Bions, Confinement and Resurgence

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Introduction
Gauge theory on compactified spacetime

QCD on $\mathbb{R}^4$

$\downarrow$

QCD on $\mathbb{R}^3 \times S^1$ w/ compact length $L$

Theory can be analytically studied at $L \ll 1/\Lambda_{\text{QCD}}$. Is it related to strong-coupling physics for $L \to \infty$.
QCD(adj.) on $\mathbb{R}^3 \times S^1$

Fermion BC: $\lambda(\vec{x}, x_4 + L) = e^{i\phi} \lambda(\vec{x}, x_4)$ \hspace{1cm} $\phi=\pi$ (ABC) $\rightarrow$ Finite temperature 
$\phi=0$ (PBC) $\rightarrow$ spatial compact

I. QCD(adj.) for $T \neq 0$ \hspace{1cm} ($\phi=\pi$)
- Exact center symmetry
- Confining phase transition

II. Hosotani mechanism + Bion mechanism \hspace{1cm} ($\phi=0$)
- Mass gap based on topological objects
- Resurgent expansion for IR-renormalon problem

(consider Dirac adjoint fermions only)
I. QCD(adj.) for finite $T$ ($\phi = \pi$)

- Exact $Z_N$ center symmetry
- Polyakov loop = order parameter
- $Z_N$ center phase transition (deconfining transition)

$\langle |P| \rangle \sim \exp[-F_q/\beta]$ quark free energy

$\langle |P| \rangle = 0 \rightarrow F_q = \infty$ confined

$\langle |P| \rangle \neq 0 \rightarrow F_q \neq \infty$ deconfined
II. Hosotani mechanism for QCD(adj) \( (\phi = 0) \)

Polyakov-loop holonomy in \( S^1 \) affects physics

\[ W = P \exp \left\{ ig \int_C dy A_y \right\} \]

1. Polyakov loop = \( \text{diag}[e^{2\pi i q_1}, e^{2\pi i q_2}, \ldots, e^{2\pi i q_N}] \)
2. Invariant under gauge transformation keeping B.C.
3. Not gauged away, and contributes to physics

Adjoint Higgs mechanism \( q_i \neq q_j \)

KK Spectrum \( m_n^2 = \frac{1}{L^2} (n + q_i - q_j)^2 \rightarrow \) massive gauge boson

\( cf. \) \( q_1 \neq q_2 \) in SU(2) \( \rightarrow \) SU(2)→U(1) \( m_{W^\pm} \sim 1/L \)
Phase diagram for center and gauge symmetry ($\phi = 0$)

- $L \ll 1/\Lambda_{QCD}$: $SU(N) \rightarrow U(1)^{N-1}$ (Z$_N$ center) 
  $\langle A_4 \rangle = (0, 2\pi/N, \ldots, 2(N-1)\pi/N)$ 
  $\rightarrow \langle TrP \rangle = 0$

- $L \sim 1/\Lambda_{QCD}$: $SU(N) \rightarrow SU(M) \times \ldots$ (Z$_N$ broken) 
  $\rightarrow \langle TrP \rangle \neq 0$

Pure Yang-Mills

$T_c = 1/L_c$

Schematic phase diagram

- Cossu, D'Elia (08)
- Cossu, Hatanaka, Hosotani, Noaki (13)
- Kashiwa, TM (13)
Topological objects in $U(1)^{N-1}$ center-symmetric phase

BPS solutions: $1/N$-fractional instantons ($Q=1/N$)

§ Polyakov loop

$$P = \text{diag}[1, e^{2\pi i/N}, e^{4\pi i/N}, \ldots, e^{2\pi (N-1)i/N}] \quad (P^N = 1)$$

- equivalent to imposing $\mathbb{Z}_N$ twisted B.C.
- Elementary BPS solutions map to $1/N$ of target space

cf.) SU(2) case ($\mathbb{Z}_2$ case)

half of target space
**Topological objects in $U(1)^{N-1}$ center-symmetric phase**

BPS solutions: $1/N$-fractional instantons ($Q=1/N$)

cf.) SU(2) QCD(adj.)

\[ \pi_2(SU(2)/U(1)) = \mathbb{Z} \rightarrow \text{Monopoles} \]

\[ \begin{align*}
Q=1 & \quad \text{SU}(2) \\
Q=1/2 & \quad \text{SU}(2) \\
Q=1/2 & \quad \text{SU}(3) \\
Q=1/3 & \quad \text{SU}(3)
\end{align*} \]

Lee, Yi(97)
Lee, Lu(97)
Kraan, van Baal(97)
Eto, et.al. (04)
Bruckmann (07)
**Types of fractional instantons**

\[
\left( \int F, \int F \tilde{F} \right) = \text{(magnetic charge, instanton charge)}
\]

cf.) SU(2) QCD adj.

