Resurgence and Non-Perturbative Physics: Decoding the Path Integral

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GD, M. Ünsal, G. Başar, 1210.2423, 1210.3646, 1306.4405, 1401.5202, 1501.05671, 1505.07803, 1509.05046, 1511.05977

GD, introductory lectures at CERN 2014 Winter School

GD, introductory lectures at Schladming 2015 Winter School

Some Physical Motivation

- improved asymptotics in QFT
- infrared renormalon puzzle in asymptotically free QFT
- non-perturbative physics without instantons:
  physical meaning of non-BPS saddle configurations

Bigger Picture

- non-perturbative definition of nontrivial QFT in continuum
- analytic continuation of path integrals (Lefschetz thimbles)
- dynamical and non-equilibrium physics from path integrals

Some Mathematical Motivation

Resurgence: 'new' idea in mathematics (Écalle, 1980; Stokes, 1850)

resurgence = unification of perturbation theory and non-perturbative physics

- perturbation theory $\longrightarrow$ divergent (asymptotic) series
- formal series expansion $\longrightarrow$ *trans-series* expansion
- trans-series 'well-defined under analytic continuation'
- perturbative and non-perturbative physics entwined
- applications: ODEs, PDEs, fluids, QM, Matrix Models, QFT, String Theory, ...
- philosophical shift:

  view asymptotics/semiclassics as potentially exact

Series $\longrightarrow$ Trans-Series

- trans-series expansion in QM and QFT applications:

$$f(g^2) = \sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=1}^{k-1} \underbrace{c_{k,l,p}\, g^{2p}}_{\text{perturbative fluctuations}} \underbrace{\left(\exp\left[-\frac{c}{g^2}\right]\right)^k}_{\text{k}-\text{instantons}} \underbrace{\left(\ln\left[\pm\frac{1}{g^2}\right]\right)^l}_{\text{quasi-zero-modes}}$$

- Écalle (1980): functions 'closed under all operations':

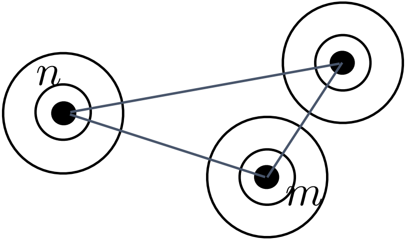
(Borel transform) + (analytic continuation) + (Laplace transform)

- *trans-monomial elements*: g^2, $e^{-\frac{1}{g^2}}$, $\ln(g^2)$, are familiar
- "multi-instanton calculus" in QFT
- new: analytic continuation encoded in trans-series
- new: trans-series coefficients $c_{k,l,p}$ highly correlated
- new: exponential asymptotics (Olver, Kruskal, Segur, Costin, ...)

Resurgence

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle, 1980



local analysis encodes more global information than one might naïvely think

(i) divergent, alternating:

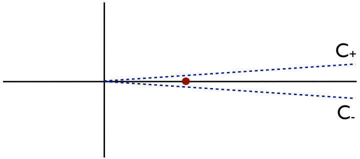
$$\sum_{n=0}^{\infty}(-1)^n\, n!\, g^{2n} = \int_0^{\infty} dt\, e^{-t}\, \frac{1}{1+g^2\, t}$$

(ii) divergent, non-alternating:

$$\sum_{n=0}^{\infty} n!\, g^{2n} = \int_0^{\infty} dt\, e^{-t}\, \frac{1}{1-g^2\, t}$$

$\Rightarrow$ ambiguous imaginary non-pert. term: $\pm \frac{i\,\pi}{g^2}\, e^{-1/g^2}$

avoid singularities on $\mathbb{R}^+$: *directional* Borel sums:



$\theta = 0^{\pm} \longrightarrow$ non-perturbative ambiguity: $\pm\text{Im}[\mathcal{B}f(g^2)]$

challenge: use physical input to resolve ambiguity

Resurgent relations within Trans-series

- trans-series (neglecting logs for now for simplicity)

