficient condition for instability but not the necessary condition.

Analysis on the full AdS-RN-BH geometry:

• consider normalizable modes,
• impose “in-going” boundary condition at the horizon,
• see whether its amplitude decays or grows along time.

The behavior depends on $k$ and $\alpha$ and $T$.

We found that the numerical analysis on the full RN-BH geometry yields the same critical value $\alpha_c=0.2896,...$. 
Comment

Although $\alpha_c$ is not modified in the full-geometry analysis, the momentum-range for instability is corrected.

The instability range goes until some upper limit of $k$ like but only the lower curves are shown.

The red curve: the BF bound from the NH analysis. Curve I: the lower bound for the instability region. Curve II: onset of higher-excited unstable modes.
b: “unstable region” from the near-horizon analysis
Decoupling modes

There are unstable modes which are **not detected** in the near-horizon analysis.

They become **non-normalizable mode** in the near-horizon geometry.

Intuitive but **not precise** cartoon:

- mode for curve I: no node
- mode for curve II: 1 node
  This goes down faster.
More precisely:

An appropriate combination of the gauge field and the graviton near the horizon (r=0):

$$\phi(r) \approx r^{-\frac{1}{2} + \sqrt{\frac{m^2 + 3}{12}}} + \ldots.$$ 

“Normalizable” means

$$\int_0^\infty dr \phi^* \phi = \text{finite}.$$ 

Full-geometry analysis: the normalizability is checked numerically. 

Near-horizon analysis: take the above profile as if it is valid until r=∞.

$$\phi^* \phi = r^{-1} \left| e^{\sqrt{\frac{m^2 + 3}{12}} \log r} \right|^2 = r^{-1} \cos \left( 2 \sqrt{\frac{m^2 + 3}{12}} \log r \right) \rightarrow \text{const.} \times \cos \left( 2 \sqrt{\frac{m^2 + 3}{12}} \log r \right)$$

implies divergence

but, if the BF bound is violated \((m^2 < -3)\),

integration oscillating but finite

The modes outside the curve II satisfy the BF bound and are non-normalizable in the near-horizon analysis.
Other geometries

We have also analyzed the following cases at the near-horizon limit.

- 5d extremal AdS-RN-BH with the boundary geometry $S^3$.
- 3-charge solutions

As far as we have analyzed, the BF bound is always barely satisfied (within 0.4%) if we employ the CS coupling coming from type IIB super-gravity.

Are there any unknown reason?
A Phenomenological Approach
CS coupling at arbitrary values

Presence of CS term is quite general for gravity theories coming from superstring theory through various compactifications.

Investigation of general nature of the gravity theory with Maxwell+CS is important, being motivated by the possible correspondence to yet to known CFT.

Cf. Holographic superconductor

Let us go a little bit further to investigate the nature of Einstein+Maxwell+CS theory at arbitrary CS coupling.
A possible phase transition

Suppose that the CS coupling is a parameter which we can choose phenomenologically.

In other words, let us start with 5d Einstein + Maxwell + CS theory with electric field.

Then we have unstable RN-AdS BH’s at sufficiently large CS couplings.

\( \alpha_c \) is a function of the temperature.

We have a phase transition into a modulated phase when we cool down the temperature towards zero.
A holographic interpretation

\[ ds^2 = \tilde{g}_{\mu\nu}(z)dx^\mu dx^\nu + dz^2 \]

\[ A_\mu(z) = A^{(0)}_\mu + z^2 A^{(2)}_\mu + \ldotsd \]

non-normalizable mode \hspace{1cm} normalizable mode

source \hspace{2cm} <current>

\( A^{(0)}_0 = \mu \)

Condensation of the gauge field:

development of the non-zero expectation value of the current without the external source.

Finite momentum \( \rightarrow \) spatially modulated current (circularly polarized helical current)
Circularly-polarized current

- The current has a Helical structure
- Spontaneous breaking of translational and rotational symmetries.
The boundary of the two regions: **onset** of the instability. This **may not** be the real phase boundary which is determined by the competition of the free energies of the two phases, since we do not know the exact free energy for the modulated phase.
**Brazovskii model**

Brazovskii model: a model for phase transitions to inhomogeneous phases.

The non-standard dispersion relation in which the spectrum has the minimum at a finite momentum is postulated there.

Applied to various physics, such as

- weakly anisotropic antiferromagnets
- cholesteric liquid crystals
- pion condensates in neutron stars
- Rayleigh-Bénard convection
- symmetric diblock copolymers

In our model, the non-standard dispersion relation is realized by the CS term.
A cholesteric liquid crystal is a type of liquid crystal with a helical structure and which is therefore chiral. Cholesteric liquid crystals are also known as chiral nematic liquid crystals.

Taken from Wikipedia
Even if we do not have instability, the dispersion relation of the circularly-polarized modes is non-standard.

At the minimum of the spectrum, the density of states per unite energy diverges: Van Hove singularity.
Van Hove singularity

Example: Plasmino in QCD

Di-lepton production rate diverges at the Van Hove singularity.

Even for N=4 SYM at finite R-charge, the anomaly induced Van Hove singularity exists.
Summary

• 5d gravity theories reduced from type IIB SUGRA have a **CS term**.

• The circularly-polarized modes of gauge fields in Einstein+Maxwell+CS theory with electric flux have non-standard dispersion relations.

• If the CS interaction is **strong enough**, the circularly-polarized modes become **tachyonic**.

• As far as we have analyzed, the gravity theories coming from SUGRA are barely **stable** although SUSY is broken.

• Einstein(Λ<0)+Maxwell+CS theory is an **interesting** theory, which has a potential possibility to be a gravity dual of phase transitions to inhomogeneous phases.