- BPS: \((1, 1/2)\)
- KK: \((-1, 1/2)\)

- BPS: \((-1, -1/2)\)
- KK: \((1, -1/2)\)

**Nye-Singer index theorem:** 2Nf zero modes to each monopole

\[
I_{\text{adj}}^{(0)}[n_1, \ldots, n_N] = 2Nn_N - \sum_{i, j=1}^{N} \frac{q_i - q_j}{2\pi} \left\{(n_i - n_{i-1}) - (n_j - n_{j-1})\right\}
\]
**Types of fractional instantons**

\[
\left( \int F, \int F \tilde{F} \right) = \text{ (magnetic charge, instanton charge) }
\]

cf.) SU(2) QCD adj.

\[
\begin{array}{c}
\begin{array}{c}
\text{BPS} \\
(1, 1/2)
\end{array} \\
\begin{array}{c}
\text{BPS} \\
(-1, -1/2)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{KK} \\
(-1, 1/2)
\end{array} \\
\begin{array}{c}
\text{KK} \\
(1, -1/2)
\end{array}
\end{array}
\]

\[\text{Q}=0 \text{ combinations play important roles!}\]
**Neutral bions as renormalons**

\[
\left( \int F, \int F \tilde{F} \right) = \text{(magnetic charge, instanton charge)}
\]

cf.)\( SU(2) \) QCD adj.

\[
\begin{array}{c}
\text{BPS} \\
(1, 1/2) \\
\end{array}
\quad
\begin{array}{c}
\text{BPS} \\
(-1, -1/2) \\
\end{array}
\quad
\begin{array}{c}
\text{KK} \\
(-1, 1/2) \\
\end{array}
\quad
\begin{array}{c}
\text{KK} \\
(1, -1/2) \\
\end{array}
\]

\[ \text{Im} [B_{ii}]_{\theta=0\pm} \sim e^{-2n S_1/N} \]

→ cancellation of "IR-renormalon ambiguity"
**Confinement via magnetic bions**

\[
\left( \int F, \int F \tilde{F} \right) = \text{(magnetic charge, instanton charge)}
\]

cf.) SU(2) QCD adj.

Zero-mode exchange

\[ V_{\text{eff}}(r) = \frac{4\pi L}{g^2 r} + 4N_f^W \log \frac{r}{L} \]

repulsive \hspace{1cm} attractive

``Magnetic Bions``
**Bion-induced confinement mechanism** \( (\phi = 0) \)  

Magnetic bions

Confinement \( (L \ll 1/\Lambda) \)

- Similar to 3D Polyakov model (Yang-Mills + Higgs + adj quarks)

Similarities : Higgs SU(2)→U(1), Index theorem
Differences : Chiral anomaly \( U(1)_A \rightarrow \mathbb{Z}_{4N_f} \)

Tools : Abelian duality + Symmetry
**Bion-induced confinement mechanism \( (\phi = 0) \)**

Magnetic bions

Confinement \((L \ll 1/\Lambda)\)

\[
L_{\text{QCD}}^{d} = \frac{1}{2}(\partial \sigma)^2 - b e^{-S_{0}} \cos 2\sigma + i \bar{\psi}^{I} \gamma_{\mu} \partial_{\mu} \psi^{I} + c \ e^{-S_{0}} \cos \sigma (\det \psi^{I} \psi^{J} + \text{c.c.})
\]

\begin{align*}
\text{bion contributions} & \quad \text{monopole contributions} \\
& \quad \text{Dual photon mass} \rightarrow \text{Chromoelectric flux tube} \rightarrow \text{Confinement}
\end{align*}

\[
m_{D} \sim \Lambda (\Lambda L)^{b_{0}-1} = \Lambda (\Lambda L)^{8-2n_{f}}/3
\]

Mass gap for dual photon
Analogue of type-II Superconductor

Type-II superconductor

• Cooper pair condensates
• Magnetic field is confined

Bion-induced confinement

• Monopole pair condensates
• Chromoelectric field is confined

• generalization of dual SC \( \text{monopole} \rightarrow \text{pair condensate} \)
• No specific gauge, but SU(2) broken to U(1)
$\phi=0$ : Periodic b.c. phase diagram

$T_c = 1/L_c$

Confined (SC)

Deconfinement

Confined (WC)

$N_f > 1$
Two confined phases are smoothly connected or intervened? (No order parameter to tell from two confining phases)
How bion confining phase transforms

\( \phi = 0 \) : Periodic b.c.

\( \phi = \pi \) : Anti-Periodic b.c.

Hosotani-Unsal regime

Finite-temperature
How bion confining phase transforms

\( \phi = 0 : \) Periodic b.c.

\( \phi = \pi : \) Anti-Periodic b.c.

Hosotani-Unsal regime

Finite-temperature
Continuity of center phases exists?
Topics related to QCD(adj) on $\mathbb{R}^3 \times S^1$
Table of contents

1. Confinement & Phase diagram
   - Phase structure for $0<\Phi<\pi$
   - Bion confinement for $0<\Phi<\pi$

2. Resurgence & Neutral bions
   - CpN & Grassmann model
   - Cancellation of ambiguity

3. Summary and future works
I. Phase diagram & confinement
One-loop-based effective potential

variables: \( \langle A_4 \rangle = \frac{1}{L} \text{diag}(q_1, q_2, \ldots, q_{N_c}) \) with \( \sum_{k=1}^{N_c} q_k = 0 \)

Gauge: \( V_g^{np}(N_c, M, L; \{q\}) = -\frac{2}{\pi^2 L^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \left(1 - \frac{M^2 L^2}{4n^2}\right) \sum_{i,j=1}^{N_c} \left(1 - \frac{1}{N_c} \delta_{ij}\right) \cos(nq_{ij}) \)