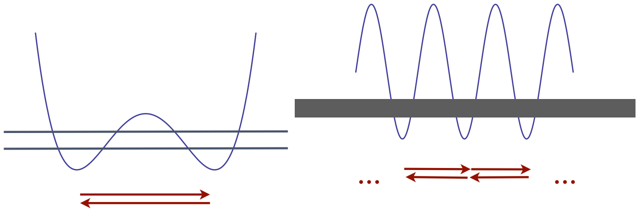
$$
\begin{aligned}
F(g^2) \quad \sim \quad & \left(c_0^{(0)} + c_1^{(0)} g^2 + c_2^{(0)} g^4 + \dots\right) \\
& + \sigma\, e^{-S/g^2} \left(c_0^{(1)} + c_1^{(1)} g^2 + c_2^{(1)} g^4 + \dots\right) \\
& + \sigma^2\, e^{-2S/g^2} \left(c_0^{(2)} + c_1^{(2)} g^2 + c_2^{(2)} g^4 + \dots\right) \\
& + \dots
\end{aligned}
$$

- basic idea: ambiguous imaginary non-perturbative contributions from Borel summation of non-alternating divergent series in one sector must cancel against terms in some other non-perturbative sector

- implies very strong relations between trans-series expansion coefficients in different non-perturbative sectors

Hint of Resurgence in QM Spectral Problems

- QM analog of IR renormalon problem in QFT



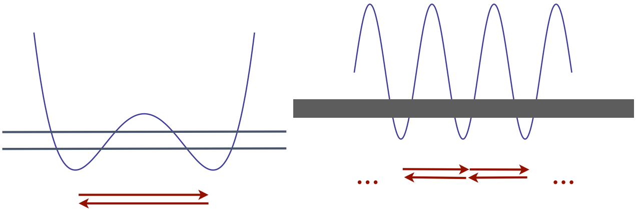
- degenerate vacua: double-well, Sine-Gordon, ...

splitting of levels: a real one-instanton effect: $\Delta E \sim e^{-\frac{S}{g^2}}$

surprise: pert. theory non-Borel-summable: $c_n \sim \frac{n!}{(2S)^n}$

- stable systems
- ambiguous imaginary part
- $\pm i\, e^{-\frac{2S}{g^2}}$, a 2-instanton effect

"Bogomolny/Zinn-Justin mechanism" Bogomolny 1980; Zinn-Justin 1980



- degenerate vacua: double-well, Sine-Gordon, ...
  1. perturbation theory non-Borel-summable: ill-defined/incomplete
  2. instanton gas picture ill-defined/incomplete: $\mathcal{I}$ and $\bar{\mathcal{I}}$ attract
- regularize *both* by analytic continuation of coupling

$\Rightarrow$ ambiguous, imaginary non-perturbative terms cancel !

"resurgence" $\Rightarrow$ cancellation to all orders

Resurgence in Matrix Models, Topological Strings

Mariño, Schiappa, Weiss: *Nonperturbative Effects and the Large-Order Behavior of Matrix Models and Topological Strings* 0711.1954; Mariño, *Nonperturbative effects and nonperturbative definitions in matrix models and topological strings* 0805.3033

• resurgent Borel-Écalle analysis of matrix models

$$Z(g_s, N) = \int dU \, \exp\left[\frac{2}{g_s} \text{tr}\, V(U)\right]$$

• two variables: g_s and N ('t Hooft coupling: $\lambda = g_s N$)

• e.g. Gross-Witten-Wadia: $V = U + U^{-1}$

• 3rd order phase transition at $\lambda = 1$, associated with condensation of instantons (Neuberger)

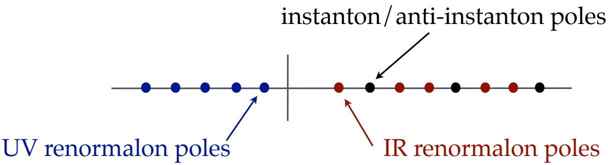
• double-scaling limit: Painlevé II

IR Renormalon Puzzle in Asymptotically Free QFT

QFT: new source of divergence in perturbation theory

IR renormalons (pert theory): $\longrightarrow$ $\pm i\, e^{-\frac{2S}{\beta_0\, g^2}}$

non-pert instantons ($\mathbb{R}^2$ or $\mathbb{R}^4$): $\longrightarrow$ $\pm i\, e^{-\frac{2S}{g^2}}$



appears that BZJ cancellation cannot occur

asymptotically free theories remain inconsistent

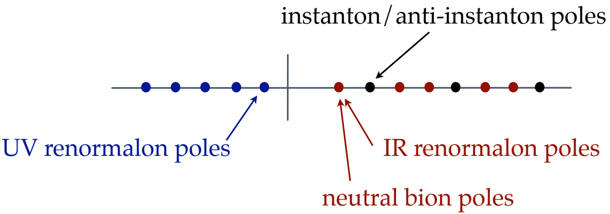
't Hooft, 1980; David, 1981

IR Renormalon Puzzle in Asymptotically Free QFT

(Argyres, Ünsal 1206.1890; GD, Ünsal, 1210.2423, 1210.3646; Misumi, Nitta, Sakai, 2014, 2015)

resolution: there is another problem with the non-perturbative instanton gas analysis: scale modulus of instantons

- spatial compactification and principle of continuity
- 2 dim. $\mathbb{CP}^{N-1}$ model: $S_{\rm inst} \to \frac{S_{\rm inst}}{N} = \frac{S_{\rm inst}}{\beta_0}$



cancellation occurs !

$\rightarrow$ semiclassical realization of IR renormalons

Non-perturbative Physics Without Instantons Dabrowski/GD 1306.0921

- 2d $O(N)$ & principal chiral model have no instantons !

- but: have non-BPS finite action solutions
- negative fluctuation modes

twisted b.c.s → fractionalize (Cherman et al, 1308.0127, 1403.1277; GD, Unsal, 1505.07803, Nitta et al, ...): saddles = bions in resurgent structure

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{g^2}S[A_{\text{saddle}}]} \times (\text{fluctuations}) \times (\text{qzm})$$

- Yang-Mills, $\mathbb{CP}^{N-1}$, $U(N)$ PCM, $O(N)$, $Gr(N,M)$, ... have 'unstable' finite action non-BPS saddles

- what do these mean physically ?

resurgence: ambiguous imaginary non-perturbative terms should cancel ambiguous imaginary terms from Borel summation of perturbation theory

open problem: non-BPS saddle classification/fluctuations

Resurgence and Localization

Mariño, 1104.0783; Aniceto, Russo, Schiappa, 1410.5834; Wang, Wang, Huang, 1409.4967; Grassi, Hatsuda, Mariño, 1410.7658, ...)

- certain protected quantities in especially symmetric QFTs can be reduced to matrix models ⇒ resurgent asymptotics
- 3d Chern-Simons on $\mathbb{S}^3$ → matrix model

$$Z_{CS}(N,g)=\frac{1}{\mathrm{vol}(U(N))}\int dM\,\exp\left[-\frac{1}{g}\,\mathrm{tr}\left(\frac{1}{2}\left(\ln M\right)^2\right)\right]$$

- ABJM: $\mathcal{N}=6$ SUSY CS, $G=U(N)_k\times U(N)_{-k}$

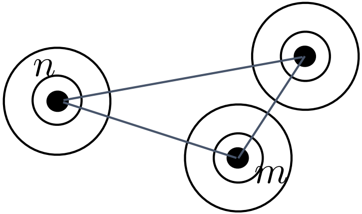
$$Z_{ABJM}(N,k)=\sum_{\sigma\in S_N}\frac{(-1)^{\epsilon(\sigma)}}{N!}\int\prod_{i=1}^{N}\frac{dx_i}{2\pi k}\,\frac{1}{\prod_{i=1}^{N}2\mathrm{ch}\left(\frac{x_i}{2}\right)\,\mathrm{ch}\left(\frac{x_i-x_{\sigma(i)}}{2k}\right)}$$

- $\mathcal{N}=4$ SUSY Yang-Mills on $\mathbb{S}^4$

$$Z_{SYM}(N,g^2)=\frac{1}{\mathrm{vol}(U(N))}\int dM\,\exp\left[-\frac{1}{g^2}\,\mathrm{tr}M^2\right]$$

The Bigger Picture: Decoding the Path Integral

what is the origin of resurgent behavior in QM and QFT ?



to what extent are (all?) multi-instanton effects encoded in perturbation theory? And if so, why?