QCD scale: \( M \sim \Lambda_{QCD} \)

Adj quark: \( V_{adj}(N_c, N_f^D, L, m, \phi; \{q\}) \)

\[
= \frac{2N_f^D m^2}{\pi^2 L^2} \sum_{n=1}^{\infty} \frac{K_2(nLm)}{n^2} \sum_{i,j=1}^{N_c} \left(1 - \frac{1}{N_c} \delta_{ij}\right) \cos(n(q_{ij} + \phi))
\]

For \( 1/L \gg \Lambda \), the potential reduces to one-loop one.
SU(2) for $m=0$, $1/L \gg M$

\[
\langle P_F \rangle = \frac{1}{2} \text{Tr} \begin{pmatrix} e^{\pm i\pi/2} & 0 \\ 0 & e^{\mp i\pi/2} \end{pmatrix} = 0
\]

\[
\langle P_F \rangle = \frac{1}{2} \text{Tr} \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} = \pm 1
\]

center symmetric \hspace{2cm} \text{center broken}
SU(3) for m=0, 1/L≫M
SU(2) for $N_f^D=1$

$\phi = 0$

$m \over M$

$1 / LM$

Hosotani-Unsal regime

Red asterisk for 1st-order
Blue triangle for 2nd-order

cf.)

T_c=1/L_c
SU(2) Deconfined
SU(2) Confined
U(1) Confined

$0 \over 1/L \over \infty$
be probed by Instantons further break $U(1)$ $U(1)$ in QCD(adj) on twisted boundary conditions using a high-temperature expansion. In section appropriate model building in section the possible patterns of symmetry breaking. The insights obtained here are used for an impact of chiral symmetry breaking.

In this section we will incorporate the $e$ $3$ figure 6

\begin{align*}
\langle \lambda \rangle & = \frac{1}{2} \\
\langle \det N \rangle & = 1 \\
\text{Tr} & = 2
\end{align*}

\begin{align*}
\lambda & = 0 \\
\phi & = 0.2 \pi
\end{align*}

Phase diagram for $SU(N)$ with $R$ $D$ $\triangle$ $\lambda$ $N$ $\langle$ $D$ $\rangle$ $\langle$ $0$ $f$ $D$ $0$ $\rangle$ $= 2$ and $\langle$ $\pi$ $\rangle$ $f$ $D$ $\pi$. Where we analyze an NJL-type model with $\frac{1}{\text{uniE000}}$. $JHEP06(2014)181$
In QCD(adj) on twisted boundary conditions using a high-temperature expansion. In section 3.1 Chiral symmetry in QCD(adj) model is solved numerically and its phase diagram is presented. In this section we will incorporate the easterisks (\*\*\*). The spontaneous breaking of continuous and discrete chiral symmetries can be probed by instantons further break $U(1)\times U(1)\times U(1)$. Phase diagram for $SU(N)\times SU(N)\times SU(N)$ with $f_D = 2$ and $f_W = 32$, respectively, where $g$ is larger than the standard symmetry $U(1)$. In the chiral limit, the flavor symmetry of adjoint fermions is given by $SU(N)\times SU(N)\times SU(N)$.

\[ \langle A \rangle \text{ with } f_L \] denotes a second-order phase transition, and the red line with triangles (\triangledown) denotes a first-order phase transition. The PNJL model is used to study the interplay of center and chiral symmetry on the phase diagram of QCD(adj).

\[ \phi = 0.32\pi \]
In QCD(adj) on 3.1 Chiral symmetry in QCD(adj) model is solved numerically and its phase diagram is presented. The spontaneous breaking of continuous and discrete chiral symmetries can be probed by theory \[ U(1) \times U(1) \]. The blue line with triangles (\( \phi = 0.4\pi \)) denotes a first-order phase transition, and the red line with asterisks (\( \phi \in \{0, \pi\} \)) denotes a second-order phase transition, and the red line with crosses denotes a transition down to \( \phi = 1 \) with \( f_\pi, g_\pi \). In the chiral limit, the flavor symmetry of adjoint fermions is given by SU(\( f \times g \)), respectively, where

\[
\langle D \rangle \rightarrow \langle A \rangle \text{ owing to the reality of the adjoint representation}.
\]

In section 3.2 we study the interplay of center and chiral symmetry on the phase diagram of QCD(adj). In section 3.3 the possible patterns of symmetry breaking. The insights obtained here are used for an appropriate model building in section 2.1, where we analyze an NJL-type model with explicit, for the case of SU(\( f \times g \)).

\[\text{Figure 6}\]
be probed by theory. Instantons further break $U(1)$ down to $U(1)_{Z2}$. In QCD(adj) on $\mathbb{R}^4$, this symmetry breaking is explicitly given by $SU(N) \times SU(N)$, which is larger than the standard symmetry $U(1)_{Z2}$. In this section, we will incorporate the effects of dynamical chiral symmetry breaking to $SU(N)_{W}$ and $SU(N)_{D}$, respectively, where we analyze an NJL-type model with appropriate model building in section 2.1.

In section 3.2, where we analyze an NJL-type model with explicit, for the case of $SU(N)_{W}$, we review global symmetries of QCD(adj) and comment on restrictions on the possible patterns of symmetry breaking. The insights obtained here are used for an analysis of the PNJL model in section 3.3.

The blue line with triangles denotes a second-order phase transition, and the red line with asterisks denotes a first-order phase transition. In the chiral limit, the flavor symmetry of adjoint fermions is given by $SU(\mathbb{C}^N)$. Phase diagram for for $SU(N)_{W}$

The spontaneous breaking of continuous and discrete chiral symmetries can be expressed as $\langle \lambda \rangle$, $\langle \phi \rangle$, and $\langle \phi \rangle$, respectively, where $\phi = 0.6\pi$. The PNJL model is solved numerically and its phase diagram is presented.
3.1 Chiral symmetry in QCD(adj) model is solved numerically and its phase diagram is presented. In section appropriate model building in section the possible patterns of symmetry breaking. The insights obtained here are used for an study the interplay of center and chiral symmetry on the phase diagram of QCD(adj). In this section we will incorporate the e 3 Impact of chiral symmetry breaking approximation schemes.

Figure 6

We only consider even $\Lambda \equiv (\mathcal{N} \times \mathcal{N} \times \mathcal{N} \times \mathcal{N})_{\text{uni}}$.

Phase diagram for for $\mathcal{N} \times \mathcal{N} \times \mathcal{N} \times \mathcal{N}$ we review global symmetries of QCD(adj) and comment on restrictions on $f_D \equiv \langle \mathcal{D} \rangle$.

In the chiral limit, the flavor symmetry of adjoint fermions is given by $\mathcal{F} = \mathcal{G}_f$ explicitly, for the case of $\mathcal{G}_f = \mathcal{SU}(2)$.

Deconfined $\mathcal{SU}(2)$

Finite-$T$
Center phase diagram for nonzero b.c. twist

For large $m$ it results in finite-temperature PYM.

Exact results for large $1/L$ while model results for small $1/L$.

Hosotani-Unsal regime

Finite-T Pure Yang-Mills

Finite-T

$1/LM$

$T_c = 1/L_c$

SU(2) Confined

SU(2) Deconfined

U(1) Confined

$\phi/\pi$
Center phase diagram for nonzero b.c. twist

Figure 3: 3D phase diagram for $1/L$, $m_{\text{adj}}$, $\phi$. The black points are the tricritical points (line).
Phase diagram for center and chiral symmetries \((N_c=2, \ N_f^D=1)\)

- PNJL-type model reflecting the symmetry

\[
\begin{align*}
&= \text{Tr}[\bar{\Psi}D_{\text{adj}}\Psi] + \frac{G}{2} \left[ (\bar{\Psi}\Psi)^2 + |\bar{\Psi}^C\gamma_5\Psi|^2 \right]
\end{align*}
\]

zero bare mass \((m=0)\)

- Red for center transition lines
- Blue for chiral transition lines
- \(\ast\) for 1st-order
- \(\triangle\) for 2nd-order

variables: \(\langle A_4 \rangle = \frac{1}{L} \text{diag}(q_1, q_2, \ldots, q_{N_c})\)

\(m_{\text{dyn}} \equiv -G\langle \text{Tr} \bar{\Psi}\Psi \rangle\)
Continuity depends on model parameter choices

I. \( G\Lambda_{UV}^2 = 6.8 \) and \( \Lambda_{UV} = 20M \) \( \rightarrow \) \( T_\gamma/T_d \sim 5, \ m_{\text{dyn}} = 2.4M \)

II. \( G\Lambda_{UV}^2 = 6.8 \) and \( \Lambda_{UV} = 15M \) \( \rightarrow \) \( T_\gamma/T_d \sim 4.5, \ m_{\text{dyn}} = 1.8M \)

- Continuity realizes when dynamical mass \( m_{\text{dyn}} < \sim 2M \).
- Continuity is sensitive to ratio of dynamical mass v.s. QCD scale.
Continuity depends on ratio of dynamical mass to QCD scale.
How bion confining phase transforms

\[ \phi = 0 : \text{Periodic b.c.} \]

\[ \phi = \pi : \text{Anti-Periodic b.c.} \]

Hosotani-Unsal regime

Finite-temperature
Magnetic Bion amplitude for $U(1)$ phase for $\phi \neq 0$

1. $\phi \neq 0$ works as $m_\psi = \phi / L$ in quark propagator

$$S_F(\vec{r}) \equiv \int \frac{d^3 p}{(2\pi)^3} \frac{e^{i \vec{p} \cdot \vec{r}}}{\sigma_i p_i + im_\psi} \bigg|_{m_\psi = \phi / L} = \frac{i}{4\pi} e^{-m_\psi r} \left\{ \left( \frac{1}{r^3} + \frac{m_\psi}{r^2} \right) \sigma_i x_i - \frac{m_\psi}{r} 1 \right\}$$

The real mass produces exponential fall-off factor.