- QM & QFT: basic property of all-orders steepest descents integrals

- Lefschetz thimbles: analytic continuation of path integrals

All-Orders Steepest Descents: Darboux Theorem

- all-orders steepest descents for contour integrals:

  <u>*hyperasymptotics*</u> (Berry/Howls 1991, Howls 1992)

$$I^{(n)}(g^2) = \int_{C_n} dz\, e^{-\frac{1}{g^2} f(z)} = \frac{1}{\sqrt{1/g^2}}\, e^{-\frac{1}{g^2} f_n}\, T^{(n)}(g^2)$$

- $T^{(n)}(g^2)$: beyond the usual Gaussian approximation
- asymptotic expansion of fluctuations about the saddle n:

$$T^{(n)}(g^2) \sim \sum_{r=0}^{\infty} T_r^{(n)}\, g^{2r}$$

All-Orders Steepest Descents: Darboux Theorem

- universal resurgent relation between different saddles:

$$T^{(n)}(g^2) = \frac{1}{2\pi\, i} \sum_m (-1)^{\gamma_{nm}} \int_0^\infty \frac{dv}{v} \frac{e^{-v}}{1 - g^2 v/(F_{nm})} T^{(m)}\left(\frac{F_{nm}}{v}\right)$$

- exact resurgent relation between fluctuations about n^{th} saddle and about neighboring saddles m

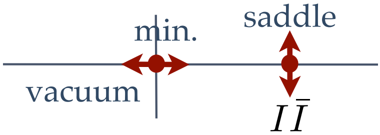
$$T_r^{(n)} = \frac{(r-1)!}{2\pi\, i} \sum_m \frac{(-1)^{\gamma_{nm}}}{(F_{nm})^r} \left[T_0^{(m)} + \frac{F_{nm}}{(r-1)} T_1^{(m)} + \frac{(F_{nm})^2}{(r-1)(r-2)} T_2^{(m)} + \dots \right.$$

- universal factorial divergence of fluctuations (Darboux)
- fluctuations about *neighboring* saddles explicitly related!

$d = 0$ partition function for periodic potential $V(z) = \sin^2(z)$

$$I(g^2) = \int_0^\pi dz\, e^{-\frac{1}{g^2} \sin^2(z)}$$

two saddle points: $z_0 = 0$ and $z_1 = \frac{\pi}{2}$.



All-Orders Steepest Descents: Darboux Theorem

- large order behavior about saddle z_0:

$$T_r^{(0)} = \frac{\Gamma\left(r+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma(r+1)}$$

$$\sim \frac{(r-1)!}{\sqrt{\pi}}\left(1 - \frac{\frac{1}{4}}{(r-1)} + \frac{\frac{9}{32}}{(r-1)(r-2)} - \frac{\frac{75}{128}}{(r-1)(r-2)(r-3)} + \dots\right)$$

- low order coefficients about saddle z_1:

$$T^{(1)}(g^2) \sim i\sqrt{\pi}\left(1 - \frac{1}{4}g^2 + \frac{9}{32}g^4 - \frac{75}{128}g^6 + \dots\right)$$

- fluctuations about the two saddles are explicitly related
- could something like this work for path integrals ?
- multi-dimensional case is already non-trivial and interesting

Picard/Lefschetz; Pham (1965); Delabaere/Howls (2002); ...

Resurgence in (Infinite Dim.) Path Integrals (GD, Ünsal, 1401.5202)

- periodic potential: $V(x) = \frac{1}{g^2} \sin^2(g\,x)$
- vacuum saddle point

$$c_n \sim n! \left(1 - \frac{5}{2} \cdot \frac{1}{n} - \frac{13}{8} \cdot \frac{1}{n(n-1)} - \dots \right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\, E \sim \pi\, e^{-2\frac{1}{2g^2}} \left(1 - \frac{5}{2} \cdot g^2 - \frac{13}{8} \cdot g^4 - \dots \right)$$

- double-well potential: $V(x) = x^2(1 - gx)^2$
- vacuum saddle point

$$c_n \sim 3^n n! \left(1 - \frac{53}{6} \cdot \frac{1}{3} \cdot \frac{1}{n} - \frac{1277}{72} \cdot \frac{1}{3^2} \cdot \frac{1}{n(n-1)} - \dots \right)$$

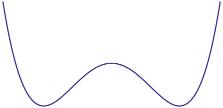
- instanton/anti-instanton saddle point:

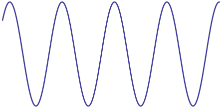
$$\mathrm{Im}\, E \sim \pi\, e^{-2\frac{1}{6g^2}} \left(1 - \frac{53}{6} \cdot g^2 - \frac{1277}{72} \cdot g^4 - \dots \right)$$

there is even more resurgent structure ...