2. Index theorem is slightly deformed

$$I^{(\phi)}[n_1, \ldots, n_N] = 2N n_N - \sum_{i, j=1}^N \frac{q_i - q_j + \phi}{2\pi} \left( n_i - n_{i-1} - (n_j - n_{j-1}) \right)$$

Every fractional instanton carries $2Nf$ zero modes in center symmetric vacua ($q = \pi / 2$).
Comparison between $\phi=0$ and $\phi>0$

$$Z_{\text{bion}} \sim \frac{1}{g^8} \exp \left( -\frac{2}{N} \frac{8\pi^2}{g^2} (1 + cg) \right) \int_{r_{\text{min}}}^{\infty} dr \ r^2 \ \exp(-V_{\text{eff}}(r)) \quad g \ll 1 \quad \text{for} \quad L \ll 1/\Lambda_{\text{QCD}}$$

Magnetic bion size

- $\phi = 0$ \quad $V_{\text{eff}}(r) = \frac{4\pi L}{g^2 r} + 4N_f^W \log r/L \quad \Rightarrow \quad r_b \sim \frac{L}{g^2}$
- $\phi > 0$ \quad $V_{\text{eff}}(r) = \frac{4\pi L}{g^2 r} + 2N_f^W m_\psi r$ \quad \text{linear potential} \quad $r_b \sim \frac{L}{g\sqrt{\phi}}$

Magnetic bion amplitude & Mass gap

- $\phi > 0$ \quad $Z_{\text{bion}} \sim \frac{1}{\phi^{7/4} g^{21/2}} \exp \left( -\frac{2}{N} \frac{8\pi^2}{g^2} - \frac{4\sqrt{2\pi N_f^W \phi}}{g} \right)$

$$\mathcal{M} \sim \left( \log \frac{1}{L\Lambda} \right)^{65/8} \exp \left( -\left(2\pi + \sqrt{\frac{8\phi}{\pi}}\right) \sqrt{\log \frac{1}{L\Lambda}} \right)$$

Bion-induced confinement exists but is suppressed at $\phi \neq 0$. 
Further investigation including lattice study required!
2. Resurgence and Neutral bions
**Neutral bions as renormalons**

\[
\left( \int F, \int F\tilde{F} \right) = \text{(magnetic charge, instanton charge)}
\]

cf.) SU(2) QCD adj.

![Diagram showing BPS and KK states with their charges and their relationships](image)

**Perturbative vacuum = Neutral bions**

Its amplitude leads to imaginary ambiguity

\[\rightarrow \text{cancellation of } \text{``renormalon ambiguity''}\]
**Renormalon ambiguity in Borel resummation**  

\[
P(g^2) = \sum_{q=0}^{\infty} a_q g^{2q} \quad \text{(Perturbation = Divergent series)}
\]

\[BP(t) := \sum_{q=0}^{\infty} \frac{a_q t^q}{q!} \quad \text{(Singualarities in Borel plane)}\]

\[
\mathbb{B}(g^2) = \frac{1}{g^2} \int_0^\infty BP(t) e^{-t/g^2} dt \quad \text{(Imaginary ambiguities)}
\]

\[
\mathbb{B}_0(|g^2| \pm i\epsilon) = \text{Re} \mathbb{B}_0(|g^2|) \pm i \text{Im} \mathbb{B}_0(|g^2|)
\]

\[
\text{Im} \mathbb{B}_0(|g^2|) \sim e^{-2n S_1} \quad \& \quad e^{-2n S_1/N}
\]

1. Instanton ambiguity  
2. IR-renormalon ambiguity
Renormalon ambiguity in Borel resummation

\[ \text{CP}(N-1) \text{ on } \mathbb{R}^1 \times S^1 \]

UV renormalons: \( t = -8\pi n/\beta_0 \)

Instanton—anti-instanton

singularity: \( t = 8\pi, 16\pi, ... \)

Neutral bions:

\( t = 8\pi n/N \) \( (n=1,2,...) \)

1. Instanton ambiguity

\[ [\mathcal{I}\mathcal{I}]_{\theta=0}^{\pm} = \Re [\mathcal{I}\mathcal{I}] \pm i \Im [\mathcal{I}\mathcal{I}]_{\theta=0}^{\pm} \]

\[ \Im [\mathcal{I}\mathcal{I}]_{\theta=0}^{\pm} \sim e^{-2nS_I} \]

cancellation!
**Renormalon ambiguity in Borel resummation**  

\[ \text{CP}(N-1) \text{ on } \mathbb{R}^1 \times S^1 \]

- **UV renormalons:** \( t = \frac{-8\pi n}{\beta_0} \)
- **Instanton—anti-instanton singularities:** \( t = 8\pi, 16\pi, \ldots \)
- **Neutral bions:** \( t = \frac{8\pi n}{N} \quad (n=1,2,\ldots) \)

1. **Instanton ambiguity**

\[ [\mathcal{I} \bar{\mathcal{I}}]_{\theta=0}^\pm = \text{Re} [\mathcal{I} \bar{\mathcal{I}}] \pm i \text{Im} [\mathcal{I} \bar{\mathcal{I}}]_{\theta=0}^\pm \]

\[ \text{Im} [\mathcal{I} \bar{\mathcal{I}}]_{\theta=0}^\pm \sim e^{-2nS_I} \]

Cancellation!

2. **IR-renormalon ambiguity**

.....expected to reflect IR physics, but, its identity not yet revealed.
1. Instanton ambiguity

\[ [\mathcal{I}\overline{I}]_{\theta=0}^{\pm} = \text{Re} [\mathcal{I}\overline{I}] \pm i \text{Im} [\mathcal{I}\overline{I}]_{\theta=0}^{\pm} \]

\[ \text{Im} [\mathcal{I}\overline{I}]_{\theta=0}^{\pm} \sim e^{-2nS_I} \]

cancellation!

2. IR-renormalon ambiguity

neutral-bion amplitude???

\[ [\mathcal{B}_{ii}]_{\theta=0}^{\pm} = \text{Re} [\mathcal{B}_{ii}] \pm i \text{Im} [\mathcal{B}_{ii}]_{\theta=0}^{\pm} \]

\[ \text{Im} [\mathcal{B}_{ii}]_{\theta=0}^{\pm} \sim e^{-2nS_I/N} \]

cancellation!!

Sum of expansions around all critical points (Resurgence) \( \rightarrow \) Physical results?
Questions

• Is there an explicit ansatz of bions? (beyond dilute gas approx.)

• If exists, is it a solution of EOM?

• Cancellation of IR renormalons occurs?
Examples in $\mathbb{C}p^{N-1}$ models
◆ Notation in $\mathbb{C}p^N$-1 model

\[
S = \frac{1}{g^2} \int d^2x (D_\mu n)^\dagger (D_\mu n), \quad n(x) = \omega(x)/|\omega(x)|
\]

\[
Q = \int d^2x \ i\epsilon_{\mu\nu} (D_\mu n)^\dagger (D_\nu n)
\]

$A_\mu(x) \equiv -in^\dagger \partial_\mu n$

- Spatial compactification $S^1$

\[-\infty \leq x_1 \leq \infty \quad \text{and} \quad 0 \leq x_2 \leq L\]

- $\mathbb{Z}_N$ twisted b.c. in $S^1$ direction

\[
\omega(x_1, x_2 + L) = \Omega \omega(x_1, x_2), \quad \Omega = \text{diag.} \left[1, e^{2\pi i/N}, e^{4\pi i/N}, \ldots, e^{2(N-1)\pi i/N}\right]
\]
Renormalon ambiguity in \( \text{Cp}^N\text{-I} \) on \( \mathbb{R}^1 \times S^1 \)

- Lowest KK-mode Hamiltonian for small \( L \)

\[
H^\text{zero}_{\alpha_k} = \frac{g^2}{2} P^2_\theta + \frac{\xi^2}{2g^2} \sin^2 \theta + \frac{g^2}{2 \sin^2 \theta} P^2_\phi, \quad \xi = \frac{2\pi}{N},
\]

\[\rightarrow \text{Full expression of perturb. expansion is known}\]

\[
\mathcal{E}(g^2) \equiv E_0 \xi^{-1} = \sum_{q=0}^{\infty} a_q (g^2)^q, \quad a_q \sim -\frac{2}{\pi} \left( \frac{1}{4\xi} \right)^q q! \left( 1 - \frac{5}{2q} + O(q^{-2}) \right)
\]

\[
B \mathcal{E}(t) = -\frac{2}{\pi} \sum_{q=0}^{\infty} \left( \frac{t}{4\xi} \right)^q = -\frac{2}{\pi} \frac{1}{1 - \frac{t}{4\xi}}
\]

(Borel resummation)

\[
S_{0\pm} \mathcal{E}(g^2) = \frac{1}{g^2} \int_{C_\pm} dt \, B \mathcal{E}(t) \, e^{-t/g^2} = \text{Re} \mathcal{B}_0 \mp i \frac{16\pi}{g^2N} e^{-\frac{8\pi}{g^2N}}
\]

Leading IR-renormalon ambiguity in \( \text{Cp}^N\text{-I} \) on \( \mathbb{R}^1 \times S^1 \)
Properties of neutral bions

- BPS and anti-BPS fractional instantons locally
- Attractive interaction
- Solution in large separation limit
**Ansatz for neutral bions in $\mathbb{C}p^{N-1}$ model**

cf.) $\mathbb{C}p^{I}$  \[ \omega = \left( 1 + \lambda_2 e^{i\theta_2} e^{\pi (z + \bar{z})}, \lambda_1 e^{i\theta_1} e^{\pi z} \right)^T \]

\[ z = x_1 + ix_2 \]
Ansatz for neutral bions in $C_p^{N-1}$ model

$$\omega = \begin{pmatrix} 1 + \lambda_2 e^{i\theta_2} e^{\pi(z + \bar{z})} & \lambda_1 e^{i\theta_1} e^{\pi z} \end{pmatrix}^T$$

$$z = x_1 + ix_2$$

Separated fractional instanton and anti-instanton
(approximate solution at infinite separation)
Ansatz for neutral bions in \( \text{Cp}^\text{N-1} \) model

\[
\omega = \left( 1 + \lambda_2 e^{i\theta_2} e^{\pi(z + \bar{z})}, \lambda_1 e^{i\theta_1} e^{\pi z} \right)^T
\]

\( z = x_1 + i x_2 \)

\( S = 1 \) to \( S = 0 \)

Attractive interaction between instantons
**Ansatz for neutral bions in Cp^N-1 model**

cf.) Cp^N-1  \( \omega = (1 + \lambda_2 e^{i\theta_2} e^{\pi(z + \bar{z})}, \lambda_1 e^{i\theta_1} e^{\pi z})^T \)