Uniform WKB & Resurgent Trans-series (GD/MÜ:1306.4405, 1401.5202)

$$-\frac{\hbar^2}{2}\frac{d^2}{dx^2}\psi + V(x)\psi = E\,\psi$$





- weak coupling: degenerate harmonic classical vacua

$\Rightarrow$ uniform WKB: $\quad \psi(x) = \dfrac{D_\nu\left(\frac{1}{\sqrt{\hbar}}\varphi(x)\right)}{\sqrt{\varphi'(x)}}$

- non-perturbative effects: $\quad g^2 \leftrightarrow \hbar \quad \Rightarrow \quad \exp\left(-\frac{S}{\hbar}\right)$

- trans-series structure follows from exact quantization condition $\rightarrow E(N,\hbar) =$ trans-series

- Zinn-Justin, Voros, Pham, Delabaere, Aoki, Takei, Kawai, Koike, ...

Misumi, Nitta, Sakai, 2015

Connecting Perturbative and Non-Perturbative Sector

Zinn-Justin/Jentschura conjecture:
generate *entire trans-series* from just two functions:

(i) perturbative expansion $E = E_{\rm pert}(\hbar, N)$
(ii) single-instanton fluctuation function $\mathcal{P}(\hbar, N)$
(iii) rule connecting neighbouring vacua (parity, Bloch, ...)

$$E(\hbar, N) = E_{\rm pert}(\hbar, N) \pm \frac{\hbar}{\sqrt{2\pi}} \frac{1}{N!} \left(\frac{32}{\hbar}\right)^{N+\frac{1}{2}} e^{-S/\hbar}\, \mathcal{P}(\hbar, N) + \dots$$

in fact ... (GD, Ünsal, 1306.4405, 1401.5202) fluctuation factor:

$$\mathcal{P}(\hbar, N) = \frac{\partial E_{\rm pert}}{\partial N} \exp\left[S \int_0^{\hbar} \frac{d\hbar}{\hbar^3}\left(\frac{\partial E_{\rm pert}(\hbar, N)}{\partial N} - \hbar + \frac{\left(N+\frac{1}{2}\right)\hbar^2}{S}\right)\right]$$

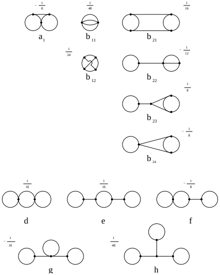
$\Rightarrow$ perturbation theory $E_{\rm pert}(\hbar, N)$ encodes everything !

Resurgence at work

- fluctuations about $\mathcal{I}$ (or $\bar{\mathcal{I}}$) saddle are determined by those about the vacuum saddle, to all fluctuation orders

- "QFT computation": 3-loop fluctuation about $\mathcal{I}$ for double-well and Sine-Gordon:

Escobar-Ruiz/Shuryak/Turbiner 1501.03993, 1505.05115



$$\text{DW}: \quad e^{-\frac{S_0}{\hbar}}\left[1-\frac{71}{72}\,\hbar-0.607535\,\hbar^2-\dots\right]$$

$$\text{resurgence}: e^{-\frac{S_0}{\hbar}}\left[1+\frac{1}{72}\,\hbar\left(-102N^2-174N-71\right)\right.$$

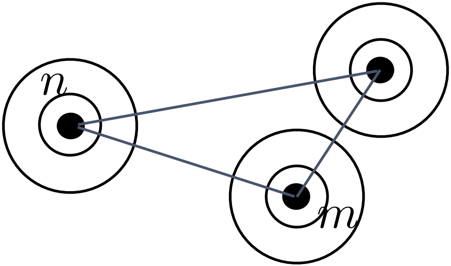
$$\left.+\frac{1}{10368}\hbar^2\left(10404N^4+17496N^3-2112N^2-14172N-6299\right)+\dots\right]$$

- known for all N and to essentially any loop order, directly from perturbation theory !