\( z = x_1 + ix_2 \)

\[ S(\tau) \]

\( N = 2 \)

Attractive interaction between instantons
**Interaction energy of neutral bions**

\[ S_{\text{int}} = S_{\text{total}} - (S_{\nu=1/N} + S_{\nu=-1/N}) \]

\[ S_{\text{int}}(N, \tau) \sim -\frac{4}{N} e^{-\xi \tau}, \quad \xi = \frac{2\pi}{N} \]

Valid approximation for a broad range of separation
**Ambiguity from the interaction energy**

Neutral bion amplitude: 

\[ B_{ij} \propto -e^{-2S_I/N} \int_0^\infty d\tau e^{-V_{ij}^{\text{eff}}(\tau)} \]

\[ V_{ij}^{\text{eff}}(\tau) = S_{\text{int}}(\tau) + 2N_f\xi\tau \]

- Amplitude gets divergent due to attractive interaction
- Semiclassical dilute-gas description is broken down

<table>
<thead>
<tr>
<th>BZJ prescription</th>
<th>( g^2 \rightarrow -g^2 )</th>
<th>Bogomolny(79) Zinn-Justin(81)</th>
</tr>
</thead>
</table>

\[ \tilde{B}_{ii}(-g^2, N_f) \propto (-g^2N/8\pi)^{2N_f} \Gamma(2N_f)e^{-2S_I/N} \]

\[ B_{ii}(g^2, 0) = \left( \log \left( \frac{g^2N}{8\pi} \right) - \gamma \right) \frac{16}{g^2N} e^{-2S_I/N} \]

\[ \pm i \frac{16\pi}{g^2N} e^{-\frac{8\pi}{g^2N}} \]
Renormalon ambiguity in $\mathbb{C}P^{N-1}$ on $R^1 \times S^1$

- Lowest KK-mode Hamiltonian for small $L$

$$H_{\alpha_k}^{\text{zero}} = \frac{q^2}{2} P^2_{\theta} + \frac{\xi^2}{2g^2} \sin^2 \theta + \frac{g^2}{2 \sin^2 \theta} P^2_{\phi}, \quad \xi = \frac{2\pi}{N},$$

→ Full expression of perturbative expansion is known

$$\mathcal{E}(g^2) \equiv E_0 \xi^{-1} = \sum_{q=0}^{\infty} a_q(g^2)^q, \quad a_q \sim -\frac{2}{\pi} \left( \frac{1}{4\xi} \right)^q q! \left( 1 - \frac{5}{2q} + O(q^{-2}) \right)$$

$$B\mathcal{E}(t) = -\frac{2}{\pi} \sum_{q=0}^{\infty} \left( \frac{t}{4\xi} \right)^q = -\frac{2}{\pi} \frac{1}{1 - \frac{t}{4\xi}} \quad \text{(Borel resummation)}$$

$$S_{0^\pm}\mathcal{E}(g^2) = \frac{1}{g^2} \int_{C_{\pm}} dt \ B\mathcal{E}(t) \ e^{-t/g^2} = \Re \mathbb{B}_0 \mp i \frac{16\pi}{g^2 N} e^{-\frac{8\pi}{g^2 N}} \quad \text{Leading IR-renormalon ambiguity in } \mathbb{C}P^{N-1}$
Ambiguity from the interaction energy

Neutral bion amplitude: \( B_{ij} \propto e^{-2S_I/N} \int_0^\infty d\tau e^{-V_{\text{eff}}^{ij}(\tau)} \) \( V_{\text{eff}}^{ij}(\tau) = S_{\text{int}}(\tau) + 2N_f \xi \tau \)

- Amplitude gets divergent due to attractive interaction
- Semiclassical description is broken down

BZJ prescription \( g^2 \to -g^2 \) Bogomolyn(79) Zinn-Justin(81)

\[ \tilde{B}_{ii}(g^2, N_f) \propto (-g^2 N/8\pi)^{2N_f} \Gamma(2N_f) e^{-2S_I/N} \]

\[ B_{ii}(g^2, 0) = \left( \log \left( \frac{g^2 N}{8\pi} \right) - \gamma \right) \frac{16}{g^2 N} e^{-2S_I/N} \pm i \frac{16\pi}{g^2 N} e^{-\frac{8\pi}{g^2 N}} \]

\[ [B_{ii}]_{\theta=0}^\pm = \text{Re} [B_{ii}] \pm i \text{Im} [B_{ii}]_{\theta=0}^\pm \] cancellation!

Neutral bion is IR renormalon at least for small L.
Neutral bions as IR-RENORMALON!

**CP(N–1) on R x S\(^{1}\)**

- **UV renormalons:** \( t = -8\pi n/\beta_0 \)
- **Neutral bions:** \( t = 8\pi n/N \quad (n=1,2,...) \)
- **Instanton—anti-instanton singularities:** \( t = 8\pi, 16\pi, ... \)

- Figure 10
  - Upper figure: The conjectured structure of the Borel plane for CP\(^{N–1}\) on R\(^2\)
  - Lower figure: The semisclassical singularities associated with the neutral bion molecules in CP\(^{N–1}\) on small R \(\times\) S\(^{1}\)

- For \( N_f = v \) the weak coupling regime has an extra singularity closer to the origin than the leading renormalon pole on R\(^2\).
- For \( N_f \geq w \) the location of the semisclassical and nonssemisclassical renormalon singularities coincide.
- Although the theory moves from a weakly coupled description to a strongly coupled one, the structure of the Borel plane singularities either do not change at all or change extremely mildly.
- We take this as evidence that the neutral bion molecules are the semisclassical realization of renormalons.
- This also gives us hope that even the theory on R\(^2\) may potentially be solvable at arbitrary N.

- In other words, when \( N_f = v \) we first see the nonperturbative ambiguities in the neutral bion amplitude \( B_{ii} \), while for \( N_f \geq w \) the nonperturbative ambiguities first arise in the neutral correlators of two bions.
- The location of the ambiguities in the semisclassical molecules matches the location of the renormalon singularities on R\(^2\) for \( N_f \geq w \) theories, and for \( N_f = v \) the semisclassical has an extra singularity closer to the origin than the leading renormalon pole on R\(^2\).
- See Figure \( wv \).

- The elegance of this analysis is that a very difficult problem in QFT, tied with the renormalon singularities, reduces to a relatively simpler problem in quantum mechanics without...
Neutral bions as IR-RENORMALON!

$$0 = \text{Im} \left( \mathbb{B}_{[0,0], \theta=0^\pm} + \mathbb{B}_{[2,0], \theta=0^\pm} [B_{ii}]_{\theta=0^\pm} + \mathbb{B}_{[4,0], \theta=0^\pm} [B_{ij}B_{ji}]_{\theta=0^\pm} + \mathbb{B}_{[6,0], \theta=0^\pm} [B_{ij}B_{jk}B_{ki}]_{\theta=0^\pm} + \ldots \right)$$

$$f(\lambda h) \sim \sum_{k=0}^\infty c_{(0,k)} (\lambda h)^k + \sum_{n=1}^\infty (\lambda h)^{-\beta n} e^{-n A/(\lambda h)} \sum_{k=0}^\infty c_{(n,k)} (\lambda h)^k$$

consistent definition of semiclassical QFT?

Need higher-order bion-molecule amplitude!
Classification of bions in Grassmann $\sigma$ models

- Full classification by $D$-brane picture

- Interaction energy and ambiguity

$$S_{\text{int}}(N_F, \tau) \sim -\frac{4}{N_F} e^{-(2\pi/N_F)\tau}$$
Status of Resurgence in quantum theories

1. Quantum mechanics
   - Instanton ambiguities are canceled
   - No other ambiguities

2. Matrix models, String and SQFT
   - Expansions about D-brane or Membrane instantons
   - Consistent with exact results by localization

3. QCD and its cousins
   - May work at the weak-coupling regime (small L)
   - Not clear if it continues to strong-coupling
Summary

• Deeper understanding on QCD(adj) phase diagram and Bion confinement

• Explicit ansatz for neutral bions in CP^N-1 models are found out.

• Leading-order cancellation of renormalon ambiguity is confirmed.
Low-energy theory for $1/L \gg \Lambda_{QCD}$ in center $U(1)$ phase

- Off-diagonal components of gauge and adj have $m \sim 1/L$
- $g(\mu)$ ceases to run for $\mu < 1/L \rightarrow g \ll 1$ for $1/L \gg \Lambda_{QCD}$

\[
S_{\text{eff}} = \int_{\mathbb{R}^3} d^3 x \frac{L}{g^2} \sum_{\ell=1}^{N-1} \left[ \frac{1}{4} F_{ij}^{(\ell)} \right]^2 + \sum_{f=1}^{N_f^D} \overline{\Psi}_f^{(\ell)} \left( \gamma_i \partial_i + i \frac{\phi}{L} \gamma_4 \right) \Psi_f^{(\ell)}
\]

3D $U(1)$ gauge theory with free adjoint fermions
**Chiral symmetry of QCD(adj.) for \( \phi \neq 0 \)**

- Study on center is not sufficient for understanding QCD.
- Chiral and center properties are affected by each other.

- Classical symmetry due to reality of adj

\[
\begin{align*}
\phi &= 0, \pi \\
U(1)_A \times SU(2N_f^D) &\quad \rightarrow \quad U(1)_A \times U(1)_B \times SU(N_f^D)_R \times SU(N_f^D)_L
\end{align*}
\]

- Discrete quantum symmetry

\[
\begin{align*}
N_f = 1 & \quad \frac{(Z_{4N_c})_A \times SU(2)}{Z_2} \quad \text{for } \phi = 0 \text{ or } \pi \\
& \quad \frac{(Z_{4N_c})_A \times U(1)_B}{Z_2} \quad \text{for } 0 < \phi < \pi \\
N_f = 2 & \quad \frac{(Z_{8N_c})_A \times SU(4)}{Z_4} \quad \text{for } \phi = 0 \text{ or } \pi \\
& \quad \frac{(Z_{8N_c})_A \times U(1)_B \times SU(2)_R \times SU(2)_L}{Z_4 \times (Z_2)_{R+L}} \quad \text{for } 0 < \phi < \pi
\end{align*}
\]