- diagramatically mysterious ...

Connecting Perturbative and Non-Perturbative Sector

all orders of multi-instanton trans-series are encoded in perturbation theory of fluctuations about perturbative vacuum



$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{g^2}S[A_{\text{saddle}}]} \times (\text{fluctuations}) \times (\text{qzm})$$

Analytic Continuation of Path Integrals: Lefschetz Thimbles

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{thimbles}\, k} \mathcal{N}_k\, e^{-\frac{i}{g^2}\, S_{\text{imag}}[A_k]} \int_{\Gamma_k} \mathcal{D}A\, e^{-\frac{1}{g^2}S_{\text{real}}[A]}$$

Lefschetz thimble = "functional steepest descents contour"

remaining path integral has real measure:
(i) Monte Carlo
(ii) semiclassical expansion
(iii) exact resurgent analysis

resurgence: asymptotic expansions about different saddles are closely related

requires a deeper understanding of complex configurations and analytic continuation of path integrals ... gradient flow

Stokes phenomenon: intersection numbers $\mathcal{N}_k$ can change with phase of parameters

Thimbles from Gradient Flow

gradient flow to generate steepest descent thimble:

$$\frac{\partial}{\partial \tau} A(x;\tau) = -\overline{\frac{\delta S}{\delta A(x;\tau)}}$$

- keeps $Im[S]$ constant, and $Re[S]$ is monotonic

$$\frac{\partial}{\partial \tau}\left(\frac{S-\bar{S}}{2i}\right) = -\frac{1}{2i}\int\left(\frac{\delta S}{\delta A}\frac{\partial A}{\partial \tau} - \overline{\frac{\delta S}{\delta A}}\,\overline{\frac{\partial A}{\partial \tau}}\right) = 0$$

$$\frac{\partial}{\partial \tau}\left(\frac{S+\bar{S}}{2}\right) = -\int\left|\frac{\delta S}{\delta A}\right|^2$$

- Chern-Simons theory (Witten 2001)
- complex Langevin (Aarts 2013; ... ; Hayata, Hidaka, Tanizaki, 2015)
- lattice (Cristoforetti et al 2013, 2014; Fujii, Honda et al, 2013; Nishimura et al, 2014; Mukherjee et al 2014; Fukushima et al, 2015; Kanazawa et al, 2015; ...)

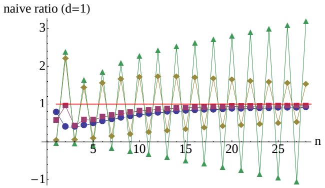
resurgence: asymptotics about different saddles related

Ghost Instantons: Analytic Continuation of Path Integrals

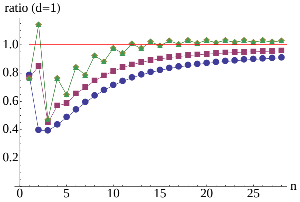
$$\mathcal{Z}(g^2|m) = \int \mathcal{D}x\, e^{-S[x]} = \int \mathcal{D}x\, e^{-\int d\tau \left(\frac{1}{4}\dot{x}^2 + \frac{1}{g^2}\,\mathrm{sd}^2(g\,x|m)\right)}$$

- doubly periodic potential: *real* & *complex* instantons

$$a_n(m) \sim -\frac{16}{\pi} n! \left(\frac{1}{(S_{\mathcal{I}\bar{\mathcal{I}}}(m))^{n+1}} - \frac{(-1)^{n+1}}{|S_{\mathcal{G}\bar{\mathcal{G}}}(m)|^{n+1}} \right)$$



without ghost instantons



with ghost instantons

- complex instantons directly affect perturbation theory, even though they are not in the original path integral measure

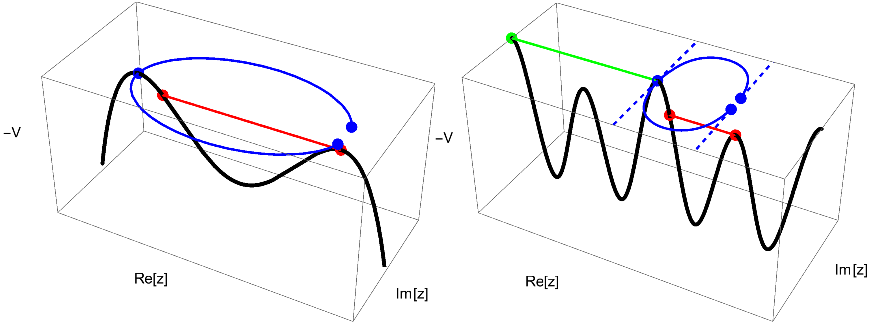
(Başar, GD, Ünsal, arXiv:1308.1108)

Necessity of Complex Saddles

(Behtash, GD, Schäfer, Sulejmanpasic, Ünsal (1510.00978), (1510.03435)

SUSY QM: $g\,\mathcal{L} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}(W')^2 \pm \frac{g}{2} W''$

- $W = \frac{1}{3}x^3 - x \rightarrow$ tilted double-well
- $W = \cos\frac{x}{2} \rightarrow$ double Sine-Gordon
- new (exact) complex saddles ($=$ neutral bions)



Necessity of Complex Saddles

(Behtash, GD, Schäfer, Sulejmanpasic, Ünsal (1510.00978), (1510.03435)

SUSY QM: $g\,\mathcal{L} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\left(W'\right)^2 \pm \frac{g}{2}W''$

- complex saddles have complex action:

$$S_{\text{complex bion}} \sim 2S_I + i\pi$$

- $W = \cos\frac{x}{2} \rightarrow$ double Sine-Gordon

$$E_{\text{ground state}} \sim 0 - 2\,e^{-2S_I} - 2\,e^{-i\pi}e^{-2S_I} = 0 \qquad \checkmark$$

- $W = \frac{1}{3}x^3 - x \rightarrow$ tilted double-well

$$E_{\text{ground state}} \sim 0 - 2\,e^{-i\pi}e^{-2S_I} > 0 \qquad \checkmark$$

semiclassics: complex saddles required for SUSY algebra

Connecting weak and strong coupling

important physics question:

does weak coupling analysis contain enough information to extrapolate to strong coupling ?

... even if the degrees of freedom re-organize themselves in a very non-trivial way?

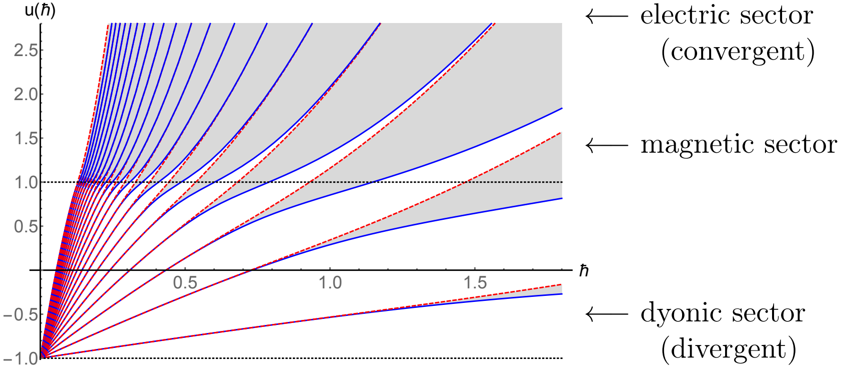
what about a QFT in which the vacuum re-arranges itself in a non-trivial manner?

classical (Poincaré) asymptotics is clearly not enough:

is resurgent asymptotics enough?

Resurgence in $\mathcal{N}=2$ and $\mathcal{N}=2^*$ Theories (Başar, GD, 1501.05671)

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2}+\cos(x)\,\psi=u\,\psi$$



- energy: $u=u(N,\hbar)$; 't Hooft coupling: $\lambda\equiv N\,\hbar$
- very different physics for $\lambda\gg 1$, $\lambda\sim 1$, $\lambda\ll 1$

- Mathieu, Lamé encode Nekrasov-Shatashvili superpotential

Resurgence of $\mathcal{N}=2$ SUSY SU(2)

- moduli parameter: $u = \langle \mathrm{tr}\, \Phi^2 \rangle$
- electric: $u \gg 1$; magnetic: $u \sim 1$; dyonic: $u \sim -1$
- $a = \langle \text{scalar} \rangle$, $a_D = \langle \text{dual scalar} \rangle$, $a_D = \frac{\partial \mathcal{W}}{\partial a}$
- Nekrasov-Shatashvili twisted superpotential $\mathcal{W}(a, \hbar, \Lambda)$:
- Mathieu equation:

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \Lambda^2 \cos(x)\,\psi = u\,\psi \quad , \quad a \equiv \frac{N\hbar}{2}$$

- Matone relation:

$$u(a, \hbar) = \frac{i\pi}{2}\Lambda \frac{\partial \mathcal{W}(a, \hbar, \Lambda)}{\partial \Lambda} - \frac{\hbar^2}{48}$$

Mathieu Equation Spectrum

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = u\,\psi$$

- small N: divergent, non-Borel-summable $\rightarrow$ trans-series

$$u(N,\hbar) \sim -1 + \hbar\left[N + \frac{1}{2}\right] - \frac{\hbar^2}{16}\left[\left(N + \frac{1}{2}\right)^2 + \frac{1}{4}\right] - \frac{\hbar^3}{16^2}\left[\left(N + \frac{1}{2}\right)^3 + \frac{3}{4}\left(N + \frac{1}{2}\right)\right] - \ldots$$

- large N: convergent expansion: $\longrightarrow$?? trans-series ??

$$u(N,\hbar) \sim \frac{\hbar^2}{8}\left(N^2 + \frac{1}{2(N^2-1)}\left(\frac{2}{\hbar}\right)^4 + \frac{5N^2+7}{32(N^2-1)^3(N^2-4)}\left(\frac{2}{\hbar}\right)^8 + \frac{9N^4+58N^2+29}{64(N^2-1)^5(N^2-4)(N^2-9)}\left(\frac{2}{\hbar}\right)^{12} + \ldots\right)$$

Resurgence of $\mathcal{N}=2$ SUSY SU(2) (Başar, GD, 1501.05671)

- $N\hbar \ll 1$, deep inside wells: resurgent trans-series

$$u^{(\pm)}(N,\hbar) \sim \sum_{n=0}^{\infty} c_n(N)\hbar^n \pm \frac{32}{\sqrt{\pi}\, N!}\left(\frac{32}{\hbar}\right)^{N-1/2} e^{-\frac{8}{\hbar}} \sum_{n=0}^{\infty} d_n(N)\hbar^n + \dots$$

- Borel poles at two-instanton location
- $N\hbar \gg 1$, far above barrier: convergent series

$$u^{(\pm)}(N,\hbar) = \frac{\hbar^2 N^2}{8}\sum_{n=0}^{N-1}\frac{\alpha_n(N)}{\hbar^{4n}} \pm \frac{\hbar^2}{8}\frac{\left(\frac{2}{\hbar}\right)^{2N}}{(2^{N-1}(N-1)!)^2}\sum_{n=0}^{N-1}\frac{\beta_n(N)}{\hbar^{4n}} + \dots$$

(Basar, GD, Ünsal, 2015)

- coefficients have poles at O(two-(complex)-instanton)
- $N\hbar \sim \frac{8}{\pi}$, near barrier top: "instanton condensation"

$$u^{(\pm)}(N,\hbar) \sim 1 \pm \frac{\pi}{16}\hbar + O(\hbar^2)$$

Conclusions

- Resurgence systematically unifies perturbative and non-perturbative analysis
- trans-series 'encode' all information, and expansions about different saddles are intimately related
- local analysis encodes more than one might think
- matrix models, large N, strings, SUSY QFT
- IR renormalon puzzle in asymptotically free QFT
- multi-instanton physics from perturbation theory
- $\mathcal{N} = 2$ and $\mathcal{N} = 2^*$ SUSY gauge theory
- hydrodynamical equations
- fundamental property of steepest descents expansion
- analytic continuation for path integrals